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Research article

Digital products with PN_k-adjacencies and the almost fixed point property in $DTC_k^{\blacktriangle}$

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Abstract: Given two digital images $(X_i, k_i), i \in \{1, 2\}$, first of all we establish a new PN_k-adjacency relation in a digital product $X_1 \times X_2$ to obtain a relation set $(X_1 \times X_2, PN_k)$, where the term "PN" means "pseudo-normal". Indeed, a PN-k-adjacency is softer or broader than a normal k-adjacency. Next, the present paper initially develops both PN-k-continuity and PN-k-isomorphism. Furthermore, it proves that these new concepts, the PN-k-continuity and PN-k-isomorphism, need not be equal to the typical k-continuity and a k-isomorphism, respectively. Precisely, we prove that none of the typical k-continuity (*resp.* typical k-isomorphism) and the PN-k-continuity (*resp.* PN-k-isomorphism) implies the other. Then we prove that for each $i \in \{1, 2\}$, the typical projection map $P_i : X_1 \times X_2 \to X_i$ preserves a PN_k-adjacency relation in $X_1 \times X_2$ to the k_i-adjacency relation in (X_i, k_i). In particular, using a PN-k-isomorphism, we can classify digital products with PN_k-adjacencies. Furthermore, in the category of digital products with PN_k-adjacencies and PN-k-continuous maps between two digital products with PN_k-adjacencies, denoted by $DTC_k^{\blacktriangle}$, we finally study the (almost) fixed point property of $(X_1 \times X_2, PN_k)$.

Keywords: normal adjacency; pseudo-normal adjacency; PN-k-continuity; PN-k-isomorphism; almost fixed point property
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1. Introduction

Given two digital images, the study of a Cartesian product with a normal k-adjacency plays an important role in applied topology including digital topology and digital geometry [3, 5, 6]. Indeed, the paper [3] initially developed the notion of a normal k-adjacency of a digital product [3] which is a digital topological version of the strong adjacency of a graph product in typical graph theory [1]. However, it is clear that these two versions have their own features which need not be equivalent to each other (see Remarks 3.2, 3.5, 3.8, and 3.11, and Example 3.1). A normal k-adjacency of a digital product is essential for studying the multiplicative property of a digital k-fundamental group (see [3]). Besides, it has been often used to examine some product properties of digital topological invariants [10]. Given two digital images (X_i, k_i) in $\mathbb{Z}^{n_i}, i \in \{1, 2\}$, consider a Cartesian product $X_1 \times X_2$. Then, not every $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ has a normal k-adjacency (see Example 3.2 and Remark 3.11). As a matter of fact, a normal k-adjacency relation between two elements in $X_1 \times X_2$ is very rigid so that we alternatively have the following research goals.

• Is there a certain relation in a digital product $X_1 \times X_2$ such that

(1) it is softer than a normal k-adjacency relation,

(2) for each $i \in \{1, 2\}$, the typical projection map $P_i : X_1 \times X_2 \to X_i$ preserves the established relation in $X_1 \times X_2$ to k_i-adjacency relation in (X_i, k_i) ?

In case this relation is formulated, it can certainly make the earlier works in the literature more generalized and vivid from the viewpoints of digital topology and digital geometry. Hence, in this paper we will develop the so-called PN_k-adjacency on $X_1 \times X_2$, denoted by $(X_1 \times X_2, PN_k)$, supporting this initiative. To be precise, this new PN_k-adjacency need not be a typical k-adjacency of $\mathbb{Z}^n$, *i.e.*, it is a new relation in a digital product associated with the typical k-adjacency (see Remark 3.5).

Next, given two digital products with PN_k- and $PN_{k'}$-adjacency, $(X_1 \times X_2, PN_k)$ and $(Y_1 \times Y_2, PN_{k'})$, we naturally pose the following queries.

• How to introduce the notion of PN-(k, k')-continuity of a map or a PN-(k, k')-isomorphism between $(X_1 \times X_2, PN_k)$ and $(Y_1 \times Y_2, PN_{k'})$?

• What differences are there between the typical (k, k')-continuity and the PN-(k, k')-continuity and further, between a typical (k, k')-isomorphism and a PN-(k, k')-isomorphism ?

• Given digital products with PN_k-adjacencies, how to classify them by using a PN-k-isomorphism ?

• Let $DTC_k^{\blacktriangle}$ be the category of digital products $(X_1 \times X_2, PN_k)$ and PN-k-continuous maps (for details, see Section 3). Then, how can we establish the (almost) fixed point property of a given $(X_1 \times X_2, PN_k)$ in $DTC_k^{\blacktriangle}$?

• Under what condition do we have the almost fixed point property of $(X_1 \times X_2, PN_k)$ in $DTC_k^{\blacktriangle}$?

To address these issues, first of all we will introduce the notion of a PN_k-neighborhood of a point (see the property of (3.3)) in a given digital product with a PN_k-adjacency (see Definition 3.4 in the present paper). Besides, for $(X_1 \times X_2, PN_k)$, we will investigate various properties of a PN-k-continuous self-map and a PN-k-isomorphism.

This paper is organized as follows. Section 2 provides some basic notions which will be used in the paper. Section 3 establishes a pseudo-normal k-adjacency (PN_k-adjacency, for short) of a digital product and further, intensively studies some properties of the PN-k-continuity. Section 4 studies various properties of a PN-k-isomorphism. Using a PN-k-isomorphism, we can classify digital products with PN_k-adjacencies. Section 5 refers to the (almost) fixed point property of $(X_1 \times X_2, PN_k)$ in the category of digital products with PN_k-adjacencies and PN-k-continuous maps, denoted by $DTC_k^{\blacktriangle}$. Finally, after establishing the notion of the almost fixed point property (*AFPP* for brevity) in $DTC_k^{\blacktriangle}$, we establish some conditions supporting the *AFPP* in $DTC_k^{\blacktriangle}$. Section 6 concludes the paper with some remarks and a further work. In the paper we will deal with digital products $X \times Y$ such that each of the cardinalities of the sets X and Y is greater than or equal to 2, *i.e.*, $|X \times Y| \geq 4$.

2. Preliminaries

This section recalls basic notions of the graph-theoretical approach of digital topology, *i.e.*, Rosenfeld model [12, 13]. In relation to the study of digital images in $\mathbb{Z}^n$, in the case we follow the Rosenfeld model, a digital picture is usually represented as a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$, where $n \in \mathbb{N}$, a black points set $X \subset \mathbb{Z}^n$ is the set of points we regard as belonging to the image depicted, k represents as an adjacency relation for X and $\bar{k}$ is used for the elements in $\mathbb{Z}^n \setminus X$. We say that the pair (X, k) is a digital image in a quadruple $(\mathbb{Z}^n, k, \bar{k}, X)$. Owing to the *digital k-connectivity paradox* of a digital image (X, k) [11], we remind the reader that $k \neq \bar{k}$ except the case $(\mathbb{Z}, 2, 2, X)$. However, the present paper is not concerned with the $\bar{k}$-adjacency of $\mathbb{Z}^n \setminus X$.

Motivated by the digital k-connectivity for low dimensional digital image in $\mathbb{Z}^3$ [11, 12], the k-adjacency relations of $\mathbb{Z}^n$ were initially established to study a high dimensional digital image, as follows [3]: For a natural number m with $1 \leq m \leq n$, the two distinct points in $\mathbb{Z}^n$

$$p = (p_i)_{i \in [1,n]_{\mathbb{Z}}} \text{ and } q = (q_i)_{i \in [1,n]_{\mathbb{Z}}},$$

are $k(m, n)$-adjacent if at most m of their coordinates differ by ± 1 and the others coincide. According to this statement, the $k(m, n)$-adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$, were formulated [3, 7]), as follows:

$$k := k(m, n) = \sum_{i=1}^{m} 2^i C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.1}$$

For instance,

$$(n, m, k) \in \left\{ \begin{array}{l} (1, 1, 2), \\ (2, 1, 4), (2, 2, 8), \\ (3, 1, 6), (3, 2, 18), (3, 3, 26), \\ (4, 1, 8), (4, 2, 32), (4, 3, 64), (4, 4, 80), \text{and} \\ (5, 1, 10), (5, 2, 50), (5, 3, 130), (5, 4, 210), (5, 5, 242). \end{array} \right\} \tag{2.2}$$

Hereinafter, for our purposes, for $\{a, b\} \subset \mathbb{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbb{Z}}$ is assumed to be the set $\{m \in \mathbb{Z} \,|\, a \leq m \leq b\}$. For a digital image (X, k), two points $x, y \in X$ are k-connected (or k-path connected) if there is a finite k-path from x to y in $X \subset \mathbb{Z}^n$ [11]. We say that a digital image (X, k) is k-connected (or k-path connected) if any two points $x, y \in X$ is k-connected (or k-path connected). Indeed, in a digital image (X, k) the two notions, k-connectedness and k-path connectedness are equivalent to each other. Besides, a digital image (X, k) with a singleton is assumed to be k-connected for any k-adjacency. Given a k-adjacency relation of (2.1), a *simple k-path* from x to y on $X \subset \mathbb{Z}^n$ is assumed to be the sequence

$(x_i)_{i\in[0,l]_{\mathbb{Z}}} \subset X \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$ [11] and further, $x_0 = x$ and $x_l = y$. The *length* of this simple k-path, denoted by $l_k(x, y)$, is the number l. To be precise, $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x. Besides, a simple closed k-curve with l elements in $\mathbb{Z}^n$, denoted by $SC_k^{n,l}$ [3, 11], is a sequence $(x_i)_{i\in[0,l-1]_{\mathbb{Z}}}$ in $\mathbb{Z}^n$, where x_i and x_j are k-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$ [11].

As a matter of fact, this $l_k(x, y)$ induces a certain metric function d_k on a digital image (X, k) [5, 8]. To be specific, assume a function on a digital image (X, k) which need not be k-connected, as follows:

$$d_k : (X, k) \times (X, k) \to \mathbb{N} \cup \{0, \infty\} \tag{2.3}$$

such that

$$d_k(x, x') := \begin{Bmatrix} 0, \text{ if } x = x', \\ l_k(x, x'), \text{ if } x \neq x' \text{ and } x \text{ is } k\text{-connected with } x' \text{ and} \\ \infty, \text{ if } x \text{ is not } k\text{-connected with } x'. \end{Bmatrix} \tag{2.4}$$

Owing to (2.3) and (2.4), the map d_k is obviously a metric function [5, 8] and further, we can see that $d_k(x, x') \geq 1$ if $x \neq x'$. In addition, in (2.3) and (2.4), in the case (X, k) is k-connected, the codomain of the map d_k of (2.3) can be replaced with $\mathbb{N} \cup \{0\}$. Thus, we can represent a digital k-neighborhood of the point x_0 with radius 1 [5, 8] in the following way [8]

$$N_k(x_0, 1) = \{x \in X \,|\, d_k(x_0, x) \leq 1\}. \tag{2.5}$$

Besides, we can also generalize it to formulate a digital k-neighborhood of the point x_0 with radius $\epsilon \in \mathbb{N}$, as follows:

$$N_k(x_0, \varepsilon) = \{x \in X \,|\, d_k(x_0, x) \leq \varepsilon\}. \tag{2.6}$$

Indeed, we can represent the typical digital (k_0, k_1)-continuity of a map by using the above digital k-neighborhood (see Proposition 2.1). Let us investigate some properties of maps between digital images. To map every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1), the paper [13] established the notion of digital continuity. To represent the digital continuity more conveniently and mathematically, we have the following.

Proposition 2.1. *[3,4] Let (X, k_0) and (Y, k_1) be digital images on $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 2.1, in the case $n_0 = n_1$ and $k_0 = k_1$, we call it k_0*-continuous*.

Based on these concepts, let us consider a digital topological category, denoted by *DTC*, consisting of the following two data [3] (see also [5]):

- the set of (X, k) on $\mathbb{Z}^n$ as objects;
- for every ordered pair of objects (X, k_0) and (Y, k_1), the set of (k_0, k_1)-continuous maps as morphisms.

In *DTC*, in the case $n_0 = n_1$ and $k_0 = k_1 := k$, we will particularly use the notation *DTC(k)* [5].

To classify digital images, we have often used the terminology a "(k_0, k_1)-isomorphism" as used in [4] rather than a "(k_0, k_1)-homeomorphism" as proposed in [2]. For two digital images (X, k_0) on $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. In particular, in the case $n_0 = n_1$ and $k_0 = k_1$, we can call it a k_0*-isomorphism*.

3. Characterization of the *PN*-*k*-continuity and comparison between the *PN*-*k*-continuity and the typical *k*-continuity

For two digital images (X, k_1) on $\mathbb{Z}^{n_1}$ and (Y, k_2) on $\mathbb{Z}^{n_2}$, this section formulates the so-called PN_k-adjacency relation in a digital product $X\times Y$ derived from the given (X, k_1) and (Y, k_2) so that we obtain a new relation set $(X\times Y, PN_k)$. Hereinafter, each digital image (X, k) is assumed to be k-connected. Then, we initially establish the notion of PN-k-continuity of a map between two digital products with PN_k-adjacencies and compare it with the typical k-continuity. Since this work is associated with a normal k-adjacency of a digital product in [3], let us recall it as follows: Motivated by the strong adjacency in [1] from the viewpoint of graph theory, the following notion was initially established in [3].

Definition 3.1. *[3] For two digital images (X, k_1) on $\mathbb{Z}^{n_1}$ and (Y, k_2) on $\mathbb{Z}^{n_2}$, a certain k-adjacency on the Cartesian product $X \times Y \subset \mathbb{Z}^{n_1+n_2}$ is defined as follows: For two points $(x, y), (x', y') \in X \times Y$, (x, y) is k-adjacent to (x', y') if and only if*
(1) y is equal to y' and x is k_1-adjacent to x', or
(2) x is equal to x' and y is k_2-adjacent to y', or
(3) x is k_1-adjacent to x' and y is k_2-adjacent to y'.
Then, we say that the relation set $(X \times Y, k)$ is a digital product with a normal k-adjacency derived from the given two digital images (X, k_1) and (Y, k_2).

Remark 3.2. *(1) One of the important things is that the number k for a normal k-adjacency of Definition 3.1 is one of the number k in (2.1).*
(2) A normal k-adjacency is a little bit different from the strong adjacency in graph theory in [1]. More precisely, given any two graphs G_1, G_2, we always have a strong adjacency in [1] for a graph product $G_1 \times G_2$. However, as stated in Definition 3.1, for two digital images (X, k_1) and (Y, k_2), not every normal k-adjacency exists on the Cartesian product $X \times Y$ (see also Proposition 3.3 below). For instance, consider the case $SC_4^{2,8} \times SC_8^{2,6}$. Then any k-adjacency of $\mathbb{Z}^4$ is not eligible to be a normal k-adjacency for $SC_4^{2,8} \times SC_8^{2,6} \subset \mathbb{Z}^4$.

Based on the product adjacency relation in $X\times Y \subset \mathbb{Z}^{n_1+n_2}$ stated in Definition 3.1, the papers [3,5,6] studied various properties of digital products with normal k-adjacencies. Indeed, a digital product with a normal k-adjacency $(X \times Y, k)$ is kind of relation set and further, the normal adjacency relation is symmetric in $X \times Y$. Thus $(X \times Y, k)$ is a kind of digital space [9], where we say that a *digital space* is a nonempty, π-connected, symmetric relation set, denoted by (X, π), if it is k-connected [9]. Thus a digital product with a normal k-adjacency can be represented as a relation set as follows:

Proposition 3.3. *[5, 6] Assume two digital images (X, k_1) on $\mathbb{Z}^{n_1}$ and (Y, k_2) on $\mathbb{Z}^{n_2}$. For the Cartesian product $X \times Y \subset \mathbb{Z}^{n_1+n_2}$, the following are equivalent.*
(1) The relation set $(X \times Y, k)$ is a digital product with a normal k-adjacency.
(2) For two distinct points $p := (x, y), q := (x', y') \in X \times Y$,

$$q \in N_k(p, 1) \text{ if and only if both } x' \in N_{k_1}(x, 1) \text{ and } y' \in N_{k_2}(y, 1). \tag{3.1}$$

(3) For any point $p := (x, y) \in X \times Y$,

$$N_k(p, 1) = N_{k_1}(x, 1) \times N_{k_2}(y, 1). \tag{3.2}$$

Indeed, the paper [5] called the normal k-adjacency using the property of (3.2) an S-compatible k-adjacency. As for the normal k-adjacency of a Cartesian product, we have many examples [5, 6] including $([a,b]_{\mathbb{Z}} \times [c,d]_{\mathbb{Z}}, 8), (SC_8^{2,6} \times [a,b]_{\mathbb{Z}}, 26), (SC_8^{2,6} \times SC_8^{2,6}, 80), (SC_{18}^{3,6} \times SC_8^{2,4}, 242)$, and so on. Furthermore, neither in $SC_4^{2,8} \times SC_8^{2,6}$ nor in $SC_4^{2,8} \times SC_4^{2,8}$ there exists any k-adjacency in $\mathbb{Z}^4$ satisfying Definition 3.1. Namely, given two digital images (X, k_1) and (Y, k_2), not every Cartesian product $X \times Y \subset \mathbb{Z}^{n_1+n_2}$ has a normal k-adjacency. As mentioned in Remark 3.2, the notion of a normal k-adjacency is rigid. However, its utilities are very huge when studying some product properties of digital topological invariants. Hence, motivated by the several types of equivalent representations of the normal k-adjacency in Proposition 3.3, we may establish the following relation to address the first query posed in Section 1.

Definition 3.4. *For two digital images $(X, k_1 := k_1(m_1, n_1))$ on $\mathbb{Z}^{n_1}$ and $(Y, k_2 := k_2(m_2, n_2))$ on $\mathbb{Z}^{n_2}$, we say that a point $(x, y) \in X \times Y$ is related to $(x', y') \in X \times Y$ on $\mathbb{Z}^{n_1+n_2}$ if*
they are $k := k(m, n_1 + n_2)$-adjacent using a certain k-adjacency of $\mathbb{Z}^{n_1+n_2}$, where m is a certain number with $m \in [m_1 + m_2, n_1 + n_2]_{\mathbb{Z}}$, such that only
(1) in the case $y = y'$, x is k_1-adjacent to x', and
(2) in the case $x = x'$, y is k_2-adjacent to y', and
(3) in the case of neither $y = y'$ nor $x = x'$, x is k_1-adjacent to x' and y is k_2-adjacent to y'.
Then we say that these two related points (x, y) and (x', y') are pseudo-normally k-adjacent (PN_k-adjacent, for brevity).

After comparing the two adjacencies in Definitions 3.1 and 3.4, we can observe some distinctions between them.

Remark 3.5. *(1) The relation of Definition 3.4 is softer or broader than that of Definition 3.1. Indeed, Definition 3.1 is used the condition "if and only if" for establishing a normal adjacency. More precisely, as mentioned above, given two digital images (X, k_1) on $\mathbb{Z}^{n_1}$ and (Y, k_2) on $\mathbb{Z}^{n_2}$ (see Definition 3.1), not every normal k-adjacency exists on the digital product $X \times Y$. However, according to Definition 3.4, we always have a certain PN_k-adjacency relation in a digital product as a subset of $\mathbb{Z}^{n_1+n_2}$.*
(2) Two typical k-adjacent points in $X \times Y$ need not be PN_k-adjacent. For instance, see the point p_3 in $SC_4^{2,4} \times [0, 1]_{\mathbb{Z}}$ (see Figure 1(c)). To be specific, while the point p_1 is typically 18-adjacent to p_3, it is not PN_{18}-adjacent to p_3.
(3) Given the hypothesis of Definition 3.4, not every $m \in [m_1 + m_2, n_1 + n_2]_{\mathbb{Z}}$ is used for formulating a certain PN_k-adjacency. The number m is taken in $[m_1 + m_2, n_1 + n_2]_{\mathbb{Z}}$ depending on the situation. Indeed, there is at least a number $m := m_1 + m_2$ determining a PN_k-adjacency, where $k := k(m_1 + m_2, n_1 + n_2)$.

In view of Definitions 3.1 and 3.4, comparing this approach with the normal k-adjacency relation of Definition 3.1, we clearly observe that a PN_k-adjacency relation is relatively weak and soft, as follows:

Remark 3.6. *A normal k-adjacency implies a PN_k-adjacency. However, the converse does not hold.*

Example 3.1. *Assume a digital image represented via the sequence $MSC_{18} := (x_0, x_1, x_2, x_3, x_4, x_5)$ which is called a minimal simple closed 18-curve with six elements [5], where $x_0 = (0, 0, 0), x_1 = (1, -1, 0), x_2 = (1, -1, 1), x_3 = (2, 0, 1), x_4 = (1, 1, 1), x_5 = (1, 1, 0)$. Furthermore, consider the digital image as a sequence $SC_{26}^{3,5} := (y_0, y_1, y_2, y_3, y_4)$ [8], where $y_0 = (0, 0, 0), y_1 = (-1, -1, 1), y_2 = (-1, 0, 2), y_3 = (0, 1, 2), y_4 = (1, 1, 1)$. Then assume the digital* *product* $MSC_{18} \times SC_{26}^{3,5} \subset \mathbb{Z}^6$ *as the following* (6×5)*-matrix*

$$[c_{i,j}]_{i\in[0,5]_{\mathbb{Z}}, j\in[0,4]_{\mathbb{Z}}},$$

where $c_{i,j} := (x_i, y_j) \in \mathbb{Z}^6$. *Let us now take the point* $c_{3,3} = (x_3, y_3)$. *Then we obtain*

$$N^{\blacktriangle}_{k(5,6)}(c_{3,3}, 1) = \{c_{i,j} \mid i, j \in [2,4]_{\mathbb{Z}}\}$$

which is different from $N_{k(5,6)}(c_{3,3}, 1)$, *where* $k(5,6)$ *is the adjacency of (2.1).*
In particular, we need to remind that no normal k-adjacency exists on $MSC_{18} \times SC_{26}^{3,5}$ *(see Proposition 3.3).*

Let us now intensively characterize the relation of Definition 3.4 using a certain neighborhood of a point of $X \times Y$. Based on the relation of Definition 3.4, we now establish the following notation.

Definition 3.7. *For a point* $p \in X \times Y$,

$$N^{\blacktriangle}_k(p) := \{q \in X \times Y \mid q \text{ is } PN_k\text{-adjacent to } p\} \tag{3.3}$$

and further,

$$N^{\blacktriangle}_k(p, 1) := N^{\blacktriangle}_k(p) \cup \{p\}. \tag{3.4}$$

Then we call $N^{\blacktriangle}_k(p, 1)$ *a* PN_k*-neighborhood of p.*

Owing to this feature, given a PN_k-adjacency relation in a Cartesian product of Definition 3.4, unlike the approach of Definition 3.1, we also have a certain digital space [9] derived from a PN_k-adjacency relation.

Remark 3.8. *Owing to this structure of (3.3), we obtain the following:*
(1) $N^{\blacktriangle}_k(p, 1)$ *always and uniquely exists in* $X \times Y$.
(2) $N^{\blacktriangle}_k(p)$ *need not be equal to* $N^*_k(p)$, *where* $N^*_k(p) := \{q \in X \times Y \mid q \text{ is } k\text{-adjacent to } p\}$.
(3) Not every $N_k(p, 1)$ *is equal to* $N^{\blacktriangle}_k(p, 1), p \in X \times Y$.
(4) The adjacency k for establishing an $N^{\blacktriangle}_k(p)$ *need not be unique. For instance, given* $SC_8^{2,4}$ *and* $SC_{18}^{3,6}$, *we can obtain the k-adjacency for establishing an* $N^{\blacktriangle}_k(p, 1) \subset SC_8^{2,4} \times SC_{18}^{3,6} \subset \mathbb{Z}^5, k \in \{210, 242\}$ *(see (2.2)). More precisely, assume a digital product* $X_1 \times X_2$ *with a normal k-adjacency derived from two digital images* $(X_i, k_i), i \in \{1, 2\}$. *Then, for each point p of* $X_1 \times X_2$, *we can establish* $N_k(p, 1)$ *such that* $N^{\blacktriangle}_k(p, 1) = N_k(p, 1)$.

Let us now characterize $N^{\blacktriangle}_k(p)$ with some examples more precisely.

Example 3.2. *(1) Let us consider the digital images* $X := SC_4^{2,4}$ *and* $(Y := [0,1]_{\mathbb{Z}}, 2)$ *(see Figure 1 (1), (a) and (b)). Then we can consider an* $N^{\blacktriangle}_{18}(p)$ *in the digital product* $(X \times Y, PN_{18})$ *(see Figure 1 (2)), where* $X := SC_4^{2,4} := \{x_1 = (0,0), x_2 = (1,0), x_3 = (1,1), x_4 = (0,1)\}$. *Let us now assume the set* $P := X \times Y = \{p_1 = (0,0,0), p_2 = (1,0,0), p_3 = (1,1,0), p_4 = (0,1,0), p_5 = (0,0,1), p_6 = (1,0,1), p_7 = (1,1,1), p_8 = (0,1,1)\}$. *Then, for the point* $p_1 := (0,0,0)$, *we obtain the following (see Figure 1(2)):*

$$N^{\blacktriangle}_{18}(p_1, 1) = (X \times Y) \setminus \{p_3, p_7\}.$$

Naively, we strongly need to observe that the point p_1 *is not* PN_{18}*-adjacent to each of* p_3 *and* p_7.

(2) Unlike the situation of (1) above, let us consider the set X in (1) above with 8*-adjacency,* $(X, 8)$*, instead of* $(X := SC_4^{2,4}, 4)$ *and* $(Y, 2)$*. Then we obtain a digital product* $X \times Y$ *with a normal* 26*-adjacency, i.e.,* $(X \times Y, 26)$ *such that for any point* $p_i \in X \times Y$*, we obtain* $N_{26}^{\blacktriangle}(p_i, 1) = X \times Y = N_{26}(p_i, 1), i \in [1, 8]_{\mathbb{Z}}$.

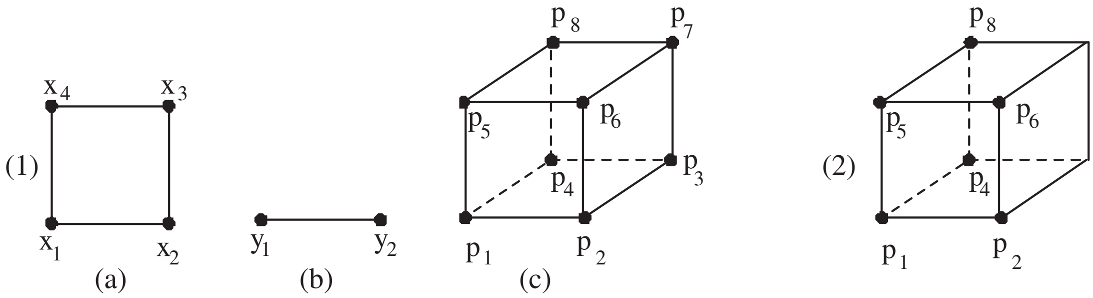


Figure 1. (1) Given two digital images $(X := SC_4^{2,4}, 4)$ and $(Y := [0, 1]_{\mathbb{Z}}, 2)$ in (a) and (b), the digital product $X \times Y$ with an 18-adjacency is assumed. (2) Consider $N_{18}^{\blacktriangle}(p_i, 1), i \in [1, 8]_{\mathbb{Z}}$. Based on the digital product $(X \times Y, PN_{18})$ in Figure 1(c), the PN_{18}-neighborhood of a given point p_1 is determined to be the set $N_{18}^{\blacktriangle}(p_1, 1) \neq N_{18}(p_1, 1) = (X \times Y) \setminus \{p_7\}$ in $X \times Y$ as mentioned in Example 3.2(1).

In view of Definition 3.4, using $N_k^{\blacktriangle}(p)$ of (3.3), we can observe the following.

Remark 3.9. *Given two digital images* (X_i, k_i) *on* $\mathbb{Z}^{n_i}$*,* $i \in \{1, 2\}$*, for two distinct points* $p, q \in X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$*, the following are equivalent because the* PN_k*-adjacency relation is symmetric.*
(1) p and q are PN_k*-adjacent.*
(2) $q \in N_k^{\blacktriangle}(p)$*.*

As mentioned in Remark 3.9, owing to the symmetric relation of a PN_k-adjacency, we can clearly obtain the following:

Based on the PN_k-adjacency of a digital product, motivated by the classical notions in a typical digital image in [11] (see the previous part in Section 2), we now define some terminology used in $DTC_k^{\blacktriangle}$. For a digital product with a certain PN_k-adjacency, $(X \times Y, PN_k)$, we say that two points $z, w \in X \times Y$ are PN_k-connected (or PN_k-path connected) if there is a finite PN_k-path $(z_0, z_1, \cdots, z_m) \subset X \times Y$ from z to w on $X \times Y$ such that $z_0 = z$ and $z_m = w$, where we say that a PN_k-path from z to w in $X \times Y$ means a finite sequence $(z_0, z_1, \cdots, z_m) \subset X \times Y$ such that z_i is PN_k-adjacent to z_j if $j = i + 1$ or $i = j + 1$. We say that a digital product $(X \times Y, PN_k)$ is PN_k-connected (or PN_k-path connected) if any two points $z, w \in X \times Y$ are PN_k-connected (or PN_k-path connected). In $DTC_k^{\blacktriangle}$, a singleton as a subset of $X \times Y$ is assumed to be PN_k-connected. Indeed, in a digital product $(X \times Y, PN_k)$ the two notions, PN_k-connectedness and PN_k-path connectedness are equivalent to each other. Given a PN_k-adjacency relation in $X\times Y$, a *simple* PN_k*-path* from z to w in $X\times Y$ is assumed to be the PN_k-path $(z_i)_{i\in[0,l]_{\mathbb{Z}}} \subset X\times Y$ such that z_i and z_j are PN_k-adjacent if and only if either $j = i + 1$ or $i = j + 1$ and further, $z_0 = x$ and $z_l = y$. Besides, a simple closed PN_k-curve with l elements in $X \times Y \subset \mathbb{Z}^n$, denoted by $SC_{PN_k}^{n,l}$, is a sequence $(z_i)_{i\in[0,l-1]_{\mathbb{Z}}}$ in $X \times Y$, where z_i and z_j are PN_k-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.

Proposition 3.10. *The relation set* $(X_1 \times X_2, PN_k)$ *is a digital space, where* (X_i, k_i) *is* k_i*-connected,* $i \in \{1, 2\}$*.*

Proof: By Remark 3.9, since the relation PN_k in $X_1 \times X_2$ is symmetric, we need to examine if $(X_1 \times X_2, PN_k)$ is PN_k-connected. Take any two distinct points $p := (x_1, x_2)$ and $q := (x_1', x_2')$ in $X_1 \times X_2$. Then, without loss of generality, we may assume the case $x_1 \lneq x_1'$ and $x_2 \leq x_2'$ or the case $x_1 \leq x_1'$ and $x_2 \lneq x_2'$. For our purposes, we may take the first case, *i.e.*, $x_1 \lneq x_1'$ and $x_2 \leq x_2'$. Then consider the differences $|\,x_1 - x_2| \gneq 0$ and $|\,y_1 - y_2| \geq 0$. Depending on these finite differences, we can take a finite set

$$\{p := p_1, p_2, p_3, \cdots, p_n := q\} \subset X_1 \times X_2 \tag{3.5}$$

such that p_i is PN_k-adjacent to p_{i+1} in $(X_1 \times X_2, PN_k), i \in [1, n-1]_{\mathbb{Z}}$ and

$$p, q \in \bigcup_{i \in [1,n]_{\mathbb{Z}}} N_k^{\blacktriangle}(p_i, 1) \subset X_1 \times X_2. \tag{3.6}$$

Owing to (3.5) and (3.6), we can conclude that $(X_1 \times X_2, PN_k)$ is PN_k-connected. □

Remark 3.11. *Comparing the two relations of Definitions 3.1 and 3.4, we can confirm some differences between them using a PN_k-neighborhood. For instance, for the given digital images $(SC_4^{2,4}, 4)$ and $(Y, 2)$ as mentioned in Example 3.2, we can follow Definition 3.4 to obtain a PN_{18}-adjacency relation in $SC_4^{2,4} \times Y$. Then, while each point $p \in SC_4^{2,4} \times Y$ has an $N_{18}^{\blacktriangle}(p, 1)$ in $SC_4^{2,4} \times Y$, the Cartesian product $SC_4^{2,4} \times Y$ does not have a normal 18-adjacency because the 18-adjacency does not satisfy Definition 3.1. Thus $(SC_4^{2,4} \times Y, PN_{18})$ is a digital space.*

In view of these notions, we can take the following:

Remark 3.12. *Given an $(X \times Y, PN_k)$ in $\mathbb{Z}^n$, we have the following:*
(1) A PN_k-path need not be equal to a k-path.
(2) $SC_{PN_k}^{n,l}$ need not be equal to $SC_k^{n,l}$.
(3) A PN-(k, k')-continuous map $f : (X \times Y, PN_k) \to (X' \times Y', PN_{k'})$ implies that any PN_k-connected subset of $(X \times Y, PN_k)$ is preserved onto a $PN_{k'}$-connected subset of $(X' \times Y', PN_{k'})$ by the map f.

Based on the relation set $(X \times Y, PN_k)$ established in Definition 3.4, we obtain the following:

Lemma 3.13. *Assume a digital product with a PN_k-adjacency, $(X_1 \times X_2, PN_k)$, derived from two digital images $(X, k_i), i \in \{1, 2\}$. Then, given a point $p \in X_1 \times X_2$, we always obtain $N_k^{\blacktriangle}(p, 1) \subset N_k(p, 1)$. However, $N_k^{\blacktriangle}(p, 1)$ need not be equal to $N_k(p, 1)$, i.e., $|\,N_k^{\blacktriangle}(p, 1)\,| \leq |\,N_k(p, 1)\,|$.*

Proof: It is clear that for any point $q \in N_k^{\blacktriangle}(p, 1)$, according to the property (1) of Definition 3.4, we obtain $q \in N_k(p, 1)$. However, in view of Example 3.2(1) as a counterexample, we can disprove $N_k(p, 1) \subset N_k^{\blacktriangle}(p, 1)$. Naively, consider the digital product $(SC_4^{2,4} \times Y, PN_{18})$ in Example 3.2. Then we obviously obtain the following property.

$$N_{18}^{\blacktriangle}(p_1, 1) = (SC_4^{2,4} \times Y) \setminus \{p_3, p_7\} \text{ and } N_{18}(p_1, 1) = (SC_4^{2,4} \times Y) \setminus \{p_7\}$$

so that $N_{18}(p_1, 1) \subsetneq N_{18}^{\blacktriangle}(p_1, 1)$, which implies that $N_k^{\blacktriangle}(p, 1)$ need not be equal to $N_k(p, 1)$. □

Motivated by a relation preserving mapping, we introduce the following map between two digital products with PN_k- and $PN_{k'}$-adjacency.

Definition 3.14. *Consider two digital products $(X_1 \times X_2, PN_k)$ and $(Y_1 \times Y_2, PN_{k'})$. A function $f :$ $(X_1 \times X_2, k) \to (Y_1 \times Y_2, k')$ is PN-(k, k')-continuous at a point $p := (x_1, x_2)$ if for any point $q \in X_1 \times X_2$*

such that $q \in N_k^{\blacktriangle}(p)$ (denoted by $p \leftrightarrow_{k^{\blacktriangle}} q$), we obtain $f(q) \in N_{k'}^{\blacktriangle}(f(p), 1)$ (denoted by $f(p) \Leftrightarrow_{k'}^{\blacktriangle} f(q)$). In the case the map f is PN-(k, k')-continuous at each point $p \in X_1 \times X_2$, we call the map f is PN-(k, k')-continuous.

In Definition 3.14, in the case $k = k'$, we say that the map f is a *PN-k*-continuous map. Besides, it is clear that the property "$f(q) \in N_{k'}^{\blacktriangle}(f(p), 1)$" implies that $f(p)$ is equal to $f(q)$ or $f(p)$ is $PN_{k'}$-adjacent to $f(q)$.

Using both PN_k- and $PN_{k'}$-neighborhood of $(X_1 \times X_2, PN_k)$ and $(Y_1 \times Y_2, PN_{k'})$, we can mathematically represent the *PN*-(k, k')-continuity of a map, as follows:

Proposition 3.15. *Assume $(X_1 \times X_2, PN_k)$ and $(Y_1 \times Y_2, PN_{k'})$. A function $f : (X_1 \times X_2, k) \to (Y_1 \times Y_2, k')$ is PN-(k, k')-continuous if and only if for every $p \in X_1 \times X_2$, $f(N_k^{\blacktriangle}(p, 1)) \subset N_{k'}^{\blacktriangle}(f(p), 1)$.*

Proof: Assume any point $p \in X_1 \times X_2$ and take any point $q \in X_1 \times X_2$ such that $q \in N_k^{\blacktriangle}(p)$. By the hypothesis, we obtain $f(p) \Leftrightarrow_{k^{\blacktriangle}} f(q)$ which implies that $f(N_k^{\blacktriangle}(p, 1)) \subset N_{k'}^{\blacktriangle}(f(p), 1)$. The converse clearly holds. □

Lemma 3.16. *The PN-(k, k')-continuity has the transitive property.*

Proof: Assume $f : (X_1 \times X_2, k) \to (Y_1 \times Y_2, k')$ which is *PN*-(k, k')-continuous and $g : (Y_1 \times Y_2, k') \to (Z_1 \times Z_2, k'')$ which is *PN*-(k', k'')-continuous. For any point $p \in X_1 \times X_2$ since $f(N_k^{\blacktriangle}(p, 1)) \subset N_{k'}^{\blacktriangle}(f(p), 1))$ and $g(N_{k'}^{\blacktriangle}(f(p), 1)) \subset N_{k''}^{\blacktriangle}(g(f(p)), 1))$, the proof is completed. □

Theorem 3.17. *Assume $(X_1 \times X_2, PN_k)$ derived from two digital images $(X_i, k_i), i \in \{1, 2\}$, where $k_i := k(m_i, n_i)$. Then, each of the projection maps P_1 and P_2 preserve the PN_k-adjacency of $X_1 \times X_2$ onto the k_1- and k_2-adjacency, respectively.*

Proof: Assume a digital product with a PN_k-adjacency, $(X_1 \times X_2, PN_k)$ derived from two digital images $(X, k_i), i \in \{1, 2\}$. For each point $p := (x_1, x_2) \in X_1 \times X_2$, consider the PN_k-neighborhood of p, $N_k^{\blacktriangle}(p, 1)$. Next, consider each of the projection maps P_1 and P_2 as follows:

$$\left\{ \begin{matrix} P_1 : X_1 \times X_2 \to X_1 \text{ given by } P_1(p) = x_1, \\ P_2 : X_1 \times X_2 \to X_2 \text{ given by } P_2(p) = x_2 \end{matrix} \right\} \tag{3.7}$$

Then, since $N_k^{\blacktriangle}(p, 1) = N_{k_1}(x_1, 1) \times N_{k_2}(x_2, 1)$, we obtain

$$P_1(N_k^{\blacktriangle}(p, 1)) = N_{k_1}(x_1, 1) \text{ and } P_2(N_k^{\blacktriangle}(p, 1)) = N_{k_2}(x_2, 1),$$

which implies the preservation of the PN_k-adjacency of $X_1 \times X_2$ onto the k_1- and k_2-adjacency by each projection map. □

Let us now compare the typical (k, k')-continuity and the *PN*-(k, k')-continuity

Theorem 3.18. *None of the typical (k, k')-continuity and the PN-(k, k')-continuity implies the other.*

Proof: For the sake of contradiction, we will use counterexamples for the proof. Consider the two digital images $(X, 4)$ and $(Y, 2)$, where $X = \{x_1 = (0, 0), x_2 = (1, 0), x_3 = (1, 1), x_4 = (0, 1)\}$. Then we may assume a PN_{18}-adjacency of digital product $X \times Y$ with such as $(P := X \times Y, PN_{18})$ and further,

($Q := SC_4^{2,4} \times Y, PN_{18}$) as in Figure 2. Let us now consider the following two cases.
(1) Let us assume the map $f : P \to Q$ given by

$$f(p_i) = q_i, i \in [1,8]_{\mathbb{Z}} \setminus \{3,7\} \text{ and } f(p_3) = q_7 = f(p_7).$$

Naively, we have $f(P) = Q \setminus \{q_3\}$.

Then, for the points $p_i \in X \times Y, i \in \{1,5\}$ we obtain $f(N_{18}^{\blacktriangle}(p_i,1)) = N_{18}^{\blacktriangle}(f(p_i),1)$. In particular, we see

$$f(N_{18}^{\blacktriangle}(p_3,1)) = Q \setminus \{q_1,q_3,q_5\} \subsetneq N_{18}^{\blacktriangle}(f(p_3),1) = Q \setminus \{q_1,q_5\} = N_{18}^{\blacktriangle}(q_7,1).$$

This implies the PN-18-continuity of f.
However, the map f is not an 18-continuous map in $DTC(18)$ because while $p_3 \in N_{18}(p_1,1)$, we have

$$f(p_3) = q_7 \notin N_{18}(f(p_1,1)) = N_{18}(q_1,1),$$

which implies the typical non-18-continuity of f (see Proposition 2.1).
(2) Let us consider the map $g : P \to Q$ given by (see Figure 2(2))

$$g(p_i) = q_i, i \in [1,8]_{\mathbb{Z}} \setminus \{3,5\} \text{ and } g(p_3) = q_5, g(p_5) = q_3, \tag{3.8}$$

where $Q := SC_4^{2,4} \times [0,1]_{\mathbb{Z}}$. In $DTC(18)$, since $g(N_{18}(p_i,1) = N_{18}(g(p_i),1), i \in [1,8]_{\mathbb{Z}}$, g is a typically 18-continuous map in $DTC(18)$. However, consider the points $p_1, p_5 \in X \times Y$. For the point $p_5 \in N_{18}^{\blacktriangle}(p_1,1)$, we obtain $g(p_5) \notin g(N_{18}^{\blacktriangle}(p_1,1))$, which implies the non-$PN$-18-continuity of the map g at the point p_1. □

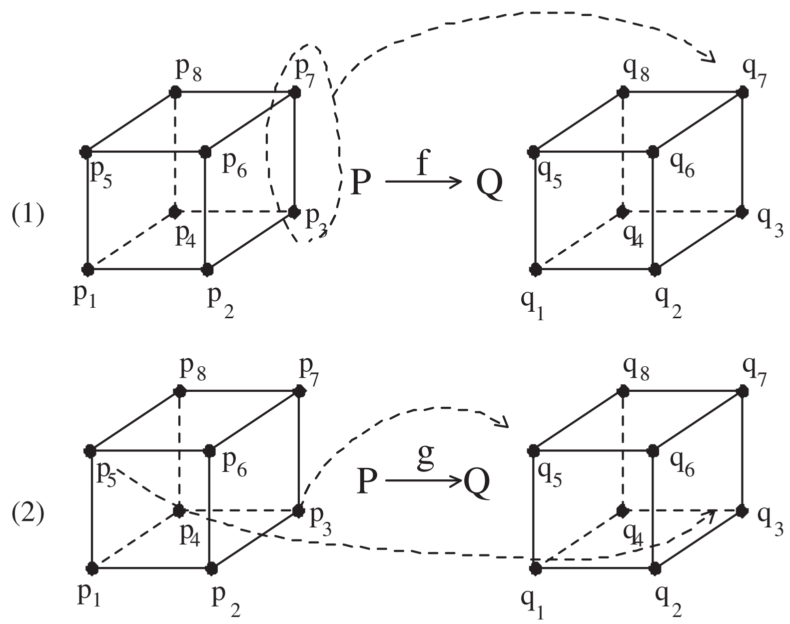


Figure 2. A comparison between the PN-8-continuity and the 8-continuity of the given maps f and g. The map f in (1) is the map used in the proof (1) of Theorem 3.18 and the map g in (2) is the map related to the proof (2) of Theorem 3.18

Based on these concepts, let us establish the category for digital products with a PN_k-adjacency, denoted by $DTC^{\blacktriangle}$, consisting of the following two data:
• The set of digital products $(X_1 \times X_2, PN_k)$ as objects;
• For every ordered pair of objects $(X_1 \times X_2, PN_k)$ and $(Y_1 \times Y_2, PN_{k'})$, the set of PN-(k, k')-continuous maps as morphisms.
In $DTC^{\blacktriangle}$, in the case $k = k'$, we will particularly use the notation $DTC^{\blacktriangle}_k$.

By Theorem 3.18, we obtain the following:

Remark 3.19. *$DTC^{\blacktriangle}_k$ need not be equal to the category consisting of digital products with a normal k-adjacency or a typical k-adjacency, and their k-continuous maps.*

4. A PN-k-isomorphism between two digital products with a PN_k-adjacency

To classify digital products with a PN_k-adjacency, we use the following notion. Assume a digital product $X_1 \times X_2$ with a PN_k-adjacency, denoted by $(X_1 \times X_2, PN_k)$, and a digital product $Y_1 \times Y_2$ with a $PN_{k'}$-adjacency, denoted by $(Y_1 \times Y_2, PN_{k'})$. We may pose the following query: Is there a PN-(k, k')-continuous bijection $f : (X_1 \times X_2, PN_k) \to (Y_1 \times Y_2, PN_{k'})$ such that the inverse of f is not PN-(k', k)-continuous ?

Example 4.1. *(1) Using digital images $(X_1 := SC^{2,4}_4, 4)$, $(X_2 := SC^{2,4}_8, 8)$ and digital interval $(Y := [0, 1]_{\mathbf{Z}}, 2)$, consider the digital products $(X_1 \times Y, PN_{18})$ and $(X_2 \times Y, PN_{26})$. Let $f : (X_1 \times Y, PN_{18}) \to (X_2 \times Y, PN_{26})$ be a map defined by $f(p_i) = q_i$ where $i \in [1, 8]_{\mathbb{Z}}$. Then, f is a PN-$(18, 26)$-continuous bijection. However, since $f^{-1}(N^{\blacktriangle}_{26}(q_1, 1)) \not\subseteq N^{\blacktriangle}_{18}(f^{-1}(q_1), 1) = N^{\blacktriangle}_{18}(p_1, 1)$ so that the map f^{-1} is not a PN-$(26, 18)$-continuous map.*
(2) Consider digital images $SC^{2,l}_4 := (x_i)_{i\in[0,l-1]_{\mathbb{Z}}}, l \in \mathbb{N}_0 \setminus \{2, 6\}$ and $([0, 1]_{\mathbb{Z}}, 2)$. Then we obtain two relation sets $([0, l-1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}}, PN_8)$ and $(SC^{2,l}_4 \times [0, 1]_{\mathbb{Z}}, PN_{18})$ (see Figure 4(1)). Then consider the map $f : ([0, l-1]_{\mathbb{Z}}, 2) \to SC^{2,l}_4$ given by $f(i) = x_i$ which is a $(2, 4)$-continuous bijection. Next, further define the map

$$g : [0, l-1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \to SC^{2,l}_4 \times [0, 1]_{\mathbb{Z}}$$

given by $g(i, t) = (f(i), t) = (x_i, t)$. Then the map g is clearly a PN-$(8, 18)$-continuous bijection. However, it is clear that the inverse of g, g^{-1}, is not PN-$(18, 8)$-continuous because since $g^{-1}(N^{\blacktriangle}_{18}(q, 1)) \not\subseteq N^{\blacktriangle}_8(p, 1)$, where $p := (0, 0) \in [0, l-1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}}$ and $q := (x_0, 0) \in SC^{2,l}_4 \times [0, 1]_{\mathbb{Z}}$.
For instance, consider the digital products $[0, 7]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}}$ and $SC^{2,8}_4 \times [0, 1]_{\mathbb{Z}}$ (see Figure 4(1)), where $SC^{2,8}_4 := (x_i)_{i\in[0,7]_{\mathbb{Z}}}, x_0 := (0, 0), x_1 := (0, -1), x_2 := (1, -1), x_3 := (2, -1), \cdots, x_7 := (0, 1)$. Define the map

$$g : [0, 7]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \to SC^{2,8}_4 \times [0, 1]_{\mathbb{Z}}$$

given by $g(i, t) = (f(i), t) = (x_i, t)$. Then the map g is clearly a PN-$(8, 18)$-continuous bijection. However, it is clear that the inverse of g, g^{-1}, is not PN-$(18, 8)$-continuous because since $g^{-1}(N^{\blacktriangle}_{18}(q, 1)) \not\subseteq N^{\blacktriangle}_8(p, 1)$, where $p := (0, 0) \in [0, 7]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}}$ and $q := (x_0, 0) \in SC^{2,8}_4 \times [0, l]_{\mathbb{Z}}$.

Motivated by Example 4.1, we now establish the following notion.

Definition 4.1. *Assume a digital product $(X_1 \times X_2, PN_k)$ derived from two digital images $(X_i, k_i), i \in \{1, 2\}$ and a digital product with $(Y_1 \times Y_2, PN_{k'})$ derived from two digital images $(Y_i, k_i), i \in \{1, 2\}$. We* *say that a map $h : (X_1 \times X_2, PN_k) \to (Y_1 \times Y_2, PN_{k'})$ is a PN-(k, k')-isomorphism if h is a PN-(k, k')-continuous bijection and the inverse of h, h^{-1}, is a PN-(k', k)-continuous map.*

Example 4.2. *Given the digital images $(X, 4)$ and $(Y, 2)$, consider two digital products $X \times Y$ with a PN_{18}-adjacency such as $(P := X \times Y, PN_{18})$ and $(Q := X \times Y, PN_{18})$ as in Figure 3. Then consider the following maps h, f, and g below.*
(1) Let us consider the bijection $h : (P, PN_{18}) \to (Q, PN_{18})$ given by (see Figure 3(1))

$$h(p_i) = q_i, i \in [1, 8]_{\mathbb{Z}} \setminus \{3, 7\} \text{ and } h(p_3) = q_7, h(p_7) = q_3.$$

Since $h(N^{\blacktriangle}_{18}(p_i, 1)) = N^{\blacktriangle}_{18}(h(p_i), 1)$ for each point $p_i \in P$, we have the PN-18-continuity of h. Furthermore, h is a PN-18-isomorphism.
(2) Let us now assume the bijection $f : (P, PN_{18}) \to (Q, PN_{18})$ given by (see Figure 3(2))

$$\left\{\begin{matrix} f(p_i) = q_i, i \in [1, 8]_{\mathbb{Z}} \setminus \{1, 3, 5, 7\} \text{ and} \\ f(p_1) = q_3, f(p_3) = q_1, f(p_5) = q_7, f(p_7) = q_5. \end{matrix}\right\}$$

Then, it is clear that f is a PN-18-isomorphism.
(3) Let us recall the map g of (3.8). While it is bijection, it is not a PN-18-isomorphism.

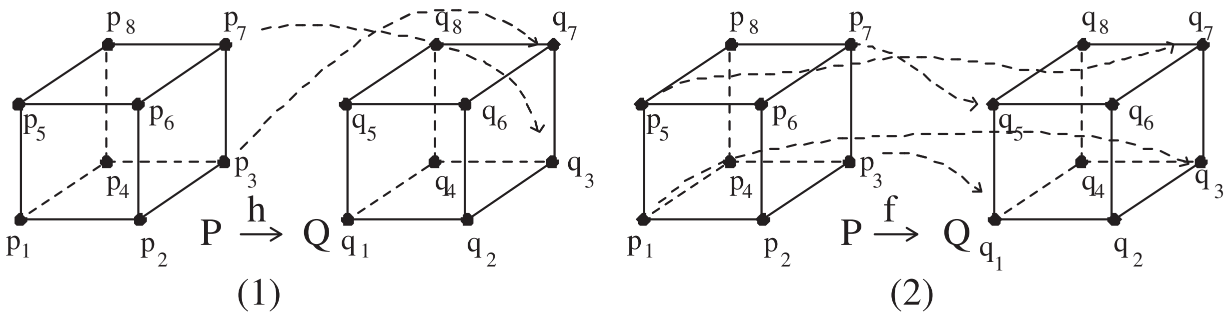


Figure 3. Comparison between a typical (k, k')-isomorphism and a *PN*-(k, k')-isomorphism. In particular, both maps h and f are *PN*-18-isomorphisms.

Theorem 4.2. *None of a typical (k, k')-isomorphism and a PN-(k, k')-isomorphism implies the other.*

Proof: For the sake of contradiction, let us consider a counterexample. Let us assume $(X_1, 4) = (Y_1, 4)$ and $(X_2, 2) = (Y_2, 2)$, where $X_1 := SC^{2,4}_4 := Y_1$ and $X_2 = I := [0, 1]_{\mathbb{Z}} := Y_2$(see Figure 3). Naively, each of $X_1 \times X_2$ and $Y_1 \times Y_2$ is assumed to be a kind of I^3. Then consider the map

$$h : P := (X_1 \times X_2, PN_{18}) \to Q := (Y_1 \times Y_2, PN_{18})$$

defined by (see the map h in Figure 3(1)), as follows:

$$h(p_i) = q_i, i \in [1, 8]_{\mathbb{Z}} \setminus \{3, 7\} \text{ and } h(p_3) = q_7, h(p_7) = q_3.$$

Then, while the bijection h is a *PN*-18-isomorphic map (see Example 4.2(1)), it is not an 18-isomorphism because

$$h(N_{18}(p_1, 1)) \not\subseteq N_{18}(h(p_1), 1) = N_{18}(q_1, 1) = Q \setminus \{q_7\}.$$

Conversely, let us recall the map g used in the proof of Theorem 3.18 (see the map of (3.8)). Then it is obvious that while the map g is an 18-isomorphism, it is not a PN-18-isomorphism. □

Motivated by the S-compatible adjacency for a Cartesian product $X_1 \times X_2$ in [6], we can represent it with the following relation between two points in $X_1 \times X_2$ (see Proposition 3.3).

Lemma 4.3. *Assume a digital product $(X_1 \times X_2, PN_k)$ derived from two digital images $(X, k_i), i \in \{1, 2\}$. In case each point $p \in X_1 \times X_2$ has the property $N^{\blacktriangle}_k(p, 1) = N_k(p, 1)$, the given PN_k-adjacency is a normal k-adjacency.*

Proof: By Proposition 3.3 and Remark 3.9, the proof is completed. □

Theorem 4.4. *For $SC^{n,l}_k$ and the digital interval $(I := [0, 1]_{\mathbb{Z}}, 2)$, consider the digital product $SC^{n,l}_k \times I \subset \mathbb{Z}^{n+1}$ with a certain k'-adjacency of $\mathbb{Z}^{n+1}$. Then we obtain the following.*
(1) In the case of $k = 2n$, no normal k'-adjacency exists.
(2) In the case of $k \neq 2n$, for $4 \leq l \in \mathbb{N}$, a $PN_{k'}$-adjacency is a normal k'-one.

Proof: (1) In the case of $k = 2n$, we can consider the following two cases.
(Case 1) In the case of $l = 4$, any k'-adjacency of $\mathbb{Z}^{n+1}$ is not a normal k'-one for $SC^{n,l}_k \times I \subset \mathbb{Z}^{n+1}$. As a counterexample, consider $(SC^{2,4}_4, 4)$ and $(I := [0, 1]_{\mathbb{Z}}, 2)$ as mentioned in Example 3.2. Then every point $p \in SC^{2,4}_4 \times I \subset \mathbb{Z}^3$, we observe $N^{\blacktriangle}_{18}(p, 1) \neq N_{18}(p, 1)$, which implies the assertion.
(Case 2) In the case $8 \leq l \in \mathbb{N}_0$, the $PN_{k'}$-adjacency is not a normal one either. Consider $(SC^{2,l}_4, 4)$ with $8 \leq l \in \mathbb{N}_0$ and $(I := [0, 1]_{\mathbb{Z}}, 2)$. Then there is a certain point $p \in SC^{2,l}_4 \times I \subset \mathbb{Z}^3$ such that

$$N^{\blacktriangle}_{18}(p, 1) \neq N_{18}(p, 1).$$

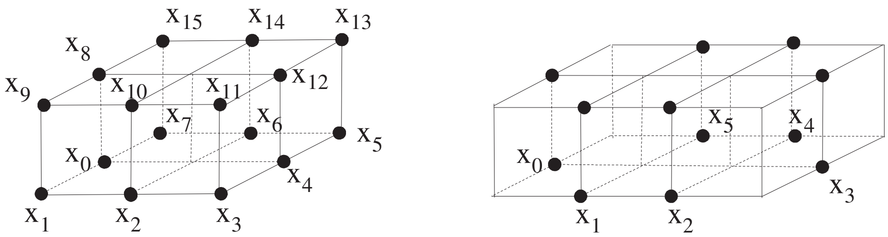
For instance, see the points $x_0, x_2, x_4, x_6, x_8, x_{10}, x_{12}, x_{14}$ in Figure 4(1), which supports the assertion. In details, consider $(SC^{2,8}_4 \times Y, PN_{18})$ or a typical image $(SC^{2,8}_4 \times Y, 18)$ in $DTC(18)$. Then we observe that

$$N^{\blacktriangle}_{18}(x_2, 1) = \{x_1, x_2, x_3, x_9, x_{10}, x_{11}\}, N_{18}(x_2, 1) = N^{\blacktriangle}_{18}(x_2, 1) \cup \{x_0, x_4\}$$

so that $N^{\blacktriangle}_{18}(x_2, 1) \neq N_{18}(x_2, 1)$.

(2) With the hypothesis, in case $4 \leq l \in \mathbb{N}$, let us consider any $SC^{n,l}_k, k \neq 2n, k := k(m, n)$ and take the Cartesian product $SC^{n,l}_k \times I$. It is clear that each point $p \in SC^{n,l}_k \times I$ has $N^{\blacktriangle}_{k'}(p, 1)$ which is equal to $N_{k'}(p, 1), k' = k'(m + 1, n + 1)$, which implies that in $SC^{n,l}_k \times I$ the $PN_{k'}$-adjacency is equal to the normal k'-adjacency. For instance, with the hypothesis, consider $SC^{2,l}_8, l \in \mathbb{N}_0 \setminus \{2, 6\}$ and $(I := [0, 1]_{\mathbb{Z}}, 2)$. Then every point $p \in (SC^{2,l}_8 \times I) \subset \mathbb{Z}^3$ (see Figure 4(2) for the case $SC^{2,6}_8 \times I$), we observe $N^{\blacktriangle}_{26}(p, 1) = N_{26}(p, 1)$. Thus we can consider a PN_{26}-adjacency as a 26-normal one. □

Example 4.3. *Let us consider $X := SC^{n,l}_k$ and $Y := [a, b]_{\mathbb{Z}}$. For instance, we may consider $X \in \{SC^{2,8}_4, SC^{2,6}_8, SC^{3,7}_{18}, SC^{3,5}_{26}\}$ and $Y := [0, 1]_{\mathbb{Z}}$. Then we obtain the corresponding PN_k-adjacency for the given digital products, $k \in \{18, 26, 64, 80\}$.*
(1) A digital product $(SC^{2,8}_4 \times Y, PN_{18})$ which does not have a normal 18-adjacency.
(2) Examples of a normal k-adjacency for the given digital products: $(SC^{2,6}_8 \times Y, 26)$, $(SC^{3,7}_{18} \times Y, 64)$, $(SC^{3,5}_{26} \times Y, 80)$.



(1) (2)

Figure 4. (1) Assume $SC_4^{2,8} \times Y$ with PN_{18}-adjacency. Then we observe that $N_{18}^{\blacktriangle}(x_2, 1) = \{x_1, x_2, x_3, x_9, x_{10}, x_{11}\}$, $N_{18}(x_2, 1) = N_{18}^{\blacktriangle}(x_2, 1) \cup \{x_0, x_4\}$ so that $N_{18}^{\blacktriangle}(x_2, 1) \neq N_{18}(x_2, 1)$. (2) For each point $p \in (SC_8^{2,6} \times Y, PN_{26})$, we obtain $N_{26}^{\blacktriangle}(p, 1) = N_{26}(p, 1)$.

5. The product property of the $AFPP$ by a PN-k-isomorphism

This section initially studies the almost fixed point property ($AFPP$, for brevity) and the fixed point property (FPP, for short) of a digital product with a PN_k-adjacency in the category $DTC_k^{\blacktriangle}$. To do this work, we initially establish these notions as follows:

Definition 5.1. *Assume a digital product $X_1 \times X_2$ with a PN_k-adjacency derived from (X_i, k_i) on $\mathbb{Z}^{n_i}$, $i \in \{1, 2\}$.*
(1) For a digital product $(X_1 \times X_2, PN_k)$ on $\mathbb{Z}^{n_1+n_2}$ and every PN-k-continuous map $f : X_1 \times X_2 \to X_1 \times X_2$, if there exists a point $p \in X_1 \times X_2$ satisfying $f(p) = p$, then we say that $(X_1 \times X_2, PN_k)$ has the FPP.
(2) For a digital product $(X \times Y, PN_k)$ on $\mathbb{Z}^{n_1+n_2}$ and every PN-k-continuous map $f : X \times Y \to X \times Y$, if there exists a point $p \in X \times Y$ satisfying the property $f(p) \in N_k^{\blacktriangle}(p, 1)$, then we say that $(X_1 \times X_2, PN_k)$ has the AFPP.

Lemma 5.2. *[10] Only the $(3^m - 1)$-adjacency of the digital product $\prod_{i=1}^{m} X_i \subset \mathbb{Z}^m$ of the digital intervals $(X_i, 2)$ is normal, $i \in [1, m]_{\mathbb{Z}}$.*

Let us consider a digital product with a normal k-adjacency, $X \times Y \subset \mathbb{Z}^{n_1+n_2}$, derived from two digital images (X, k_1) on $\mathbb{Z}^{n_1}$ and (Y, k_2) on $\mathbb{Z}^{n_2}$ as in Definition 3.1. Then, for a point $p \in X \times Y$, the paper [10] used the notation $N_k^{\star}(p, 1)$ (see Remark 4.2 of [10]), as follows:

$$\begin{Bmatrix} N_k^{\star}(p) := \{q \in X \times Y \,|\, q \text{ is normally } k\text{-adjacent to } p\} \text{ and} \\ N_k^{\star}(p, 1) := N_k^{\star}(p) \cup \{p\}. \end{Bmatrix} \tag{5.1}$$

As a matter of fact, in Theorem 4.4, Remark 4.5, Corollary 4.7, Lemma 4.8 of the paper [10], the authors just stressed on the condition "$N_k^{\star}(p, 1) = N_k(p, 1)$". However, we may not concern about it, as follows:

Remark 5.3. *In Theorem 4.4, Remark 4.5, Corollary 4.7, Lemma 4.8 of [10], the condition "$N_k^{\star}(p, 1) = N_k(p, 1)$" is redundant because each of these assertions is already assumed to have a normal k-adjacency of the given digital products.*

By Proposition 3.3, we obtain the following:

Lemma 5.4. *Assume a digital product with a normal k-adjacency, $X \times Y \subset \mathbb{Z}^{n_1+n_2}$ derived from two digital images (X, k_1) on $\mathbb{Z}^{n_1}$ and (Y, k_2) on $\mathbb{Z}^{n_2}$. Then, for any point $p := (x, y) \in X \times Y$ we have*

$$N_k^{\blacktriangle}(p, 1) = N_k^{\star}(p, 1) = N_k(p, 1) = N_{k_1}(x, 1) \times N_{k_1}(y, 1).$$

By Lemma 5.2 and Lemma 5.4, we obtain the following:

Lemma 5.5. *Assume a digital product $(\prod_{i=1}^{m} X_i, PN_k)$ on $\mathbb{Z}^n$ derived from digital intervals $(X_i := [m_i, m_i + p_i]_{\mathbb{Z}}, 2)$, $i \in [1, n]_{\mathbb{Z}}$. Then a digital product $\prod_{i=1}^{n} X_i$ has the only PN_{3^n-1}-adjacency.*

Hereafter, let us study the *AFPP* for digital n-dimensional cubes, denoted by $\prod_{i=1}^{n}[m_i, m_i + p_i]_{\mathbb{Z}} \subset \mathbb{Z}^n, m_i \in \mathbb{Z}, p_i \in \mathbb{N}$, as a Cartesian product of finite digital intervals $[m_i, m_i + p_i]_{\mathbb{Z}}$.

Corollary 5.6. *Assume finite digital intervals $(X_i, 2), i \in [1, n]_{\mathbb{Z}}$. If the digital product $\prod_{i=1}^{m} X_i \subset \mathbb{Z}^{n_1+\ldots+n_m}$ has a PN_k-adjacency considered on $\mathbb{Z}^{n_1+\ldots+n_m}$, then $(\prod_{i=1}^{m} X_i, PN_k)$ has the AFPP in $DTC_k^{\blacktriangle}$.*

Proof: The proof is completed by Lemmas 5.2 and 5.5. More precisely, for the set $X := \prod_{i=1}^{n}[m_i, m_i + p_i]_{\mathbb{Z}} \subset \mathbb{Z}^n$ with PN_k-adjacency, where $m_i \in \mathbb{Z}^n$ and $p_i \in \mathbb{N}$. Then (X, PN_k) has the *AFPP* if $k = 3^n - 1$.□

Example 5.1. *Let us consider a digital square such as $[0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}}$ derived from the given unit digital interval $([0, 1]_{\mathbb{Z}}, 2)$. Using Lemma 5.4 and Corollary 5.6, we can consider a PN-8-continuous self-map f of $Q := [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}}$. Then the digital product (Q, PN_8) has the almost fixed point property in the category $DTC_8^{\blacktriangle}$.*

Theorem 5.7. *Assume two digital images (X_1, k_1) in $\mathbb{Z}^2$ and (X_2, k_2) in $\mathbb{Z}$ and consider a digital product $X_1 \times X_2 \subset \mathbb{Z}^3$ with a certain PN_k-adjacency, $k \in \{18, 26\}$. If $k \neq 26$, then $(X_1 \times X_2, PN_k)$ does not have the AFPP in $DTC_k^{\blacktriangle}$.*

Before proving the assertion, we can consider PN_k-adjacency of this $X_1 \times X_2$, $k \in \{18, 26\}$ (see Definition 3.4)).

Proof: To disprove the assertion, we have the following counterexample. Assume $X_1 := \{(x_1, x_2) \mid x_i \in [0, m_i]_{\mathbb{Z}}$ and $3 \leq m_i \in \mathbb{N}\}$, $i \in [1, 2]_{\mathbb{Z}}$, in $\mathbb{Z}^2$. Consider $(X_1, 4)$ and $(X_2 := [0, 1]_{\mathbb{Z}}, 2)$. Then we obtain the corresponding PN_{18}-adjacency for the digital product $X_1 \times X_2$. Then take the two numbers m_1', m_2' such that $m_1' \lneq m_1$ and $m_2' \lneq m_2$. Furthermore, assume the following subsets

$$\left\{\begin{array}{l} A := \{(x_1, x_2) \in X_1 \mid x_1 \in [0, m_1']_{\mathbb{Z}}, x_2 \in [0, m_2']_{\mathbb{Z}}\}, \\ B := \{(x_1, x_2) \in X_1 \mid x_1 \in [0, m_1']_{\mathbb{Z}}, x_2 \in [m_2' + 1, m_2]_{\mathbb{Z}}\}, \\ C := \{(x_1, x_2) \in X_1 \mid x_1 \in [m_1' + 1, m_1]_{\mathbb{Z}}, x_2 \in [0, m_2']_{\mathbb{Z}}\}, \\ D := \{(x_1, x_2) \in X_1 \mid x_1 \in [m_1' + 1, m_1]_{\mathbb{Z}}, x_2 \in [m_2' + 1, m_2]_{\mathbb{Z}}\}. \end{array}\right\} \tag{5.2}$$

where $m_1' \in [1, m_1 - 2]_{\mathbb{Z}}$ and $m_2' \in [1, m_2 - 2]_{\mathbb{Z}}$. Then consider the self-map f of $X_1 \times X_2$ defined by

$$f(x_1, x_2, w) = \left\{\begin{array}{ll} (m_1' + 1, m_2' + 1, w), & (x_1, x_2) \in A \text{ and } w \in X_2 \\ (m_1' + 1, m_2', w), & (x_1, x_2) \in B \text{ and } w \in X_2 \\ (m_1', m_2' + 1, w), & (x_1, x_2) \in C \text{ and } w \in X_2 \\ (m_1', m_2', w), & (x_1, x_2) \in D \text{ and } w \in X_2. \end{array}\right\} \tag{5.3}$$

Then f is a PN-18-continuous map. However, there is no point $(x_1, x_2, w) \in X_1 \times X_2$ satisfying $f(x_1, x_2, w) \in N^{\blacktriangle}_{18}(x_1, x_2, w, 1)$. □

As a more general case, we have the following:

Corollary 5.8. *Assume a digital product $X_1 \times X_2$ with a PN_k-adjacency derived from two digital images $(X_i, k_i) \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$. If $k \neq 3^{n_1+n_2} - 1$, $(X_1 \times X_2, PN_k)$ does not have the $AFPP$ in $DTC^{\blacktriangle}_k$.*

Corollary 5.9. *Assume a digital product $(X_1 \times X_2, PN_k)$ derived from (X_i, k_i) on $\mathbb{Z}^{n_i}, i \in \{1, 2\}$. If $k \neq 3^{n_1+n_2} - 1$, $(X_1 \times X_2, PN_k)$ does not have the FPP in $DTC^{\blacktriangle}_k$.*

Theorem 5.10. *Assume that digital products $(X_1 \times X_2, PN_k)$ on $\mathbb{Z}^n$ derived from $(X_i, k_i) \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$ and $(Y_1 \times Y_2, PN_{k'})$ on $\mathbb{Z}^{n'}$ derived from $(Y_i, k_i) \subset \mathbb{Z}^{m_i}, i \in \{1, 2\}$. If $(X_1 \times X_2, PN_k)$ has the $AFPP$ in $DTC^{\blacktriangle}_k$, then a PN-(k, k')-isomorphism $h : (X_1 \times X_2, PN_k) \to (Y_1 \times Y_2, PN_{k'})$ preserves the $AFPP$ in $DTC^{\blacktriangle}_{k'}$.*

Proof: Assume any PN-k'-continuous self-map g of $(Y_1 \times Y_2, PN_{k'})$. Then consider the composition $g := h \circ f \circ h^{-1}$ as a $PN_{k'}$-continuous self-map of $(Y_1 \times Y_2, PN_{k'})$, where f is a PN-k-continuous self-map of $(X_1 \times X_2, PN_k)$. Owing to the $AFPP$ of $(X_1 \times X_2, PN_k)$, there exists a point $x := (x_1, x_2) \in X_1 \times X_2$ satisfying $f(x) \in N^{\blacktriangle}_k(x, 1)$. Owing to the given PN-(k, k')-isomorphism between $(X_1 \times X_2, PN_k)$ and $(Y_1 \times Y_2, PN_{k'})$, there is a point $y := (y_1, y_2) \in Y_1 \times Y_2$ such that $h(x) = y$. Let us consider the mapping

$$f(x) = (h^{-1} \circ g \circ h)(x) = h^{-1}(g(h(x))) = h^{-1}(g(y)). \tag{5.4}$$

From (5.4), it is clear that $h(f(x)) = g(y)$ and further, owing to the hypothesis of the $AFPP$ of $(X_1 \times X_2, PN_k)$ and the given PN-(k, k')-isomorphism h, we have

$$g(y) = h(f(x)) \in h(N^{\blacktriangle}_k(x, 1)) = N^{\blacktriangle}_{k'}(h(x), 1) = N^{\blacktriangle}_{k'}(y, 1),$$

which implies that the point $h(x)$ is an almost fixed point of the map g, which implies that $(Y_1 \times Y_2, PN_{k'})$ has the $AFPP$ in $DTC^{\blacktriangle}_{k'}$. □

6. Summary and further works

Given digital images $(X_i, k_i), i \in \{1, 2\}$ and $(Y_i, k_i), i \in \{1, 2\}$, after establishing a PN_k-adjacency of a digital product $X_1 \times X_2$ and a $PN_{k'}$-adjacency of a digital product $Y_1 \times Y_2$ and further, we have developed several key notions such as PN-(k, k')-continuity and a PN-(k, k')-isomorphism. Using these concepts, we can classify digital products with PN_k-adjacencies. Finally, after establishing the notion of $AFPP$ for a digital product with a PN_k-adjacency in the category $DTC^{\blacktriangle}_k$. Based on this work, we have addressed several issues in the fields of digital topology and digital geometry. As a further work, we can define some other adjacency relation on a digital product which can be used in the fields of pure mathematics and applied science.

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Conflict of interest

The authors declare no conflict of interest.

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