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Research article

Proof of a Dwork-type supercongruence by induction

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Abstract: In this paper we prove a Dwork-type supercongruence: for any prime $p \geq 3$ and integer $r \geq 1$,

$$\sum_{k=0}^{p^r-1} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv p \sum_{k=0}^{p^{r-1}-1} \frac{3k+1}{16^k} \binom{2k}{k}^3 \pmod{p^{3r+1-\delta_{p,3}}},$$

which extends a result of Guo and Zudilin.

Keywords: supercongruences; Dwork-type supercongruences; generalized harmonic numbers of order m; induction; Jacobsthal's binomial congruence
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1. Introduction

In the spirit of [25], the supercongruence

$$\sum_{k=0}^{p-1} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv p \pmod{p^3} \qquad \text{for } p > 2 \tag{1.1}$$

corresponds to a divergent Ramanujan-type series for $1/\pi$. In [8], Guillera and Zudilin made use of the Wilf-Zeilberger algorithmic technique to show that

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv p \pmod{p^3} \qquad \text{for } p > 2. \tag{1.2}$$

Note that (1.2) is equivalent to (1.1) since $p \mid \binom{2k}{k}$ for $\frac{p+1}{2} \leq k \leq p-1$. The supercongruence (1.2) was independently observed numerically by Sun [21, Conjecture 5.1 (ii)]).

Given a prime p and a positive integer k, for any positive integer n, we call a congruence as a Dwork-type congruence on a sequence $\{a_n\}_{n\geq 0}$ of integers if

$$a_{np} \equiv \gamma_p \cdot a_n \pmod{p^{k\nu_p(n)}},$$

where $\nu_p(n)$ is the largest integer such that $p^{\nu_p(n)} \mid n$. We refer the reader to [2, 3, 5, 14] for Dwork-type congruences and [9, 11, 16, 22] for q-analogues of Dwork-type congruences.

For any odd prime p and integer $r \geq 1$, Guo [10] gave a q-analogue of (1.2) and made the following Dwork-type supercongruence conjecture:

$$\sum_{k=0}^{p^r-1} \frac{3k+1}{16^k}\binom{2k}{k}^3 \equiv p \sum_{k=0}^{p^{r-1}-1} \frac{3k+1}{16^k}\binom{2k}{k}^3 \pmod{p^{4r-\delta_{p,3}}} \tag{1.3}$$

and

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{3k+1}{16^k}\binom{2k}{k}^3 \equiv p \sum_{k=0}^{\frac{p^{r-1}-1}{2}} \frac{3k+1}{16^k}\binom{2k}{k}^3 \pmod{p^{3r}}, \tag{1.4}$$

where the Kronecker delta function is defined as $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise. Guo and Zudilin [11] proved (1.3) and (1.4) modulo p^{3r} by establishing their q-analogues.

Recently, Sun [20] conjectured that, for any prime $p > 3$ and positive integers n, r,

$$\frac{16^{np^{r-1}}}{n^4 p^{4r}\binom{2np^{r-1}}{np^{r-1}}^4}\Big(\sum_{k=0}^{np^r-1} \frac{3k+1}{16^k}\binom{2k}{k}^3 - p\sum_{k=0}^{np^{r-1}-1} \frac{3k+1}{16^k}\binom{2k}{k}^3\Big) \equiv \frac{7}{3}B_{p-3} \pmod{p}, \tag{1.5}$$

where B_n are the Bernoulli numbers which can be defined by

$$B_0 = 1, \qquad \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0 \ (n \geq 2),$$

and

$$\begin{aligned}
&\frac{4^{np^{r-1}-1}}{n^3 p^{3r}\binom{np^{r-1}-1}{(np^{r-1}-1)/2}^3}\Big(\sum_{k=0}^{\frac{np^r-1}{2}} \frac{3k+1}{16^k}\binom{2k}{k}^3 - p\sum_{k=0}^{\frac{np^{r-1}-1}{2}} \frac{3k+1}{16^k}\binom{2k}{k}^3\Big)\\
&\equiv 2\Big(\frac{-1}{p}\Big)E_{p-3} \pmod{p},
\end{aligned} \tag{1.6}$$

where E_n are the Euler numbers, i.e.,

$$E_0 = 1, \qquad E_n = -\sum_{k=1}^{\lfloor n/2\rfloor}\binom{n}{2k}E_{n-2k} \ (n \geq 1)$$

and $(\frac{\cdot}{p})$ denotes the Legendre symbol. Clearly (1.3) and (1.4) follow from (1.5) and (1.6) by taking $n = 1$, respectively. In this paper, we shall prove the following result, which asserts that (1.3) is true modulo $p^{3r+1-\delta_{p,3}}$.

Theorem 1.1 Let $p \geq 3$ be a prime and r be a positive integer. Then

$$\sum_{k=0}^{p^r-1} \frac{3k+1}{16^k}\binom{2k}{k}^3 \equiv p \sum_{k=0}^{p^{r-1}-1} \frac{3k+1}{16^k}\binom{2k}{k}^3 \pmod{p^{3r+1-\delta_{p,3}}}. \tag{1.7}$$

The remainder of the paper is organized as follows. In the next section, we are going to show some basic lemmas for Theorem 1.1. The proof of Theorem 1.1 will be given in Section 3.

2. Some basic lemmas

From now on, for an assertion A we adopt the notation:

$$[A] = \begin{cases} 1, & \text{if } A \text{ holds,} \\ 0, & \text{otherwise.} \end{cases}$$

We know that $[m = n]$ coincides with the Kronecker delta function $\delta_{m,n}$. For positive integers n and m, the generalized harmonic numbers of order m are defined by $H_{n,m} = \sum_{j=1}^{n} \frac{1}{j^m}$. Clearly, $H_{n,1} = H_n = \sum_{j=1}^{n} \frac{1}{j}$. For convenience, we always abbreviate the Harmonic number $H_{k\in V, p\nmid k} = \sum_{k\in V, p\nmid k} \frac{1}{k}$ as $H'_{k\in V}$.

The following lemma is a result of Beukers [3, Lemma 2 (i)].

Lemma 2.1 Let n, k, r be positive integers and p be a prime. Then

$$\begin{aligned}\binom{p^r n-1}{k} \equiv& \binom{p^{r-1}n-1}{\lfloor \frac{k}{p} \rfloor}(-1)^{k-\lfloor \frac{k}{p} \rfloor}\Bigg\{1 - np^r \sum_{j=1, p\nmid j}^{k} \frac{1}{j} \\ &+ \frac{n^2 p^{2r}}{2}\Bigg(\Big(\sum_{j=1, p\nmid j}^{k} \frac{1}{j}\Big)^2 - \sum_{j=1, p\nmid j}^{k} \frac{1}{j^2}\Bigg)\Bigg\} \pmod{p^{3r}}. \end{aligned} \tag{2.1}$$

Proof. Note that

$$\begin{aligned}(-1)^k\binom{p^r n-1}{k} &= \prod_{j=1}^{k} \frac{j-p^r n}{j} = (-1)^{\lfloor \frac{k}{p} \rfloor}\binom{p^{r-1}n-1}{\lfloor \frac{k}{p} \rfloor} \prod_{j=1, p\nmid j}^{k}\left(1 - \frac{p^r n}{j}\right) \\ &\equiv (-1)^{\lfloor \frac{k}{p} \rfloor}\binom{p^{r-1}n-1}{\lfloor \frac{k}{p} \rfloor}\Bigg(1 - \sum_{j=1, p\nmid j}^{k} \frac{p^r n}{j} + \sum_{\substack{1\leq i<j\leq k \\ p\nmid ij}} \frac{n^2 p^{2r}}{ij}\Bigg) \pmod{p^{3r}}. \end{aligned}$$

This proves (2.1). □

The following result is due to Gy [12, Lemma 2.7].

Lemma 2.2 Let $n \geq 1$ be an integer and p be an odd prime such that $p > \frac{n+1}{2}$. We have the following congruence:

$$\frac{2^{p^n - p^{n-1}} - 1}{p^n} \equiv \sum_{j=0}^{n-1} (-1)^j \frac{q_p^{j+1}(2)}{j+1} p^j + \delta_{p,n+1} q_p(2) p^{n-1} \pmod{p^n},$$

where $q_r(a) = \frac{a^{\phi(r)}-1}{r}$ denotes Euler's quotient of an integer r with base a and $\phi(r)$ is Euler's totient function.

The following lemma involving Bernoulli numbers is due to Lin [13, Corollary 4.3].

Lemma 2.3 Let p be an odd prime and j be a positive integer with $j \le p-2$. Assume that $(2,d) = 1$. Then, we have

$$\sum_{k=1, p\nmid k}^{\lfloor \frac{dp}{2} \rfloor} \frac{1}{k} \equiv 2\sum_{k=1}^{j} (-1)^k \frac{q_p^k(2)p^{k-1}}{k} - \sum_{k=1}^{j-1} (2-2^{-k}) \frac{B_{p^{j-k-1}(p-1)-k}}{p^{j-k-1}+k}(-dp)^k \pmod{p^j}. \tag{2.2}$$

We also need three more lemmas.

Lemma 2.4 For primes p and positive integers a, b, r, s, we have

$$\binom{p^r a}{p^s b} \equiv \binom{p^{r-1}a}{p^{s-1}b}(-1)^{p^s b - p^{s-1}b} \pmod{p^{2r+\min\{r,s\}-\delta_{p,3}-2\delta_{p,2}}}. \tag{2.3}$$

The congruence (2.3) is called Jacobsthal's binomial congruence. Gessel [6], Granville [7] and Straub [18] gave extensions of (2.3) to nonnegative integers a, b and negative integers a, b independently.

Lemma 2.5 Let $p \ge 3$ be an odd prime and l be a nonnegative integer. For any positive integer s, we have

$$H'_{p^s l + p^s - 1, 2} \equiv 0 \pmod{p^{s-\delta_{p,3}}}. \tag{2.4}$$

$$H'_{p^s l + p^s - 1} \equiv 0 \pmod{p^{2s-\delta_{p,3}}}. \tag{2.5}$$

$$H'_{p^s l + \frac{p^s-1}{2}, 2} \equiv 0 \pmod{p^{s-\delta_{p,3}}}. \tag{2.6}$$

$$H'_{p^s l + \frac{p^s-1}{2}} \equiv H'_{\frac{p^s-1}{2}} \pmod{p^{2s-\delta_{p,3}}}. \tag{2.7}$$

$$H_{\frac{p-1}{2}} \equiv -2q_p(2) + pq_p^2(2) \pmod{p^2}. \tag{2.8}$$

Proof. The case $p \ge 5$ in (2.4) and the case $p = 3$ in (2.5) are due to Beukers [3, Lemma 2.1] and Cai [4, Lemma 1], respectively. By (2.5),

$$\sum_{k=1, p\nmid k}^{p^s-1} \frac{1}{k} = \sum_{k=1, p\nmid k}^{\frac{p^s-1}{2}} \left(\frac{1}{k} + \frac{1}{p^s - k}\right) = \sum_{k=1, p\nmid k}^{\frac{p^s-1}{2}} \frac{p^s}{p^s k - k^2} \equiv -p^s H'_{\frac{p^s-1}{2}, 2} \pmod{p^{2s}},$$

we get

$$H'_{\frac{p^s-1}{2}, 2} \equiv 0 \pmod{p^{s-\delta_{p,3}}}. \tag{2.9}$$

By (2.4) and (2.9), we immediately get

$$H'_{p^sl+\frac{p^s-1}{2},2}=\sum_{j=1,p\nmid j}^{p^sl-1}\frac{1}{j^2}+\sum_{j=1,p\nmid j}^{\frac{p^s-1}{2}}\frac{1}{(p^sl+j)^2}\equiv H'_{\frac{p^s-1}{2},2}\equiv 0\pmod{p^{s-\delta_{p,3}}}.$$

So (2.6) is valid. With the help of (2.5) and (2.9), we have

$$\begin{aligned}H'_{p^sl+\frac{p^s-1}{2}}&=\sum_{j=1,p\nmid j}^{p^sl-1}\frac{1}{j}+\sum_{j=1,p\nmid j}^{\frac{p^s-1}{2}}\frac{1}{p^sl+j}\equiv\sum_{j=1,p\nmid j}^{\frac{p^s-1}{2}}\frac{1}{j}\Big(1-\frac{p^sl}{j}\Big)\\&=H'_{\frac{p^s-1}{2}}-p^slH'_{\frac{p^s-1}{2},2}\equiv H'_{\frac{p^s-1}{2}}\pmod{p^{2s-\delta_{p,3}}}.\end{aligned}$$

This proves (2.7).

Finally, Cai [4, Theorem 1] showed that, for any odd integer $n>1$,

$$\sum_{k=1,\,(k,n)=1}^{\frac{n-1}{2}}\frac{1}{k}\equiv -2q_n(2)+nq_n^2(2)\pmod{n^2}. \tag{2.10}$$

The congruence (2.8) is just the $n=p$ case of the above congruence. □

Lemma 2.6 Let $p\geq 5$ be a prime. Then

$$\sum_{t=0}^{\frac{p-3}{2}}\frac{H_t}{2t+1}\equiv -2q_p^2(2)\pmod{p}. \tag{2.11}$$

Proof. Using the principle of mathematical induction, Alzer et al. [1] showed that

$$\sum_{k=1}^{n}\frac{H_k}{k}=\frac{1}{2}\Big(H_n^2+H_{n,2}\Big).$$

Note that

$$\sum_{t=0}^{\frac{p-3}{2}}\frac{H_t}{2t+1}=\sum_{t=1}^{\frac{p-1}{2}}\frac{H_{\frac{p-1}{2}-t}}{2(\frac{p-1}{2}-t)+1}=\sum_{t=1}^{\frac{p-1}{2}}\frac{H_{\frac{p-1}{2}-t}}{p-2t}\equiv-\frac{1}{2}\sum_{t=1}^{\frac{p-1}{2}}\frac{H_{\frac{p-1}{2}+t}}{t}\pmod{p},$$

and

$$H_{p-k}\equiv H_{k-1}\pmod{p} \tag{2.12}$$

for any positive integer $k\leq p-1$ (see Sun [19, Lemma 2.1]). Using

$$\sum_{t=1}^{\frac{p-1}{2}}\frac{H_{\frac{p-1}{2}+t}}{t}=\sum_{t=1}^{\frac{p-1}{2}}\frac{H_t}{t}+\sum_{t=1}^{\frac{p-1}{2}}\frac{1}{t}\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{t+i}$$

and

$$\sum_{t=1}^{\frac{p-1}{2}}\frac{1}{t}\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{t+i}=\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{i}\sum_{t=1}^{\frac{p-1}{2}}\left(\frac{1}{t}-\frac{1}{t+i}\right)=H_{\frac{p-1}{2}}^2-\sum_{t=1}^{\frac{p-1}{2}}\frac{1}{t}\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{t+i},$$

the congruence (2.11) follows from

$$\sum_{t=0}^{\frac{p-3}{2}}\frac{H_t}{2t+1}\equiv-\frac{1}{2}\left(\sum_{t=1}^{\frac{p-1}{2}}\frac{H_t}{t}+\frac{1}{2}H_{\frac{p-1}{2}}^2\right)=-\frac{2H_{\frac{p-1}{2}}^2+H_{\frac{p-1}{2},2}}{4}\equiv-2q_p^2(2)\pmod p$$

by (2.8) and (2.9). □

3. Proof of Theorem 1.1

We start with the following identity due to Mao and Zhang [15]:

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{3k+1}{16^k}\binom{2k}{k}^3&=\frac{n\binom{2n}{n}}{4\cdot16^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}^2\frac{(-1)^k(2k)!(4n-2k-2)!}{k!(2n-1)!(2n-k-1)!}\\&=\frac{n^2\binom{2n}{n}}{4\cdot16^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}^2\frac{(-1)^k\binom{2n}{k}\binom{4n}{2n}}{(4n-2k-1)\binom{4n}{2k}}.\end{aligned}\tag{3.1}$$

Setting $n=p^{r-1+j}$ for $j\in\{0,1\}$ in (3.1) yields

$$\begin{aligned}&\sum_{k=0}^{p^r-1}\frac{3k+1}{16^k}\binom{2k}{k}^3-p\sum_{k=0}^{p^{r-1}-1}\frac{3k+1}{16^k}\binom{2k}{k}^3\\&\equiv\frac{p^r\binom{2p^{r-1}}{p^{r-1}}}{4\cdot16^{p^{r-1}}}\Bigg(p^r\sum_{k=0}^{p^r-1}\binom{p^r-1}{k}^2\frac{(-1)^k\binom{2p^r}{k}\binom{4p^r}{2p^r}}{(4p^r-2k-1)\binom{4p^r}{2k}}-p^{r-1}\cdot16^{p^r-p^{r-1}}\\&\quad\times\sum_{k=0}^{p^{r-1}-1}\binom{p^{r-1}-1}{k}^2\frac{(-1)^k\binom{2p^{r-1}}{k}\binom{4p^{r-1}}{2p^{r-1}}}{(4p^{r-1}-2k-1)\binom{4p^{r-1}}{2k}}\Bigg)\pmod{p^{4r-\delta_{p,3}}},\end{aligned}\tag{3.2}$$

since for any odd prime p, we have $\binom{2p^r}{p^r}\equiv\binom{2p^{r-1}}{p^{r-1}}\pmod{p^{3r-\delta_{p,3}}}$ by (2.3). In view of [23, Lemma 2.3], for any odd prime p and positive integers r,s with $r\geq s$, we get

$$\frac{2^{p^r-p^{r-1}}-1}{p^r}\equiv\frac{2^{p^s-p^{s-1}}-1}{p^s}\pmod{p^s}.\tag{3.3}$$

It follows that

$$\begin{aligned}16^{p^r-p^{r-1}}&=(1+(2^{p^r-p^{r-1}}-1))^4\equiv1+4(2^{p^r-p^{r-1}}-1)+6(2^{p^r-p^{r-1}}-1)^2\\&\equiv1+4(2^{p^r-p^{r-1}}-1)+[p\geq5]\cdot6p^{2r}q_p^2(2)\pmod{p^{2r+1}}\end{aligned}$$

by Euler's theorem. For any integer $l \in \{0, 3^{r-1}-1\}$,

$$H'_{3l+1,2} \equiv \sum_{j=1,3\nmid j}^{3l-1} \frac{1}{j^2} + 1 \equiv \frac{5}{4} l + 1 \equiv 2l + 1 \pmod 3.$$

Clearly, $\binom{12}{6} \equiv \binom{4}{2} \pmod{27}$. As in [17], we know that the super Catalan numbers $S(m,n)$ are defined by $\frac{(2m)!(2n)!}{(m+n)!m!n!}$ and

$$S(n-k,k) = \frac{\binom{n}{k}\binom{2n}{n}}{\binom{2n}{2k}} = \frac{\binom{n-1}{k}\binom{2n}{n}}{\binom{2n-1}{2k}}$$

is an integer. With the help of Lemmas 2.1, 2.4, 2.5 and the above congruences, we obtain

$$\begin{aligned}
&\frac{\binom{2p^r-1}{pl+\frac{p-1}{2}}\binom{4p^r}{2p^r}}{\binom{4p^r-1}{2pl+p-1}} \\
&\equiv \frac{(-1)^{\frac{p-1}{2}}\binom{2p^{r-1}-1}{l}(1-2p^rH'_{pl+\frac{p-1}{2}} + 2p^{2r}((H'_{pl+\frac{p-1}{2}})^2 - H'_{pl+\frac{p-1}{2},2}))\binom{4p^{r-1}}{2p^{r-1}}}{\binom{4p^{r-1}-1}{2l}(1-4p^rH'_{2pl+p-1} + 8p^{2r}((H'_{2pl+p-1})^2 - H'_{2pl+p-1,2}))} \\
&\equiv \frac{(-1)^{\frac{p-1}{2}}\binom{2p^{r-1}-1}{l}\binom{4p^{r-1}}{2p^{r-1}}}{\binom{4p^{r-1}-1}{2l}} \cdot \Big(1 - 2p^rH'_{pl+\frac{p-1}{2}} + 4p^rH'_{2pl+p-1} \\
&\quad + [p \geq 5] \cdot 8p^{2r}q_p^2(2) - (2l-1)p^{2r}\delta_{p,3}\Big) \pmod{p^{2r+1}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&p^r \sum_{k=0, p\mid(2k+1)}^{p^r-1} \binom{p^r-1}{k}^2 \frac{(-1)^k\binom{2p^r}{k}\binom{4p^r}{2p^r}}{(4p^r-2k-1)\binom{4p^r}{2k}} \\
&\quad - p^{r-1} \cdot 16^{p^r-p^{r-1}} \sum_{k=0}^{p^{r-1}-1} \binom{p^{r-1}-1}{k}^2 \frac{(-1)^k\binom{2p^{r-1}}{k}\binom{4p^{r-1}}{2p^{r-1}}}{(4p^{r-1}-2k-1)\binom{4p^{r-1}}{2k}} \\
&\equiv p^{r-1} \sum_{l=0}^{p^{r-1}-1} \binom{p^{r-1}-1}{l}^2 \frac{(-1)^l\binom{2p^{r-1}-1}{l}\binom{4p^{r-1}}{2p^{r-1}}}{(4p^{r-1}-2l-1)\binom{4p^{r-1}-1}{2l}} \\
&\quad \cdot \{(1 - 2p^rH'_{pl+\frac{p-1}{2}} + 4p^rH'_{2pl+p-1} + [p \geq 5] \cdot 8p^{2r}q_p^2(2) - (2l-1)p^{2r}\delta_{p,3}) \\
&\quad \cdot (1 - p^rH'_{pl+\frac{p-1}{2}} + \frac{p^{2r}}{2}((H'_{pl+\frac{p-1}{2}})^2 - H'_{pl+\frac{p-1}{2},2}))^2 \\
&\quad - 1 - 4(2^{p^r-p^{r-1}}-1) - [p \geq 5] \cdot 6p^{2r}q_p^2(2)\} \\
&\equiv p^{2r-1} \sum_{l=0}^{p^{r-1}-1} \binom{p^{r-1}-1}{l}^2 \frac{(-1)^l\binom{2p^{r-1}-1}{l}\binom{4p^{r-1}}{2p^{r-1}}}{(4p^{r-1}-2l-1)\binom{4p^{r-1}-1}{2l}} \cdot \Big(4H'_{2pl+p-1} - 4H'_{pl+\frac{p-1}{2}} \\
&\quad - 4\frac{2^{p^r-p^{r-1}}-1}{p^r} + [p \geq 5] \cdot 26p^rq_p^2(2) - lp^r\delta_{p,3}\Big) \pmod{p^{2r+1}}
\end{aligned} \tag{3.4}$$

by Lemma 2.5. Noting that

$$\begin{aligned}
&\sum_{k=0,p\nmid(2k+1)}^{p^r-1}\binom{p^r-1}{k}^2\frac{(-1)^k\binom{2p^r-1}{k}\binom{4p^r}{2p^r}}{(4p^r-2k-1)\binom{4p^r-1}{2k}}\\
&=\sum_{s=0}^{p^{r-1}-1}\sum_{t=0,t\neq\frac{p-1}{2}}^{p-1}\binom{p^r-1}{ps+t}^2\frac{(-1)^{s+t}\binom{2p^r-1}{ps+t}\binom{4p^r}{2p^r}}{(4p^r-2ps-2t-1)\binom{4p^r-1}{2ps+2t}}\\
&\equiv\sum_{s=0}^{p^{r-1}-1}\Bigg\{\sum_{t=0}^{\frac{p-3}{2}}\binom{p^{r-1}-1}{s}^2\frac{(-1)^s\binom{2p^{r-1}-1}{s}\binom{4p^{r-1}}{2p^{r-1}}(1-p^rH'_{ps+t})^2(1-2p^rH'_{ps+t})}{(4p^r-2ps-2t-1)\binom{4p^{r-1}-1}{2s}(1-4p^rH'_{2ps+2t})}\\
&\quad-\sum_{t=\frac{p+1}{2}}^{p-1}\binom{p^{r-1}-1}{s}^2\frac{(-1)^s\binom{2p^{r-1}-1}{s}\binom{4p^{r-1}}{2p^{r-1}}(1-p^rH'_{ps+t})^2(1-2p^rH'_{ps+t})}{(4p^r-2ps-2t-1)\binom{4p^{r-1}-1}{2s+1}(1-4p^rH'_{2ps+2t})}\Bigg\}\pmod{p^{r+1}}
\end{aligned}$$

by Lemma 2.1, and

$$\binom{4p^{r-1}-1}{2s+1}=\binom{4p^{r-1}-1}{2s}\frac{4p^{r-1}-2s-1}{2s+1}, \tag{3.5}$$

we have

$$\begin{aligned}
&\sum_{k=0,p\nmid(2k+1)}^{p^r-1}\binom{p^r-1}{k}^2\frac{(-1)^k\binom{2p^r-1}{k}\binom{4p^r}{2p^r}}{(4p^r-2k-1)\binom{4p^r-1}{2k}}\\
&\equiv\sum_{s=0}^{p^{r-1}-1}\binom{p^{r-1}-1}{s}^2\frac{(-1)^s\binom{2p^{r-1}-1}{s}\binom{4p^{r-1}}{2p^{r-1}}}{\binom{4p^{r-1}-1}{2s}}\sum_{t=0,t\neq\frac{p-1}{2}}^{p-1}\frac{1-4p^rH'_{ps+t}+4p^rH'_{2ps+2t}}{4p^r-2ps-2t-1}\\
&\quad-4p^{r-1}\sum_{s=0}^{p^{r-1}-1}\binom{p^{r-1}-1}{s}^2\frac{(-1)^s\binom{2p^{r-1}-1}{s}\binom{4p^{r-1}}{2p^{r-1}}}{(4p^{r-1}-2s-1)\binom{4p^{r-1}-1}{2s}}\\
&\quad\times\sum_{t=\frac{p+1}{2}}^{p-1}\frac{1-4p^rH'_{ps+t}+4p^rH'_{2ps+2t}}{4p^r-2ps-2t-1}\pmod{p^{r+1}}.
\end{aligned}\tag{3.6}$$

In view of [24, Lemma 2.4] and (2.9), for any integer l and nonnegative integer s, we get

$$\sum_{t=0,p\nmid(2t+1)}^{p^s-1}\frac{1}{2p^sl+2t+1}=\sum_{\lfloor\frac{k}{p^s}\rfloor=l,p\nmid(2k+1)}\frac{1}{2k+1}\equiv0\pmod{p^{2s-\delta_{p,3}}} \tag{3.7}$$

and

$$\sum_{t=0,t\neq\frac{p-1}{2}}^{p-1}\frac{1}{(2t+1)^2}=\sum_{j=1}^{\frac{p-1}{2}}\left(\frac{1}{(p+2j)^2}+\frac{1}{(p-2j)^2}\right)\equiv\frac{1}{2}H_{\frac{p-1}{2},2}\equiv2\delta_{p,3}\pmod{p}. \tag{3.8}$$

By (2.9),

$$\sum_{t=\frac{p+1}{2}}^{p-1}\frac{1}{(2t+1)^2}=\sum_{t=1}^{\frac{p-1}{2}}\frac{1}{(2(\frac{p-1}{2}+t)+1)^2}\equiv\frac{1}{4}H_{\frac{p-1}{2},2}\equiv\delta_{p,3}\pmod p. \tag{3.9}$$

It follows that

$$\begin{aligned}\sum_{t=0,t\neq\frac{p-1}{2}}^{p-1}\frac{1}{4p^r-2ps-2t-1}&\equiv-\sum_{t=0,t\neq\frac{p-1}{2}}^{p-1}\frac{1}{2ps+2t+1}\Big(1+\frac{4p^r}{2ps+2t+1}\Big)\\&\equiv-\sum_{t=0,t\neq\frac{p-1}{2}}^{p-1}\Big(\frac{1}{2ps+2t+1}+\frac{4p^r}{(2t+1)^2}\Big)\\&\equiv-\sum_{t=0,t\neq\frac{p-1}{2}}^{p-1}\frac{1}{2ps+2t+1}-8p^r\delta_{p,3}\pmod{p^{r+1}}\end{aligned}$$

and

$$\begin{aligned}\sum_{t=\frac{p+1}{2}}^{p-1}\frac{1}{4p^r-2ps-2t-1}&\equiv-\sum_{t=\frac{p+1}{2}}^{p-1}\Big(\frac{1}{2ps+2t+1}+\frac{4p^r}{(2t+1)^2}\Big)\\&\equiv-\sum_{t=\frac{p+1}{2}}^{p-1}\frac{1}{2ps+2t+1}-4p^r\delta_{p,3}\pmod{p^{r+1}}.\end{aligned}$$

For any nonnegative integer s, by (2.5), (2.12), (3.8) and (3.9), we have

$$\begin{aligned}\sum_{t=0,t\neq\frac{p-1}{2}}^{p-1}\frac{H'_{2ps+2t}-H'_{ps+t}}{4p^r-2ps-2t-1}&\equiv-\Big([p\geq5]\cdot\sum_{t=0}^{\frac{p-3}{2}}\frac{H_{2t}-H_t}{2t+1}+\sum_{t=\frac{p+1}{2}}^{p-1}\frac{H_{2t-p}-H_t}{2t+1}\Big)\\&=-\Big([p\geq5]\cdot\sum_{t=0}^{\frac{p-3}{2}}\frac{H_{2t}-H_t}{2t+1}+\sum_{t=0}^{\frac{p-3}{2}}\frac{H_{p-2-2t}-H_{p-1-t}}{2(p-1-t)+1}\Big)\\&\equiv-\sum_{t=0}^{\frac{p-3}{2}}\frac{[p\geq5]\cdot H_{2t}-H_{2t+1}}{2t+1}\equiv\delta_{p,3}\pmod p.\end{aligned}$$

For any positive integer $k\leq p-1$, we have $\binom{p-1}{k}(-1)^k\equiv1-pH_k\pmod{p^2}$. By (2.5) and (2.8),

$$\begin{aligned}\sum_{t=\frac{p+1}{2}}^{p-1}\frac{H'_{2ps+2t}}{4p^r-2ps-2t-1}&\equiv-\sum_{t=\frac{p+1}{2}}^{p-1}\frac{H_{2t-p}}{2t+1}=-\sum_{t=1}^{\frac{p-1}{2}}\frac{H_{2(\frac{p-1}{2}+t)-p}}{2(\frac{p-1}{2}+t)+1}\\&\equiv-\sum_{t=1}^{\frac{p-1}{2}}\frac{H_{2t-1}}{2t}\equiv-\sum_{t=1}^{\frac{p-1}{2}}\frac{1}{2pt}\Big(1+\binom{p-1}{2t-1}\Big)\end{aligned}$$

$$= -\frac{H_{\frac{p-1}{2}}}{2p} - \frac{1}{2p^2}\sum_{t=1}^{\frac{p-1}{2}}\left(\binom{p}{2t} + \binom{p}{p-2t}\right)$$
$$= -\frac{H_{\frac{p-1}{2}} + 2q_p(2)}{2p} \equiv -\frac{q_p^2(2)}{2} \pmod{p}.$$

In view of (2.5), (2.12) and Lemma 2.6, we have

$$\sum_{t=\frac{p+1}{2}}^{p-1} \frac{H'_{ps+t}}{4p^r - 2ps - 2t - 1} \equiv -\sum_{t=\frac{p+1}{2}}^{p-1} \frac{H_t}{2t+1} = -\sum_{t=0}^{\frac{p-3}{2}} \frac{H_{p-1-t}}{2(p-1-t)+1}$$
$$\equiv \sum_{t=0}^{\frac{p-3}{2}} \frac{H_t}{2t+1} \equiv -[p \geq 5] \cdot 2q_p^2(2) \pmod{p}.$$

Combining (3.2), (3.4), (3.6) with the above results, we get

$$\sum_{k=0}^{p^r-1} \frac{3k+1}{16^k}\binom{2k}{k}^3 - p\sum_{k=0}^{p^{r-1}-1} \frac{3k+1}{16^k}\binom{2k}{k}^3$$
$$\equiv \frac{p^{2r}\binom{2p^{r-1}}{p^{r-1}}}{4 \cdot 16^{p^{r-1}}}\Bigg\{-\sum_{s=0}^{p^{r-1}-1}\binom{p^{r-1}-1}{s}^2 \frac{(-1)^s\binom{2p^{r-1}-1}{s}\binom{4p^{r-1}}{2p^{r-1}}}{\binom{4p^{r-1}-1}{2s}}$$
$$\cdot\Bigg(\sum_{t=0, t\neq\frac{p-1}{2}}^{p-1} \frac{1}{2ps+2t+1} + 4p^r\delta_{p,3}\Bigg) + \sum_{s=0}^{p^{r-1}-1}\binom{p^{r-1}-1}{s}^2$$
$$\cdot \frac{(-1)^s\binom{2p^{r-1}-1}{s}\binom{4p^{r-1}}{2p^{r-1}}p^{r-1}}{(4p^{r-1}-1-2s)\binom{4p^{r-1}-1}{2s}}\Bigg(\sum_{t=\frac{p+1}{2}}^{p-1} \frac{4}{2ps+2t+1} + 4H'_{2ps+p-1}$$
$$-4H'_{ps+\frac{p-1}{2}} - 4\frac{2^{p^r-p^{r-1}}-1}{p^r} + 2p^r q_p^2(2) + (1-s)p^r\delta_{p,3}\Bigg)\Bigg\} \pmod{p^{3r+1}}. \tag{3.10}$$

Clearly, for any nonnegative integer l and positive integer s, we have

$$\sum_{t=\frac{p^s+p^{s-1}}{2}, p\nmid(2t+1)}^{p^s-1} \frac{4}{2p^s l + 2t + 1}$$
$$= \sum_{t=\frac{p^{s-1}+1}{2}, p\nmid t}^{\frac{p^s-1}{2}} \frac{4}{2(t+\frac{p^{s-1}}{2}) + 1 + 2p^s l} = \sum_{t=\frac{p^{s-1}+1}{2}, p\nmid t}^{\frac{p^s-1}{2}} \frac{2}{t(1+\frac{p^s(2l+1)}{2t})}$$
$$\equiv 2(H'_{\frac{p^s-1}{2}} - H'_{\frac{p^{s-1}-1}{2}} \cdot [s \geq 2]) - p^s(2l+1)(H'_{\frac{p^s-1}{2},2} - H'_{\frac{p^{s-1}-1}{2},2} \cdot [s \geq 2])$$
$$\equiv 2(H'_{\frac{p^s-1}{2}} - H'_{\frac{p^{s-1}-1}{2}} \cdot [s \geq 2]) \pmod{p^{s+1-\delta_{p,3}}} \tag{3.11}$$

by (2.6). Now we will handle the case $r=1$ of (3.10), that is to say, $s=0$. With the help of (2.5), (2.8) and (3.11), we obtain

$$\begin{aligned}&\sum_{t=\frac{p+1}{2}}^{p-1}\frac{4}{2t+1}+4H_{p-1}-4H_{\frac{p-1}{2}}-4q_p(2)+2pq_p^2(2)+p\delta_{p,3}\\&\equiv -2H_{\frac{p-1}{2}}-4q_p(2)+2pq_p^2(2)\equiv 0 \pmod{p^{2-\delta_{p,3}}}.\end{aligned}\tag{3.12}$$

The congruence (1.7) with $r=1$ follows from (3.7), (3.10) and (3.12). Next suppose that $r\geq 2$. By (2.5), (2.7), (2.8), (3.3), (3.11) and Lemma 2.2 with $n=2$, we have

$$\begin{aligned}&\sum_{t=\frac{p+1}{2}}^{p-1}\frac{4}{2ps+2t+1}+4H'_{2ps+p-1}-4H'_{ps+\frac{p-1}{2}}-4\frac{2^{p^r-p^{r-1}}-1}{p^r}\\&+2p^rq_p^2(2)+(1-s)p^r\delta_{p,3}\equiv -2H_{\frac{p-1}{2}}-4\frac{2^{p^2-p}-1}{p^2}\equiv 0 \pmod{p^{2-\delta_{p,3}}}.\end{aligned}$$

For any positive integer $s\leq r$, by Lemma 2.1 and (3.7), we get

$$\begin{aligned}&\sum_{k=0}^{p^{r-s}-1}\binom{p^{r-s}-1}{k}^2\frac{(-1)^k\binom{2p^{r-s}-1}{k}\binom{4p^{r-s}}{2p^{r-s}}}{\binom{4p^{r-s}-1}{2k}}\sum_{t=0,p\nmid(2t+1)}^{p^s-1}\frac{1}{2p^sk+2t+1}\\
&=\sum_{i=0}^{p^{r-s-1}-1}\sum_{j=0}^{p-1}\binom{p^{r-s}-1}{pi+j}^2\frac{(-1)^{i+j}\binom{2p^{r-s}-1}{pi+j}\binom{4p^{r-s}}{2p^{r-s}}}{\binom{4p^{r-s}-1}{2pi+2j}}\\
&\quad\cdot\sum_{t=0,p\nmid(2t+1)}^{p^s-1}\frac{1}{2p^{s+1}i+2p^sj+2t+1}\\
&\equiv\sum_{i=0}^{p^{r-s-1}-1}\binom{p^{r-s-1}-1}{i}^2\frac{(-1)^i\binom{2p^{r-s-1}-1}{i}\binom{4p^{r-s-1}}{2p^{r-s-1}}}{\binom{4p^{r-s-1}-1}{2i}}\\
&\quad\cdot\sum_{j=0}^{\frac{p-1}{2}}\sum_{t=0,p\nmid(2t+1)}^{p^s-1}\frac{1}{2p^{s+1}i+2p^sj+2t+1}\\
&\quad-\sum_{i=0}^{p^{r-s-1}-1}\binom{p^{r-s-1}-1}{i}^2\frac{(-1)^i\binom{2p^{r-s-1}-1}{i}\binom{4p^{r-s-1}}{2p^{r-s-1}}}{\binom{4p^{r-s-1}-1}{2i+1}}\\
&\quad\cdot\sum_{j=\frac{p+1}{2}}^{p-1}\sum_{t=0,p\nmid(2t+1)}^{p^s-1}\frac{1}{2p^{s+1}i+2p^sj+2t+1}\pmod{p^{r+s-\delta_{p,3}}}.\end{aligned}$$

In view of (3.5),

$$\sum_{k=0}^{p^{r-s}-1}\binom{p^{r-s}-1}{k}^2\frac{(-1)^k\binom{2p^{r-s}-1}{k}\binom{4p^{r-s}}{2p^{r-s}}}{\binom{4p^{r-s}-1}{2k}}\sum_{t=0,p\nmid(2t+1)}^{p^s-1}\frac{1}{2p^sk+2t+1}$$

$$
\begin{aligned}
&\equiv \sum_{i=0}^{p^{r-s-1}-1} \binom{p^{r-s-1}-1}{i}^2 \frac{(-1)^i \binom{2p^{r-s-1}-1}{i}\binom{4p^{r-s-1}}{2p^{r-s-1}}}{\binom{4p^{r-s-1}-1}{2i}} \sum_{t=0, p\nmid(2t+1)}^{p^{s+1}-1} \frac{1}{2p^{s+1}i+2t+1} \\
&- \sum_{i=0}^{p^{r-s-1}-1} \binom{p^{r-s-1}-1}{i}^2 \frac{(-1)^i \binom{2p^{r-s-1}-1}{i}\binom{4p^{r-s-1}}{2p^{r-s-1}} p^{r-s-1}}{\binom{4p^{r-s-1}-1}{2i}(4p^{r-s-1}-1-2i)} \\
&\cdot \sum_{t=\frac{p^{s+1}+p^s}{2}, p\nmid(2t+1)}^{p^{s+1}-1} \frac{4}{2p^{s+1}i+2t+1} \pmod{p^{r+s-\delta_{p,3}}}.
\end{aligned}
\tag{3.13}
$$

By (2.6), for any positive integer $s \geq 2$, we have

$$
\begin{aligned}
&\sum_{i=0}^{s-2} \sum_{t=\frac{p^i+1}{2}, p\nmid t}^{\frac{p^{i+1}-1}{2}} \frac{4}{(2k+1)p^s+2t} \\
&\equiv \sum_{i=0}^{s-2} \sum_{t=\frac{p^i+1}{2}, p\nmid t}^{\frac{p^{i+1}-1}{2}} \left(\frac{2}{t} - \frac{p^s(2k+1)}{t^2}\right) \\
&= \sum_{i=0}^{s-2} \left(2H'_{\frac{p^{i+1}-1}{2}} - 2H'_{\frac{p^i-1}{2}} \cdot [i \geq 1] - p^s(2k+1)\left(H'_{\frac{p^{i+1}-1}{2},2} - H'_{\frac{p^i-1}{2},2} \cdot [i \geq 1]\right)\right) \\
&= 2H'_{\frac{p^{s-1}-1}{2}} - p^s(2k+1)H'_{\frac{p^{s-1}-1}{2},2} \equiv 2H'_{\frac{p^{s-1}-1}{2}} \pmod{p^{s+1-\delta_{p,3}}}.
\end{aligned}
\tag{3.14}
$$

Combining (2.5), (2.7), (3.11) with (3.14) gives

$$
\begin{aligned}
&\sum_{t=\frac{p^s+p^{s-1}}{2}, p\nmid(2t+1)}^{p^s-1} \frac{4}{2kp^s+2t+1} + [s \geq 2] \cdot \sum_{i=0}^{s-2} \sum_{t=\frac{p^i+1}{2}, p\nmid t}^{\frac{p^{i+1}-1}{2}} \frac{4}{(2k+1)p^s+2t} \\
&+ 2p^r q_p^2(2) + 4H'_{2kp^s+p^s-1} - 4H'_{kp^s+\frac{p^s-1}{2}} - 4\frac{2^{p^r-p^{r-1}}-1}{p^r} \\
&\equiv -2H'_{\frac{p^s-1}{2}} - 4\frac{2^{p^r-p^{r-1}}-1}{p^r} + 2p^r q_p^2(2) \pmod{p^{s+1-\delta_{p,3}}}.
\end{aligned}
\tag{3.15}
$$

In the case $1 \leq s \leq r-1$, with the help of (2.10) with $n = p^s$ and (3.3),

$$
\begin{aligned}
&-2H'_{\frac{p^s-1}{2}} - 4\frac{2^{p^r-p^{r-1}}-1}{p^r} + 2p^r q_p^2(2) \\
&\equiv 4\frac{2^{p^s-p^{s-1}}-1}{p^s} - 2p^s q_p^2(2) - 4\frac{2^{p^{s+1}-p^s}-1}{p^{s+1}} \\
&= 4\frac{2^{p^s-p^{s-1}}-1}{p^s} - 2p^s q_p^2(2) - 4\frac{\sum_{k=1}^{2}\binom{p}{k}(2^{p^s-p^{s-1}}-1)^k}{p^{s+1}} \equiv 0 \pmod{p^{s+1-\delta_{p,3}}}.
\end{aligned}
\tag{3.16}
$$

Note that

$$\sum_{k=0,p\nmid(2k+1)}^{p^{r-s}-1}\binom{p^{r-s}-1}{k}^2\frac{(-1)^k\binom{2p^{r-s}-1}{k}\binom{4p^{r-s}}{2p^{r-s}}p^{r-s}}{\binom{4p^{r-s}-1}{2k}(4p^{r-s}-1-2k)}$$
$$\cdot\Bigg\{\sum_{t=\frac{p^s+p^{s-1}}{2},p\nmid(2t+1)}^{p^s-1}\frac{4}{2p^sk+2t+1}+[s\geq 2]\cdot\sum_{i=0}^{s-2}\sum_{t=\frac{p^i+1}{2},p\nmid t}^{\frac{p^{i+1}-1}{2}}\frac{4}{(2k+1)p^s+2t}$$
$$+2p^rq_p^2(2)+4H'_{(2k+1)p^s-1}-4H'_{kp^s+\frac{p^s-1}{2}}-4\frac{2^{p^r-p^{r-1}}-1}{p^r}\Bigg\}\equiv 0\pmod{p^{r+1-\delta_{p,3}}}$$

by (3.15) and (3.16) for any integer $1\leq s\leq r-1$. In light of Lemma 2.1, we obtain

$$\sum_{k=0,p\mid(2k+1)}^{p^{r-s}-1}\binom{p^{r-s}-1}{k}^2\frac{(-1)^k\binom{2p^{r-s}-1}{k}\binom{4p^{r-s}}{2p^{r-s}}p^{r-s}}{\binom{4p^{r-s}-1}{2k}(4p^{r-s}-1-2k)}$$
$$\cdot\Bigg\{\sum_{t=\frac{p^s+p^{s-1}}{2},p\nmid(2t+1)}^{p^s-1}\frac{4}{2p^sk+2t+1}+\sum_{i=0}^{s-2}\sum_{t=\frac{p^i+1}{2},p\nmid t}^{\frac{p^{i+1}-1}{2}}\frac{[s\geq 2]\cdot 4}{(2k+1)p^s+2t}$$
$$+2p^rq_p^2(2)+4H'_{(2k+1)p^s-1}-4H'_{kp^s+\frac{p^s-1}{2}}-4\frac{2^{p^r-p^{r-1}}-1}{p^r}\Bigg\}$$
$$=\sum_{l=0}^{p^{r-s-1}-1}\binom{p^{r-s}-1}{pl+\frac{p-1}{2}}^2\frac{(-1)^{l+\frac{p-1}{2}}\binom{2p^{r-s}-1}{pl+\frac{p-1}{2}}\binom{4p^{r-s}}{2p^{r-s}}p^{r-s-1}}{\binom{4p^{r-s}-1}{2pl+p-1}(4p^{r-s-1}-1-2l)}\cdot\Bigg\{2p^rq_p^2(2)$$
$$+\sum_{t=\frac{p^s+p^{s-1}}{2},p\nmid(2t+1)}^{p^s-1}\frac{4}{2p^s(pl+\frac{p-1}{2})+2t+1}+\sum_{i=0}^{s-2}\sum_{t=\frac{p^i+1}{2},p\nmid t}^{\frac{p^{i+1}-1}{2}}\frac{[s\geq 2]\cdot 4}{(2l+1)p^{s+1}+2t}$$
$$+4H'_{(2l+1)p^{s+1}-1}-4H'_{lp^{s+1}+\frac{p^{s+1}-1}{2}}-4\frac{2^{p^r-p^{r-1}}-1}{p^r}\Bigg\}$$
$$\equiv\sum_{l=0}^{p^{r-s-1}-1}\binom{p^{r-s-1}-1}{l}^2\frac{(-1)^l\binom{2p^{r-s-1}-1}{l}\binom{4p^{r-s-1}}{2p^{r-s-1}}p^{r-s-1}}{\binom{4p^{r-s-1}-1}{2l}(4p^{r-s-1}-1-2l)}$$
$$\cdot\Bigg\{\sum_{i=0}^{s-1}\sum_{t=\frac{p^i+1}{2},p\nmid t}^{\frac{p^{i+1}-1}{2}}\frac{4}{(2l+1)p^{s+1}+2t}+2p^rq_p^2(2)$$
$$+4H'_{(2l+1)p^{s+1}-1}-4H'_{lp^{s+1}+\frac{p^{s+1}-1}{2}}-4\frac{2^{p^r-p^{r-1}}-1}{p^r}\Bigg\}\pmod{p^{r+1-\delta_{p,3}}}.$$

By (3.13) and the above two congruences, for any integer $1\leq s\leq r-1$, we have

$$\sum_{k=0}^{p^{r-s}-1}\binom{p^{r-s}-1}{k}^2\frac{(-1)^k\binom{2p^{r-s}-1}{k}\binom{4p^{r-s}}{2p^{r-s}}}{\binom{4p^{r-s}-1}{2k}}\sum_{t=0,p\nmid(2t+1)}^{p^s-1}\frac{1}{2p^sk+2t+1}$$

$$
\begin{aligned}
&-\sum_{k=0}^{p^{r-s}-1}\binom{p^{r-s}-1}{k}^2\frac{(-1)^k\binom{2p^{r-s}-1}{k}\binom{4p^{r-s}}{2p^{r-s}}p^{r-s}}{\binom{4p^{r-s}-1}{2k}(4p^{r-s}-1-2k)}\\
&\cdot\Bigg\{\sum_{t=\frac{p^s+p^{s-1}}{2},p\nmid(2t+1)}^{p^s-1}\frac{4}{2p^sk+2t+1}+[s\geq 2]\cdot\sum_{i=0}^{s-2}\sum_{t=\frac{p^i+1}{2},p\nmid t}^{\frac{p^{i+1}-1}{2}}\frac{4}{(2k+1)p^s+2t}\\
&\quad+2p^rq_p^2(2)+4H'_{(2l+1)p^s-1}-4H'_{lp^s+\frac{p^s-1}{2}}-4\frac{2^{p^r-p^{r-1}}-1}{p^r}\Bigg\}\\
&\equiv\sum_{l=0}^{p^{r-s-1}-1}\binom{p^{r-s-1}-1}{l}^2\frac{(-1)^l\binom{2p^{r-s-1}-1}{l}\binom{4p^{r-s-1}}{2p^{r-s-1}}}{\binom{4p^{r-s-1}-1}{2l}}\sum_{t=0,p\nmid(2t+1)}^{p^{s+1}-1}\frac{1}{2p^{s+1}l+2t+1}\\
&\quad-\sum_{l=0}^{p^{r-s-1}-1}\binom{p^{r-s-1}-1}{l}^2\frac{(-1)^l\binom{2p^{r-s-1}-1}{l}\binom{4p^{r-s-1}}{2p^{r-s-1}}p^{r-s-1}}{\binom{4p^{r-s-1}-1}{2l}(4p^{r-s-1}-1-2l)}\cdot\Bigg\{2p^rq_p^2(2)\\
&\quad+\sum_{t=\frac{p^{s+1}+p^s}{2},p\nmid(2t+1)}^{p^{s+1}-1}\frac{4}{2p^{s+1}l+2t+1}+\sum_{i=0}^{s-1}\sum_{t=\frac{p^i+1}{2},p\nmid t}^{\frac{p^{i+1}-1}{2}}\frac{4}{(2l+1)p^{s+1}+2t}\\
&\quad+4H'_{(2l+1)p^{s+1}-1}-4H'_{lp^{s+1}+\frac{p^{s+1}-1}{2}}-4\frac{2^{p^r-p^{r-1}}-1}{p^r}\Bigg\}\pmod{p^{r+1-\delta_{p,3}}}.
\end{aligned}\tag{3.17}
$$

Combining (2.5), (3.7), (3.10), (3.17) with (1.7), by induction, it suffices to prove that

$$
\begin{aligned}
&\sum_{t=\frac{p^r+p^{r-1}}{2},p\nmid(2t+1)}^{p^r-1}\frac{4}{2t+1}+\sum_{i=0}^{r-2}\sum_{t=\frac{p^i+1}{2},p\nmid t}^{\frac{p^{i+1}-1}{2}}\frac{4}{p^r+2t}+2p^rq_p^2(2)\\
&-4H'_{\frac{p^r-1}{2}}-4\frac{2^{p^r-p^{r-1}}-1}{p^r}\equiv 0\pmod{p^{r+1-\delta_{p,3}}}.
\end{aligned}
$$

But this congruence immediately follows from (2.10) with $n=p^r$, (3.3), (3.11) and (3.14).

4. Conclusions

The main result of this paper is a theorem. For any odd prime p and positive integer r , the Dwork-type supercongruence (1.7) is proved, which is a extended result of Guo and Zudilin. Since Guo has made a conjecture that (1.7) is true modulo $p^{4r-\delta_{p,3}}$, we hope that this work will open new inquiry opportunities in this fields for other researchers and knowledge seekers.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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