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Research article

The radius of unit graphs of rings

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Abstract: Let R be a ring with nonzero identity. The unit graph of R is a simple graph whose vertex set is R itself and two distinct vertices are adjacent if and only if their sum is a unit of R. In this paper, we study the radius of unit graphs of rings. We prove that there exists a ring R such that the radius of unit graph can be any given positive integer. We also prove that the radius of unit graphs of self-injective rings are 1, 2, 3, ∞. We classify all self-injective rings via the radius of its unit graph. The radius of unit graphs of some ring extensions are also considered.

Keywords: unit graph; radius; self-injective ring; unit sum number; ring extension
Mathematics Subject Classification: 16U60, 05C25

1. Introduction

In the last decades, there is an active research topic named rings and graphs. That is, one may associate a graph with a ring and then study their interaction. It has attracted considerable attention both in ring theory and graph theory. The first concept is the zero-divisor graph of a commutative ring. Beck [9] introduced the definition and studied the coloring of a ring in 1988 and later Anderson and Livingston [4] modified the definition such that illustrated better the structure of rings. We know that an element in a finite ring is either a zero divisor or a unit. So one can associate with a graph using units of a ring and study the interaction between the properties of the ring and the resulting graph structure. The unit graph of a ring is such a graph and the topic of this paper.

Let R be a ring with identity. The unit graph of R, denoted $G(R)$, is the simple graph whose vertex set is R itself, and two distinct vertices x and y are adjacent if and only if $x + y$ is a unit of R. The unit graph was first investigated in 1990 by Grimaldi for the ring $\mathbb{Z}_n$ in [10] where the author considered the degree of a vertex, the Hamilton cycles, the covering number, the independence number and the chromatic polynomial of the graph $G(\mathbb{Z}_n)$. In 2010, Ashrafi, et al. [8] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R and obtained various characterization results regarding connectedness, chromatic index, diameter, girth, and planarity of $G(R)$. In [21], Su and Zhou proved that the girth of

$G(R)$ for an arbitrary ring R is 3, 4, 6, or ∞ using the method of combination. Recently, Su and Wei [22] investigated the diameter of $G(R)$ and gave the complete characterization of diameter of $G(R)$ for a self-injective ring R. Many papers are devoted to this topic, see, for example, [3], [5], [11], [13], [18], [19] and [20].

Diameter is one of important invariants of a graph. Many papers are devoted to the diameter of resulting graph in this research area, see, for example, [1], [2], [6], [7], [16] and [17]. For the unit graph of a ring, Heydari and Nikmehr [11] proved that the diameter of the unit graph of an Artinian ring only has four possibilities: $1, 2, 3, \infty$ and classified all Artinian rings via its diameter of unit graphs. In 2019, Su and Wei generalized the results to self-injective rings in [22]. They also proved that there is a ring such that its diameter of unit graph is over than 3. Radius is also an important invariant in graph theory. However, there are few results about the algebraic graph structure. For the the radius of unit graphs, we prove that for any positive integer n, there is a ring R such that $G(R)$ has radius n. We also prove that the radius of unit graphs of self-injective rings are 1, 2, 3 and ∞ and classify all self-injective rings via its radius of unit graphs. We consider some ring extensions of a ring R and study their radius via the radius of $G(R)$.

2. Preliminaries

Let us recall some basic definitions in graph theory. All graphs are simple, that is, no loops and no multiedges. Let G be a graph. A walk of length k in G is an alternating sequence of vertices and edges, $v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k$, which begins and ends with vertices. A path of length k in G is a walk with all vertices are distinct. The distance of two vertices x, y in G, denoted $d(x, y)$, is the number of edges in a shortest path between x and y. If there is no path connecting the two vertices, then the distance is defined as infinite. Let x be a vertex of a graph G. The eccentricity of the vertex x, denoted $\varepsilon(x)$, is the maximum distance from x to any vertex. That is, $\varepsilon(x) = max\{d(x, y)|y \in V(G)\}$. The radius of G is the minimum eccentricity among the vertices of G, i.e., $\text{rad}(G) = min\{\varepsilon(x)|x \in V(G)\}$. The diameter of G is the maximum eccentricity among the vertices of G. Thus, $\text{diam}(G) = max\{\varepsilon(x)|x \in V(G)\}$. For any connected graph G, $\text{rad}(G) \le \text{diam}(G) \le 2\text{rad}(G)$.

Throughout, rings are associative with identity. A subring of a ring shares the same multiplicative identity. We use $J(R)$, $U(R)$ and $char(R)$ to denote the Jacobson radical, the group of units, and the characteristic of a ring R, respectively. A ring R is called a division ring if each nonzero element of R is a unit. A ring R is called a local ring if R has a unique maximal ideal. We write $\overline{R} = R/J(R)$ and $\bar{a} = a + J(R) \in \overline{R}$ for $a \in R$. We denote by $\mathbb{Z}_n$ the ring of integers modulo n and by $\mathbb{F}_p$ the field of p elements. The polynomial ring over a ring R in the indeterminate x is denoted by $R[x]$. Recall that a ring R is called *right self-injective* if, for any (principal) right ideal I of R, every homomorphism from I_R to R_R extends to a homomorphism from R_R to R_R. Note that R is right self-injective implies that $R/J(R)$ is right self-injective (see [23]).

If a graph is disconnected, then its radius is clearly infinite. To find the radius of unit graphs of rings, we mainly consider connected graph. As shown in [8], the connectedness of $G(R)$ is relative to whether the ring R is generated additively by its units. So we first recall the following definitions. Let R be a ring and k be a positive integer. An element $r \in R$ is said to be *k-good* if $r = u_1 + \cdots + u_k$ with $u_i \in U(R)$ for each $1 \le i \le k$. A ring is said to be k-good if every element of R is k-good. The *unit sum number* of a ring R, denoted by $\mathbf{u}(R)$, is defined to be

(1) $\min\{k \in \mathbb{N} \mid R$ is a k-good $\}$, if R is k-good for some $k \geq 1$;

(2) ω, if R is not k-good for every $k \geq 1$, but each element of R is k-good for some k;

(3) ∞, some element of R is not k-good for any $k \geq 1$.

For example, $\mathbf{u}(\mathbb{Z}_3) = 2$, $\mathbf{u}(\mathbb{Z}_4) = \omega$ and $\mathbf{u}(\mathbb{Z}[t]) = \infty$. It is clear that if $2 \in U(R)$, then $r \in R$ being k-good implies that r is l-good for all $l \geq k$. For the unit sum number of rings, we refer the reader to [14], [15] and [24]. We note that, however, in the previous example, every element in $\mathbb{Z}_4$ can be expressed as a sum of at most two units. So we recall another slightly different definition which was introduced in [12]. Let $usn(R)$ be the smallest number n such that every element can be written as the sum of **at most** n units. If some element of R is not k-good for any $k \geq 1$, then $usn(R)$ is defined to be ∞. Note that $usn(R)$ and $\mathbf{u}(R)$ are different. For example, $\mathbf{u}(\mathbb{Z}_4) = \omega$ and $usn(\mathbb{Z}_4) = 2$.

3. The radius of unit graphs

In this section, we focus on the radius of the unit graph of a ring. It is easy to see that $\mathrm{rad}(G(\mathbb{Z}_3)) = 1$, $\mathrm{rad}(G(\mathbb{Z}_4)) = 2$, $\mathrm{rad}(G(\mathbb{Z}_6)) = 3$ and $\mathrm{rad}(G(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \infty$. So, the radius of unit graphs of rings has at least four possibilities. We first determine when the radius of the unit graph of a ring meets one of the unit graph of its factor ring. We begin with an obvious result.

Lemma 3.1. *Let R be a ring. Then* $\mathrm{rad}(G(R)) = 1$ *if and only if R is a division ring.*

Proof. ($\Rightarrow$). Suppose that $\mathrm{rad}(G(R)) = 1$. There exists an element $a \in R$ such that $\varepsilon(a) = 1$, that is, for any $x \in R$, $d(a, x) = 1$. Now for a nonzero element $r \in R$, we have $d(a, r - a) = 1$. So $r = a + (r - a) \in U(R)$. Thus, R is a division ring.

($\Leftarrow$). Suppose that R is a division ring. Since $\varepsilon(0) = 1$, we have $\mathrm{rad}(G(R)) = 1$. □

Lemma 3.2. *Let R be a ring and $\overline{R} = R/J(R)$. If* $\mathrm{rad}(G(R)) \geq 3$*, then* $\mathrm{rad}(G(\overline{R})) = \mathrm{rad}(G(R))$.

Proof. We first prove that $\mathrm{rad}(G(\overline{R})) \leq \mathrm{rad}(G(R))$. If $\mathrm{rad}(G(R)) = \infty$, there is nothing to prove. Suppose that $\mathrm{rad}(G(R)) = n < \infty$, then there exists $a \in R$ such that $\varepsilon(a) = n$. For any $\overline{x} \in \overline{R}$, a path from a to x deduces a walk from $\overline{a}$ to $\overline{x}$, so $d(\overline{a}, \overline{x}) \leq d(a, x) \leq n$, this implies $\varepsilon(\overline{a}) \leq n$ and thus $\mathrm{rad}(G(\overline{R})) \leq n$.

Now we prove $\mathrm{rad}(G(\overline{R})) \geq \mathrm{rad}(G(R))$. If $\mathrm{rad}(G(\overline{R})) = \infty$, there is nothing to prove. Let $\mathrm{rad}(G(\overline{R})) = n < \infty$. Then there exists $\overline{a} \in \overline{R}$ such that $\varepsilon(\overline{a}) = n$. For $x \in R$, if $\overline{a} = \overline{x}$, then $d(a, x) = 2 < n$; if $\overline{a} \neq \overline{x}$, then a path from $\overline{a}$ to $\overline{x}$ deduces a path from a to x, so $d(\overline{a}, \overline{x}) = d(a, x) \leq n$. This implies $\varepsilon(a) \leq n$. So $\mathrm{rad}(G(R)) \leq n$. □

Theorem 3.3. *Let R be a ring. Then the following statements are equivalent.*

(1) $\mathrm{rad}(G(\overline{R})) < \mathrm{rad}(G(R))$.

(2) $\mathrm{rad}(G(\overline{R})) = 1$ *and* $\mathrm{rad}(G(R)) = 2$.

(3) *R is a local ring but not a division ring.*

Proof. (2) $\Rightarrow$ (1). It is clear.

(1) $\Rightarrow$ (3). Suppose that $\mathrm{rad}(G(\overline{R})) < \mathrm{rad}(G(R))$. Then by Lemma 3.2, $\mathrm{rad}(G(R)) \leq 2$. As $\mathrm{rad}(G(R)) = 1$ implies $\mathrm{rad}(G(\overline{R})) = 1$ by Lemma 3.1, we have $\mathrm{rad}(G(\overline{R})) = 1$ and $\mathrm{rad}(G(R)) = 2$. Thus $\overline{R}$ is a division ring again by Lemma 3.1. So R is a local ring. As $\mathrm{rad}(G(R)) = 2$, R is not a division ring by Lemma 3.1.

(3) ⇒ (2). Suppose that R is a local ring but not a division ring. For any $0 \neq x \in R$, if $x \in U(R)$, then $d(0, x) = 1$; if $x \notin U(R)$, then the path 0—1—x implies $d(0, x) = 2$. So $\varepsilon(0) \leq 2$ and hence $\mathrm{rad}(G(R)) \leq 2$. As R is not a division ring, we know that $\mathrm{rad}(G(R)) \neq 1$ by Lemma 3.1. So $\mathrm{rad}(G(R)) = 2$. Note that $\overline{R}$ is a division ring, so, by Lemma 3.1, we have $\mathrm{rad}(G(\overline{R})) = 1$. □

Corollary 3.4. *Let R be a ring. Then* $\mathrm{rad}(G(\overline{R})) = \mathrm{rad}(G(R))$ *if and only if one of following holds:*

(1) *R is not a local ring.*

(2) *R is a division ring.*

As we mentioned in the previous, the radius of unit graphs of rings has at least four possibilities. Using the next theorem, we prove that the radius of unit graphs of rings can be any positive integer. For the completeness, we recall a lemma.

Lemma 3.5. *[22, Lemma 2.2] Let R be a ring and $r \in R$. Then the following hold:*

(1) *If r is k-good, then $d(r, 0) \leq k$;*

(2) *If $r \neq 0$ and $d(r, 0) = k$, then r is k-good but not l-good for all $l < k$.*

Theorem 3.6. *Let R be a ring but not a division ring. For an integer $n \geq 2$, the following statements are equivalent.*

(1) $\varepsilon(0) = n$.

(2) $usn(R) = n$.

(3) $\mathrm{rad}(G(R)) = n$.

Proof. (1) ⇒ (2). For any $r \in R$, since $\varepsilon(0) = n$, we have $d(r, 0) \leq n$. This implies that r is k-good for some $k \leq n$ by Lemma 3.5(2). Again, as $\varepsilon(0) = n$, there exists $x \in R$ such that $d(x, 0) = n$. This deduces that x is n-good, but not $(n-1)$-good by Lemma 3.5. By the definition of $usn(R)$, we know $usn(R) = n$.

(2) ⇒ (1). Suppose that $usn(R) = n$. Then for any given element r in R, r is k_r-good, for some $k_r \leq n$. So $d(r, 0) \leq k_r \leq n$ by Lemma 3.5(1). So $\varepsilon(0) \leq n$. As $usn(R) = n$, there must exist an element $x \in R$ such that x is exactly n-good, but not $(n-1)$-good. Thus $d(x, 0) = n$ by Lemma 3.5(2). So, $\varepsilon(0) = n$.

(2) ⇒ (3). Assume $usn(R) = n$. For any given element $x \in R$, by assumption, x is k_r-good for some $k_r \leq n$. Then $d(x, 0) \leq k_r$ by Lemma 3.5. So $\varepsilon(0) \leq k_r \leq n$. Thus $\mathrm{rad}(G(R)) \leq n$. As $usn(R) = n \geq 2$, there exists an element $y \in R$ such that y is exactly n-good. By Lemma 3.5, $d(y, 0) = n$ and hence $\varepsilon(0) = n$. So $\mathrm{rad}(G(R)) = n$.

(3) ⇒ (2). Assume $\mathrm{rad}(G(R)) = n \geq 2$. we have $\varepsilon(0) = k \geq n$. If $\varepsilon(0) = k > n$, by the equivalencies of (1) and (2), we have $usn(R) = k > n$, a contradiction. So $\varepsilon(0) = n$. Again by the equivalencies of (1) and (2), we have $usn(R) = n$. □

In [12, Corollary 4], the author has proved that there exists a ring R such that $usn(R) = n$ for any given positive integer n.

Corollary 3.7. *For a positive integer n, there exists a ring R such that* $\mathrm{rad}(G(R)) = n$.

Proof. The result follows by Lemma 3.1, Theorem 3.6 and [12, Corollary 4]. □

4. Self-injective rings

Theorem 4.1. *[22, Theorem 3.6] Let R be a ring with $R/J(R)$ right self-injective (in particular, R is right self-injective). Then* $\mathrm{diam}(G(R)) \in \{1, 2, 3, \infty\}$.

The following result is an easy observation.

Corollary 4.2. *Let R be a ring with $R/J(R)$ right self-injective (in particular, R is right self-injective). Then* $\mathrm{rad}(G(R)) \in \{1, 2, 3, \infty\}$.

Proof. By the fact that $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$ and Theorem 4.1, the result follows. □

In [14, Theorem 6], Khurana and Srivastava determined the unit sum number $\mathbf{u}(R)$ of a regular right self-injective ring R. We use the notion $usn(R)$ to restate the theorem below.

Lemma 4.3. *[14] Let R be a regular self-injective ring. Then $usn(R) = 2$, 3 or ∞. Moreover,*

(1) *$usn(R) = 2$ if and only if R has no nonzero Boolean ring as a ring direct summand or $R \cong \mathbb{Z}_2$.*

(2) *$usn(R) = 3$ if and only if $R \ncong \mathbb{Z}_2$ and R has $\mathbb{Z}_2$, but no Boolean ring with more than two elements, as a ring direct summand.*

(3) *$usn(R) = \infty$ if and only if R has a Boolean ring with more than two elements as a ring direct summand.*

Theorem 4.4. *Let R be a right self-injective ring. Then the following hold.*

(1) $\mathrm{rad}(G(R)) = 1$ *if and only if R is a division ring.*

(2) $\mathrm{rad}(G(R)) = 2$ *if and only if R is not a division ring and one of following holds*

(i) R has no factor ring isomorphic to $\mathbb{Z}_2$

(ii) $\overline{R} \cong \mathbb{Z}_2$.

(3) $\mathrm{rad}(G(R)) = 3$ *if and only if R has exactly one factor ring isomorphic to $\mathbb{Z}_2$ and $\overline{R} \ncong \mathbb{Z}_2$.*

(4) $\mathrm{rad}(G(R)) = \infty$ *if and only if R has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. By Lemma 4.3, we have $usn(R) = 2$, 3 or ∞. To complete the proof, we just need to discuss all possibilities of $usn(R)$.

Case 1. $usn(R) = 2$. Then, by Lemma 4.3, R has no nonzero Boolean ring as a ring direct summand or $R \cong \mathbb{Z}_2$. If R is a division ring, then $\mathrm{rad}(G(R)) = 1$ by Lemma 3.1. If R is not a division ring, then by Theorem 3.6, we know $\mathrm{rad}(G(R)) = 2$.

Case 2. $usn(R) = 3$. Then R has a factor ring isomorphic to $\mathbb{Z}_2$, but has no factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by Lemma 4.3. If $\overline{R} \cong \mathbb{Z}_2$, then $G(R)$ is a complete bipartite graph with at least four vertices. So $\mathrm{rad}(G(R)) = 2$. If $\overline{R} \ncong \mathbb{Z}_2$, we claim that $\mathrm{rad}(G(R)) = 3$. To see this, by Corollary 4.2, $\mathrm{rad}(G(\overline{R})) \leq 3$. Note that $d((0, 0), (x, 1)) = 3$ if x is not a unit. So $\varepsilon(0) = 3$ and hence $\mathrm{rad}(G(R)) = 3$ by Theorem 3.6.

Case 3. $usn(R) = \infty$. Then $G(R)$ is disconnected by [8, Theorem 4.3], so $\mathrm{rad}(G(R)) = \infty$. □

Afkhami and Khosh-Ahang [5] studied the unit graphs of polynomial rings and power series rings. Concerning the diameter, they prove that

Theorem 4.5. *[5, Theorem 3.5] Let R be a commutative ring. Then the following hold.*

(1) *The unit graph $G(R[x])$ is always disconnected. In particular,* $\mathrm{diam}(G(R)) = \infty$.

(2) *If* $\mathrm{diam}(G(R)) = 1$*, then* $\mathrm{diam}(G(R[[x]])) = 2$.

(3) *If* $\mathrm{diam}(G(R)) \geq 2$*, then* $\mathrm{diam}(G(R)) = \mathrm{diam}(G(R[[x]]))$.

Similarly, we have following theorem.

Theorem 4.6. *Let R be a commutative ring. Then the following hold.*

(1) $\mathrm{rad}(G(R[x])) = \infty$.

(2) *If* $\mathrm{rad}(G(R)) = 1$*, then* $\mathrm{rad}(G(R[[x]])) = 2$.

(3) *If* $\mathrm{rad}(G(R)) \geq 2$*, then* $\mathrm{rad}(G(R)) = \mathrm{rad}(G(R[[x]]))$.

Proof. (1) As the unit graph $G(R[x])$ is always disconnected, we have $\mathrm{rad}(G(R[x])) = \infty$.

(2) By Lemma 3.1, $\mathrm{rad}(G(R)) = 1$ implies that R is a division ring. For $f(x) = \sum_i a_i x^i \in G(R[[x]])$, if $a_0 \in U(R)$, then $d(0, f(x)) = 1$. If $a_0 = 0$, then the path 0—1—$f(x)$ deduces $d(0, f(x)) = 2$. So $\varepsilon(0) = 2$ in $G(R[[x]])$. By Theorem 3.6, $\mathrm{rad}(G(R[[x]])) = 2$.

(3) If $\mathrm{rad}(G(R)) = \infty$, then $G(R)$ is disconnected. This implies $G(R[[x]])$ is also disconnected and hence $\mathrm{rad}(G(R[x])) = \infty$. Now suppose that $\mathrm{rad}(G(R)) = n \geq 2$. Then, by Theorem 3.6, $\varepsilon(0) = n$ in $G(R)$. So $\varepsilon(0) \geq n$ in $G(R[[x]])$. For $f(x) = \sum_i a_i x^i \in G(R[[x]])$, as $d(0, a_0) \leq n$, we have $d(0, f(x)) \leq n$ and thus $\varepsilon(0) \leq n$. So $\varepsilon(0) = n$ in $G(R[[x]])$ and thus $\mathrm{rad}(G(R[[x]])) = n$ by Theorem 3.6. □

5. Conclusions

The properties of unit graphs of rings are widely studied. In this paper, we obtain the radius of unit graphs of self-injective rings and completely classified the self-injective rings via the radius of its unit graphs. One may consider how large class of rings having radius 1,2,3 and ∞. We think that this question is closely relative to the unit sum number of rings. In the end of this paper, we investigated the polynomial extension of rings. For the further research, we may consider the other extensions, for example, matrix extension, that is, one may consider to determine $\mathrm{rad}(G(\mathbb{M}_n(R)))$ in terms of $\mathrm{rad}(G(R))$. The trivial extension of a ring R by an R-bimodule M is $R \propto M := \{(a, x) : a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$. In fact, $R \propto M$ is isomorphic to the subring $\{\left(\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right) : a \in R, x \in M\}$ of the formal triangular matrix ring $\left(\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix}\right)$. One may consider to determine $\mathrm{rad}(G(R \propto M))$ in terms of $\mathrm{rad}(G(R))$ and the properties of M.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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