Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 11620–11630.
DOI:10.3934/math.2021675
Received: 07 April 2021
Accepted: 09 August 2021
Published: 11 August 2021

Research article

Probabilistic (ω, γ, ϕ)-contractions and coupled coincidence point results

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Abstract: In this paper, we introduce the notion of probabilistic (ω, γ, ϕ)-contraction and establish the existence coupled coincidence points for mixed monotone operators subjected to the introduced contraction in the framework of ordered Menger PM-spaces with Hadžić type t-norm. As an application, a corresponding result in the setup of fuzzy metric space is also obtained.

Keywords: Hadžić type t-norm; probabilistic (ω, γ, ϕ)-contraction; coupled coincidence point; ordered Menger PM-space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

As a generalization of metric spaces, Menger [15] initiated the theory of probabilistic metric spaces (in short, PM-spaces) which replaces the distance function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by a distribution function $F_{p,q} : \mathbb{R} \to [0, 1]$ where for any number t, the value $F_{p,q}(t)$ describes the probability that the distance between p and q is less than t. After the novel work of Schweizer and Sklar [17], the theory received much attention of researchers. This theory has its wide range of applications in random differential as well as integral equations and thus has contributed to its importance in probabilistic functional analysis.

The theory of fixed points in PM-spaces is a part of probabilistic analysis, which has been explored by many authors. In 2010, Ćirić [5] presented a truthful probabilistic version of Banach fixed point principle for general nonlinear contractions. Jachymski [12] produced a counter-example to Ćirić's key lemma, thereby, established a corrected probabilistic version of the Banach fixed point principle. Subsequently, the fixed point theorems in the setup of probabilistic metric spaces for other contraction mappings were further investigated by numerous authors (see [1, 6, 8, 16, 18–20]).

In 1987, Guo and Lakshmikantham [9] introduced the notion of coupled fixed point (CFP), but the theory developed readily with the work of Bhaskar and Lakshmikantham [2] in 2006, where they introduced the notion of mixed monotone property (MMP) in ordered metric spaces. Lakshmikantham and Ćirić [14] extended this notion to the mixed g-monotone property (MgMP) and proved some coupled coincidence point results. Afterwards, much work has been done in this direction by different authors. Recently, the investigation of coupled fixed point and coincidence points has been extended from metric spaces to probabilistic metric spaces (see [6, 19, 20]).

Now, we state some allied definitions and results which are required for the development of the present topic. Denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of nonnegative real numbers and $\mathbb{N}$ the set of positive integers.

Definition 1.1. [11] A function $f : \mathbb{R} \to [0, 1]$ is called a distribution function if it is nondecreasing and left-continuous with $\inf_{x\in\mathbb{R}} f(x) = 0$. If, in addition, $f(0) = 0$, then f is called a distance distribution function. Furthermore, a distance distribution function f satisfying $\lim_{t\to+\infty} f(t) = 1$ is called a Menger distance distribution function.

The set of all Menger distance distribution functions is denoted by $\bigwedge^+$.

Definition 1.2. [10] Let Δ be a t-norm with $\sup_{0<t<1} \Delta(t, t) = 1$. Then Δ is said to be a Hadžić type t-norm (in short, H-type t-norm) if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = t, \quad \Delta^{m+1}(t) = t\Delta(\Delta^m(t)), \quad m = 1, 2, \cdots, t \in [0, 1].$$

The t-norm $\Delta_M = \min$ is an example of an H-type t-norm.

Remark 1.3. A t-norm Δ is an H-type t-norm if and only if for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\Delta^m(t) > 1 - \lambda$ for all $m \in \mathbb{N}$, when $t > 1 - \delta$.

Definition 1.4. [17] A menger probabilistic metric space (in short, Menger PM-space) is a triple (X, F, Δ), wher X is a nonempty set, Δ is a continuous t-norm and F is a mapping from $X \times X$ into $\bigwedge^+$ such that, for the value $F_{p,q}$ of F at the pair (p, q), the following hold:

(PM_1) $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$;
(PM_2) $F_{p,q}(t) = F_{q,p}(t)$ for all $p, q \in X$ and $t > 0$;
(PM_3) $F_{p,r}(t + s) \geq \Delta(F_{p,q}(t), F_{q,r}(s))$ for all $p, q, r \in X$ and $t > 0, s > 0$.

A Menger PM-space (X, F, Δ) endowed with the partial order $\preceq$ is called an ordered Menger PM-space and is denoted by $(X, F, \Delta, \preceq)$. A complete Menger PM-space endowed with the partial order is called an ordered complete Menger PM-spaces. See [3, 21] for more information on Menger PM-spaces and some related results.

Definition 1.5. [2, 9] An element $(x, y) \in X \times X$ is called a coupled fixed point (CFP) of the mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

Definition 1.6. [9] Let $(X, \preceq)$ be a partially ordered set. A mapping $T : X \times X \to X$ is said to have the mixed monotone property (MMP) if $T(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y.

Definition 1.7. [14] Let $(X, \leq)$ be a partially ordered set and $T : X \times X \to X$ and $g : X \to X$ be mappings. We say that T has the mixed g-monotone property (MgMP) if $T(x, y)$ is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument.

Remark 1.8. The self-mapping g on X in Definition 1.7 can be replaced by any other self-mapping A on X and in this case, T is said to have mixed A monotone property (MAMP).

Definition 1.9. [14] Mappings $T : X \times X \to X$ and $A : X \to X$ are said to be commutative if $T(Ax, Ay) = AT(x, y)$ for all $x, y \in X$.

Definition 1.10. [14] An element $(x, y) \in X \times X$ is called a coupled coincidence point (CCP) of the mappings $T : X \times X \to X$ and $A : X \to X$ if $T(x, y) = Ax$ and $T(y, x) = Ay$.

Recently, Ćirić et al. [7] considered some fixed point theorems for the following class of contractive mappings in partially ordered probabilistic metric space:

$$F_{A(x),A(y)}(kt) \geq \min\{F_{h(x),h(y)}(t), F_{h(x),A(x)}(t), F_{h(y),A(y)}(t)\},$$

where $k \in (0, 1)$.

On the other hand, Ćirić et al. [6] obtained coupled coincidence point results for the following condition:

$$F_{T(x,y),T(u,v)}(kt) \geq \min\{F_{A(x),A(u)}(t), F_{A(y),A(v)}(t), F_{A(x),T(x,y)}(t), F_{A(u),T(u,v)}(t), F_{A(y),T(y,x)}(t), F_{A(v),T(v,u)}(t)\},$$

where $k \in (0, 1)$.

It is worth mentioning that using the gauge function ϕ introduced by Jachymski [12], in their remarkable work, Wang et al. [19] obtained coupled coincidence points for the following nonlinear contraction in the setting of ordered Menger PM-spaces:

$$F_{T(x,y),T(u,v)}(\phi(t)) \geq \min\{F_{A(x),A(u)}(t), F_{A(y),A(v)}(t)\}, \tag{1.1}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is the gauge function defined by Jachymski [12] which satisfies the condition: For any $t > 0$, we have $0 < \phi(t) < t$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.

In order to obtain our main results, we further require some notions as follows:

Denote by $\mathcal{W}$ the class of all continuous nondecreasing functions $\omega : [0, 1] \to [0, 1]$ with the property that $\omega(t) = 1$ if and only if $t = 1$. Also, we denote $\mathcal{V}$ the class of all left continuous functions $\gamma : [0, 1] \to [0, 1]$.

As in [12, 19], let Φ denote the class of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $0 < \phi(t) < t$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for $t > 0$. Indeed, if $\sum_{n=1}^{\infty} \phi^n(t)$ converges for all $t > 0$, then $\lim_{n\to\infty} \phi^n(t) = 0$.

Lemma 1.11. *Let $\gamma \in \mathcal{V}$ and $\omega \in \mathcal{W}$. Assume that $\gamma(a) \geq \omega(a)$ for all $a \in [0, 1]$. Then $\gamma(1) = 1$.*

Proof. Let $\{a_n\} \subset (0, 1)$ be a nondecreasing sequence with $\lim_{n\to\infty} a_n = 1$. By hypothesis, we have $\gamma(a_n) \geq \omega(a_n)$ for all $n \in \mathbb{N}$. Using the properties of γ and ω, we can obtain that $\gamma(1) \geq \omega(1) = 1$, which implies that $\gamma(1) = 1$. This completes the proof. □

Definition 1.12. Let $(X, F, \Delta, \leq)$ be an ordered Menger PM-space, where Δ is a continuous Hadžić type t-norm.

(i) A mapping $T : X\times X \to X$ is said to be a probabilistic (ω, γ, ϕ)-contraction if there exist functions $\omega \in \mathcal{W}$, $\gamma \in \mathcal{V}$ with $\gamma(a) \geq \omega(a)$ for all $a \in [0, 1]$ and $\phi \in \Phi$ such that

$$\omega(\Delta(F_{T(x,y),T(u,v)}(\phi(t)), F_{T(y,x),T(v,u)}(\phi(t)))) \geq \gamma(\Delta(F_{x,u}(t), F_{y,v}(t))) \tag{1.2}$$

for all $t > 0$ and $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$.

(ii) A mapping $T : X \times X \to X$ is said to be a probabilistic (ω, γ, ϕ)-contraction with respect to a mapping $A : X \to X$ if there exist functions $\omega \in \mathcal{W}$, $\gamma \in \mathcal{V}$ with $\gamma(a) \geq \omega(a)$ for all $a \in [0, 1]$ and $\phi \in \Phi$ such that

$$\omega(\Delta(F_{T(x,y),T(u,v)}(\phi(t)), F_{T(y,x),T(v,u)}(\phi(t)))) \geq \gamma(\Delta(F_{Ax,Au}(t), F_{Ay,Av}(t))) \tag{1.3}$$

for all $t > 0$ and $x, y, u, v \in X$ with $Ax \leq Au$ and $Ay \geq Av$.

Definition 1.13. A Menger PM-space (X, F, Δ) is said to be regular if the following hold:

(X_1) If a nondecreasing sequence $\{x_n\} \subset X$ converges to $x \in X$, then $x_n \leq x$ for all $n \in \mathbb{N}$;
(X_2) If a nonincreasing sequence $\{y_n\} \subset X$ converges to $y \in X$, then $y \leq y_n$ for all $n \in \mathbb{N}$.

In this communication, inspired by the works of Jachymski [12] and Wang et al. [19], we establish some coupled coincidence point results for a mixed monotone operators satisfying probabilistic (ω, γ, ϕ)-contractions in the setup of ordered Menger PM-spaces with Hadžić type t-norms. Suitable example has been given in support of the presented work. Further, as an application, a corresponding result in the framework of fuzzy metric space has also been established.

2. Main results

In this section, we prove our main results.

Theorem 2.1. *Let $(X, F, \Delta, \leq)$ be an ordered Menger PM-space, where Δ is a continuous Hadžić type t-norm. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that the following hold:*

(i) One of the range subspaces $T(X \times X)$ and $A(X)$ is complete;
(ii) $T(X \times X) \subset A(X)$;
(iii) T has MAMP;
(iv) There exist two elements $x_0, y_0 \in X$ such that $Ax_0 \leq T(x_0, y_0)$ and $Ay_0 \geq T(y_0, x_0)$;
(v) T is a probabilistic $\omega, \gamma, \phi)$-contraction with respect to A.

Suppose that X is regular. Then A and T have a CCP in X.

Proof. By (iv), there exist $x_0, y_0 \in X$ such that $Ax_0 \leq T(x_0, y_0)$ and $Ay_0 \geq T(y_0, x_0)$. Using (ii), we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying $Ax_{n+1} = T(x_n, y_n)$ and $Ay_{n+1} = T(y_n, x_n)$ for $n = 1, 2, \cdots$. Again by (iv), $Ax_0 \leq T(x_0, y_0) = Ax_1$ and $Ay_0 \geq T(y_0, x_0) = Ay_1$ and then using (iii), we obtain that $Ax_1 = T(x_0, y_0) \leq T(x_1, y_1) = Ax_2$ and $Ay_1 = T(y_0, x_0) \geq T(y_1, x_1) = Ay_2$. Applying induction, we obtain that

$$Ax_{n-1} \leq Ax_n \qquad \& \qquad Ay_{n-1} \geq Ay_n \tag{2.1}$$

for all $n \in \mathbb{N}$. Since Δ is a Hadžić type t-norm, for any $\eta > 0$, there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$,

$$\frac{(1-\varepsilon)\Delta(1-\varepsilon)\Delta\cdots\Delta(1-\varepsilon)}{k} > 1-\eta. \tag{2.2}$$

By (1.3) and (2.1), for $n \in \mathbb{N}$ and $t > 0$, we have

$$\begin{aligned}
&\omega(\Delta(F_{Ax_{n+1},Ax_n}(\phi(t)), F_{Ay_{n+1},Ay_n}(\phi(t))))\\
&\quad = \omega(\Delta(F_{T(x_n,y_n),T(x_{n-1},y_{n-1})}(\phi(t)), F_{T(y_n,x_n),T(y_{n-1},x_{n-1})}(\phi(t))))\\
&\quad \geq \gamma(\Delta(F_{Ax_{n-1},Ax_n}(t), F_{Ay_{n-1},Ay_n}(t)))\\
&\quad \geq \omega(\Delta(F_{Ax_{n-1},Ax_n}(t), F_{Ay_{n-1},Ay_n}(t))).
\end{aligned}$$

By the monotone property of the function ω, for all $t > 0$, we can obtain that

$$\Delta(F_{Ax_{n+1},Ax_n}\phi(t), F_{Ay_{n+1},Ay_n}(\phi(t))) \geq \Delta(F_{Ax_{n-1},Ax_n}(t), F_{Ay_{n-1},Ay_n}(t)).$$

Then inductively, for $n \in \mathbb{N}$ and $t > 0$, we get that

$$\Delta(F_{Ax_{n+1},Ax_n}\phi^n(t), F_{Ay_{n+1},Ay_n}(\phi^n(t))) \geq \Delta(F_{Ax_0,Ax_1}(t), F_{Ay_0,Ay_1}(t)).$$

Since $\lim_{t\to+\infty} F_{x,y}(t) = 1$, for all $x, y \in X$, there exists $t_0 > 0$ such that

$$F_{Ax_0,Ax_1}(t_0) > 1-\varepsilon \qquad \& \qquad F_{Ay_0,Ay_1}(t_0) > 1-\varepsilon.$$

Since $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$, for $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that $\phi^n(t_0) < \delta$ for all $n \geq N_0$. Thus, for all $n \geq N_0$, we obtain that

$$\begin{aligned}
\Delta(F_{Ax_n,Ax_{n+1}}(\delta), F_{Ay_n,Ay_{n+1}}(\delta)) &\geq \Delta(F_{Ax_n,Ax_{n+1}}(\phi^n(t_0)), F_{Ay_n,Ay_{n+1}}(\phi^n(t_0)))\\
&\geq \Delta(F_{Ax_0,Ax_1}(t_0), F_{Ay_0,Ay_1}(t_0))\\
&> (1-\varepsilon)\Delta(1-\varepsilon) > 1-\eta
\end{aligned}$$

by (2.2). From this, it is easy to conclude that

$$\lim_{n\to\infty} \Delta(F_{Ax_n,Ax_{n+1}}(t), F_{Ay_n,Ay_{n+1}}(t)) = 1 \tag{2.3}$$

for all $t > 0$.

Next, we prove that the sequences $\{Ax_n\}$ and $\{Ay_n\}$ are Cauchy sequences. To this end, firstly, we shall show that for all $k \in \mathbb{N}$, the following inequality holds by induction:

$$\Delta(F_{Ax_n,Ax_{n+k}}(\delta), F_{Ay_n,Ay_{n+k}}(\delta)) \geq \Delta^n(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta)))). \tag{2.4}$$

For $k = 1$,

$$\begin{aligned}
\Delta(F_{Ax_n,Ax_{n+1}}(\delta), F_{Ay_n,Ay_{n+1}}(\delta)) &\geq \Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta)))\\
&= \Delta(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))), 1)\\
&\geq \Delta(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))),\\
&\qquad \Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))))
\end{aligned}$$

$$= \Delta^1(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta)))).$$

Thus (2.4) holds for $k = 1$.

We now assume that (2.4) holds for $1 \le k \le p$ and some $p \in \mathbb{N}$. We next prove that (2.4) holds for $k = p + 1$. By (PM_3), we have

$$\begin{aligned}&\Delta(F_{Ax_n,Ax_{n+p+1}}(\delta), F_{Ay_n,Ay_{n+p+1}}(\delta))\\&\quad\ge \Delta(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))),\\&\qquad \Delta(F_{Ax_{n+1},Ax_{n+p+1}}(\delta-\phi(\delta)), F_{Ay_{n+1},Ay_{n+p+1}}(\delta-\phi(\delta)))).\end{aligned} \tag{2.5}$$

By (1.3) and (2.1), we have

$$\begin{aligned}&\omega(\Delta(F_{Ax_{n+1},Ax_{n+p+1}}(\phi(\delta)), F_{Ay_{n+1},Ay_{n+p+1}}(\phi(\delta))))\\&\quad= \omega(\Delta(F_{T(x_n,y_n),T(x_{n+p},y_{n+p})}(\phi(\delta)), F_{T(y_n,x_n),T(y_{n+p},x_{n+p})}(\phi(\delta))))\\&\quad\ge \gamma(\Delta(F_{Ax_n,Ax_{n+p}}(\delta), F_{Ay_n,Ay_{n+p}}(\delta)))\\&\quad\ge \omega(\Delta(F_{Ax_n,Ax_{n+p}}(\delta), F_{Ay_n,Ay_{n+p}}(\delta))).\end{aligned}$$

By the monotone property of the function ω and the fact that (2.4) holds for $p \in \mathbb{N}$, we can obtain that

$$\begin{aligned}\Delta(F_{Ax_{n+1},Ax_{n+p+1}}(\phi(\delta)), F_{Ay_{n+1},Ay_{n+p+1}}(\phi(\delta))) &\ge \Delta(F_{Ax_n,Ax_{n+p}}(\delta), F_{Ay_n,Ay_{n+p}}(\delta))\\&\ge \Delta^p(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta)))).\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$\Delta(F_{Ax_n,Ax_{n+p+1}}(\delta), F_{Ay_n,Ay_{n+p+1}}(\delta)) \ge \Delta^{p+1}(\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta)))).$$

Thus (2.4) holds for $k = p + 1$ and so (2.4) holds for all $k \in \mathbb{N}$.

Since Δ is a Hadžić type t-norm, for $\varepsilon \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that for $t > 1 - \lambda$, we have

$$\delta^n(t) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. On the other hand, by (2.3), we have

$$\lim_{n\to\infty} \Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))) = 1,$$

which implies that there exists $N_1(\varepsilon, \delta) \in \mathbb{N}$ such that

$$\Delta(F_{Ax_n,Ax_{n+1}}(\delta-\phi(\delta)), F_{Ay_n,Ay_{n+1}}(\delta-\phi(\delta))) > 1 - \lambda$$

for all $n > N_1(\varepsilon, \delta)$. So

$$\Delta(F_{Ax_n,Ax_{n+k}}(\delta), F_{Ay_n,Ay_{n+k}}(\delta)) > 1 - \lambda$$

for all $k \ge 1$ and $n > N_1(\varepsilon, \delta)$. This implies that the sequences $\{Ax_n\}$ and $\{Ay_n\}$ are Cauchy in $A(X)$. Without loss of generality, assume that the subspace $A(X)$ is complete. Then by completeness of $A(X)$, there exist $x, y \in X$ such that $\lim_{n\to\infty} Ax_n = Ax$ and $\lim_{n\to\infty} Ay_n = Ay$. Since X has the properties (X_1)

and (X_2), we have $Ax_n \preceq Ax$ and $Ay_n \succeq Ay$ for sufficiently large n. For such large n and $t > 0$, from (1.3), we have

$$\omega(\Delta(F_{Ax_{n+1},T(x,y)}(t), F_{Ay_{n+1},T(y,x)}(t))) \geq \omega(\Delta(F_{Ax_{n+1},T(x,y)}(\phi(t)), F_{Ay_{n+1},T(y,x)}(\phi(t))))$$
$$= \omega(\Delta(F_{T(x_n,y_n),T(x,y)}(\phi(t)), F_{T(y_n,x_n),T(y,x)}(\phi(t)))) \geq \gamma(\Delta(F_{Ax_n,Ax}(t), F_{Ay_n,Ay}(t))).$$

Letting $n \to \infty$, using the properties of ω, γ and Lemma 1.11, we have

$$\omega(\Delta(F_{Ax,T(x,y)}(t), F_{Ay,T(y,x)}(t))) \geq \gamma(\Delta(1,1)) = \gamma(1) = 1$$

for $t > 0$. It follows that

$$\omega(\Delta(F_{Ax,T(x,y)}(t), F_{Ay,T(y,x)}(t))) = 1$$

and so

$$F_{Ax,T(x,y)}(t) = 1, \qquad F_{Ay,T(y,x)}(t) = 1$$

for all $t > 0$. Hence $T(x,y) = Ax$ and $T(y,x) = Ay$. This completes the proof. □

Corollary 2.2. *Let $(X, F, \Delta, \preceq)$ be a complete ordered Menger PM-space, where Δ is a continuous Hadžić type t-norm. Let $T : X \times X \to X$ be a mapping such that the following hold:*

(i) T has MMP;
(ii) There exist two elements $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$;
(iii) T is a probabilistic $\omega, \gamma, \phi)$-contraction.

Suppose that X is regular. Then T has a CFP in X.

Theorem 2.3. *Let $(X, F, \Delta, \preceq)$ be an ordered Menger PM-space, where Δ is a continuous Hadžić type t-norm. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings satisfying the conditions (i)–(iv) of Theorem 2.1 and the following condition:*

$$\omega(\Delta(F_{T(x,y),T(u,v)}(\varphi(t)), F_{T(y,x),T(v,u)}(\varphi(t)))) \geq \gamma(\Delta(F_{Ax,Au}(t), F_{Ay,Av}(t))) \tag{2.7}$$

for all $t > 0$ and $x, y, u, v \in X$ with $Ax \preceq Au$ and $Ay \succeq Av$ and for some function $\omega \in \mathcal{W}$, $\gamma \in \mathcal{V}$ with the condition that $\gamma(a) \geq \omega(a)$ for $a \in [0,1]$, where the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that $0 < \varphi(t) < t$ and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. Suppose that X is regular. Then A and T have a CCP in X.

Proof. Since $\varphi(t) < t$ and $\sum_{n=1}^{\infty} \varphi^n(t)$ converges for all $t > 0$, it follows that $\varphi \in \Phi$ and by (2.7), the mapping T becomes a probabilistic (ω, γ, ϕ)-contraction with respect to the mapping A. Hence the result follows immediately from Theorem 2.1. □

We now give an example in support of the present work.

Example 2.4. Let $(X, \preceq)$ be a partially ordered set with $X = (-1, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preceq$. Let F be a mapping from $X \times X$ into $\bigwedge^+$ defined by

$$F_{x,y}(t) = H(t - |x-y|) := \begin{cases} 0, & t \leq |x-y| \\ 1, & t > |x-y| \end{cases}$$

for $x, y \in X$. Then (X, F, Δ_M) is a Menger PM-space. Define mappings $A : X \to X$ and $T : X \times X \to X$ by $Ax = x^2$ and $T(x,y) = \frac{y^2 - x^2 + 2}{8}$ for all $x, y \in X$. Clearly, the mappings T and A are not compatible.

The mapping A is neither monotonically increasing nor monotonically decreasing on X. The mapping T satisfies the MAMP. Clearly, $Ax = [0, 1]$ is complete and $T(X \times X) \subset A(X)$. Also, for $x_0 = 0.4$ and $y_0 = 0.6$, we have $Ax_0 \preceq T(x_0, y_0)$ and $Ay_0 \succeq T(y_0, x_0)$. We define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = \frac{t}{3}$ for $t \geq 0$ and the functions $\gamma, \omega : [0, 1] \to [0, 1]$ by $\gamma(s) = s^{\frac{1}{2}}$ and $\omega(s) = s^{\frac{1}{3}}$ for $s \in [0, 1]$, respectively.

Now, we verify (1.3). For $t > 0$ and $x, y, u, v \in X$ with $Ax \preceq Au$ and $Ay \succeq Av$, the inequality (1.3) takes the following form:

$$\begin{aligned}&[\min\{H(\phi(t) - |T(x, y) - T(u, v)|), H(\phi(t) - |T(y, x) - T(v, u)|)\}]^{\frac{1}{3}}\\ &\quad \geq [\min\{H(t - |Ax - Au|), H(t - |Ay - Av|)\}]^{\frac{1}{2}}.\end{aligned}$$

That is,

$$\begin{aligned}&\left[\min\left\{H\left(\frac{t}{3} - \left|\frac{(y^2 - x^2) - (v^2 - u^2)}{8}\right|\right), H\left(\frac{t}{3} - \left|\frac{(x^2 - y^2) - (u^2 - v^2)}{8}\right|\right)\right\}\right]^{\frac{1}{3}}\\ &\quad \geq [\min\{H(t - |x^2 - u^2|), H(t - |y^2 - v^2|)\}]^{\frac{1}{2}},\end{aligned}$$

that is,

$$\left[H\left(\frac{t}{3} - \left|\frac{(x^2 - y^2) - (u^2 - v^2)}{8}\right|\right)\right]^{\frac{1}{3}} \geq [\min\{H(t - |x^2 - u^2|), H(t - |y^2 - v^2|)\}]^{\frac{1}{2}}.$$

By the definition of H, we only need to verify that

$$|(x^2 - y^2) - (u^2 - v^2)| < \frac{8}{3}t \tag{2.8}$$

if $|x^2 - u^2| < t$ and $|y^2 - v^2| < t$. Now, by (2.8), we have

$$|(x^2 - y^2) - (u^2 - v^2)| = |(x^2 - u^2) - (y^2 - v^2)| \leq |x^2 - u^2| + |y^2 - v^2| < 2t < \frac{8}{3}t.$$

Therefore, (1.3) holds. Thus all the conditions of Theorem 2.1 are satisfied. Applying Theorem 2.1, we obtain that $(\frac{1}{2}, \frac{1}{2})$ is a CCP of the mappings T and A.

3. Application to fuzzy metric spaces

In this section, as an application of the main result established in Section 2, we formulate a corresponding result in the framework of fuzzy metric spaces. First, we shall recall the concept of fuzzy metric spaces as follows:

Definition 3.1. [13] A fuzzy metric space in the sense of Kramosil and Michalek (in short, KM-fuzzy metric space) is a triple (X, M, Δ), where X is a nonempty set, Δ is a continuous t-norm and M is a fuzzy set on $X^2 \times \mathbb{R}^+$ satisfying the following conditions: for all $x, y, z \in X$ and $s > 0, t > 0$,

(KM-1) $M(x, y, 0) = 0$;
(KM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(KM-3) $M(x,y,t) = M(y,x,t)$;
(KM-4) $M(x,z,t+s) \geq \Delta(M(x,y,t), M(y,z,s))$;
(KM-5) $M(x,y,\cdot) : \mathbb{R}^+ \to [0,1]$ is left continuous.

Let (X, M, Δ) be a KM-fuzzy metric space. It is well-known that if $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$, then (X, F, Δ) is a Menger PM-space, where $F_{x,y}(t) = \begin{cases} M(x,y,t), & t \leq 0 \\ 0, & t < 0 \end{cases}$. Thus the conclusion of Theorem 2.1 holds in a KM-fuzzy metric space and so we can state the conclusion as follows:

Theorem 3.2. *Let $(X, M, \Delta, \preceq)$ be an ordered KM-fuzzy metric space, where Δ is a continuous Hadžić type t-norm and $M(x,y,t) \to 1$ as $t \to \infty$. Let $T : X \times X \to X$ and $A : X \to X$ be two mappings such that the following hold:*

(i) One of the range subspaces $T(X \times X)$ and $A(X)$ is complete;
(ii) $T(X \times X) \subset A(X)$;
(iii) T has MAMP;
(iv) There exist two elements $x_0, y_0 \in X$ such that $Ax_0 \preceq T(x_0, y_0)$ and $Ay_0 \succeq T(y_0, x_0)$;
(v) There are functions $\omega \in \mathcal{W}, \gamma \in \mathcal{V}$ with $\gamma(a) \geq \omega(a)$ for all $a \in [0,1]$ and $\phi \in \Phi$ such that

$$\begin{aligned} &\omega(\Delta(M(T(x,y), T(u,v), T(u,v), \phi(t)), M(T(y,x), T(v,u), \phi(t)))) \\ &\quad \geq \gamma(\Delta(M(Ax, Au, t), M(Ay, Av, t))) \end{aligned}$$

for all $x, y, u, v \in X$ and $t > 0$ with $Ax \preceq Au$ and $Ay \succeq Av$. Suppose that X is regular. Then A and T have a CCP in X.

Remark 3.3. By assigning different value to the functions ω, γ and ϕ, we can obtain a number of generalizations from Theorems 2.1, 2.3 and 3.2. In particular, in Theorem 3.2, taking ω, γ to be identity functions on $[0,1]$ and assigning $\phi(t) = kt$, where $k \in (0,1)$, we obtain a generalization of the recent work of Choudhury et al. [4].

4. Conclusions

The produced results in this work can generalize a number of results presented in the literature of coupled fixed point theory of Menger PM-spaces and can be clarified by using the following facts:
1). Our results do not require the condition of commutativity or compatibility of the pair (A, T) of mappings and our results do not impose the monotonic condition on the self-mapping A which is the essential requirement in almost all the coupled coincidence point results presented in the literature.
2). Furthermore, the assumption of the completeness of the space X has also been replaced by the completeness of the range subspaces of any one of the mappings T or A. Hence, it can be easily seen that the methodology presented in the current work can be applied in a much wider class of problems.

Acknowledgements

We would like to express our sincere gratitude to the anonymous referee for his/her helpful comments that will help to improve the quality of the manuscript.

Conflict of interest

The authors declare that they have no competing interests.

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