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Research article

Hermite-Hadamard type inequalities based on the Erdélyi-Kober fractional integrals

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Abstract: In the paper, based on Erdélyi-Kober fractional integrals ${}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f$ and ${}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f$ for any $\chi \in [a,b]$ with $f \in \mathfrak{X}^{p}_{c}(a,b)$, authors establish some new Hermite-Hadamard type inequalities for convex function. The obtained inequalities generalize the corresponding results for Riemann-Liouville fractional integrals by taking limits when a parameter $\rho \to 1$. As applications, the error estimations of Hermite-Hadamard type inequality are also provided.

Keywords: Hermite-Hadamard inequality; convex function; Erdélyi-Kober fractional integrals; Riemann-Liouville fractional integrals; error estimations
Mathematics Subject Classification: 26A33, 26D07, 26D10, 26D15

1. Introduction

Fractional calculus is a field of applied mathematics and deals with derivatives and integrals of arbitrary orders (including complex orders). Although the definitions for fractional integrals are inconsistent and work in some cases but not in others, there are almost practical applications and profound impact in science, engineering, mathematics, economics, and other fields.

Suppose that (a,b) is a finite or infinite interval of the real line $\mathbb{R}$, where $a < b$ and $a,b \in [-\infty,+\infty]$, and α is a complex number with $\mathcal{R}e(\alpha) > 0$. Let $\Gamma(\cdot)$ be the Euler's gamma function given by

$$\Gamma(\chi) = \int_0^{\infty} \tau^{\chi-1} e^{-\tau} \mathrm{d}\tau.$$

In [20], Podlubny introduced the left-side and right-side Riemann-Liouville fractional integrals of order α of a function f as follows:

$$\mathcal{R}^{\alpha}_{a+}f(\chi) = \frac{1}{\Gamma(\alpha)} \int_a^{\chi} (\chi-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau \tag{1.1}$$

and

$$\mathcal{R}^{\alpha}_{b-}f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{b} (\tau - \chi)^{\alpha-1} f(\tau)\mathrm{d}\tau, \tag{1.2}$$

respectively, where f is a function on the interval $[a, b]$ such that $(\chi - \tau)^{\alpha-1} f(\tau) \in L[a, b]$ for any $\chi \in [a, b]$.

In [22], Samko introduced the left-side and right-side Hadamard fractional integrals of order α of a function f as follows

$$\mathcal{H}^{\alpha}_{a+}f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{a}^{\chi} (\ln \chi - \ln \tau)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau} \tag{1.3}$$

and

$$\mathcal{H}^{\alpha}_{b-}f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{b} (\ln \tau - \ln \chi)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \tag{1.4}$$

respectively, where f is a function on the interval $[a, b]$ such that $(\ln \chi - \ln \tau)^{\alpha-1} \frac{f(\tau)}{\tau} \in L[a, b]$ for any $\chi \in [a, b]$.

Suppose that $\mathfrak{X}^{p}_{c}(a, b)$ is the space of the complex-valued Lebesgue measurable functions f on $[a, b]$ with $\|f\|_{\mathfrak{X}^{p}_{c}} < \infty$, that is

$$\mathfrak{X}^{p}_{c}(a, b) = \{f : [a, b] \to \mathbb{C} \big| \|f\|_{\mathfrak{X}^{p}_{c}} < \infty\},$$

where the norm $\|f\|_{\mathfrak{X}^{p}_{c}}$ is

$$\|f\|_{\mathfrak{X}^{p}_{c}} = \left(\int_{a}^{b} |\tau^{c} f(\tau)|^{p} \frac{\mathrm{d}\tau}{\tau} \right)^{1/p} \quad for \quad 1 \le p < \infty \quad and \quad c \in \mathbb{R}$$

and

$$\|f\|_{\mathfrak{X}^{\infty}_{c}} = \mathtt{ess}\sup_{a \le \tau \le b} [\tau^{c} |f(\tau)|] \quad for \quad p = \infty \quad and \quad c \in \mathbb{R}.$$

In the sense of the above function space, Katugampola in [16] introduced the left-side and right-side fractional integrals of order α of a function $f \in \mathfrak{X}^{p}_{c}(a, b)$ defined by

$$^{\rho}\mathcal{K}^{\alpha}_{a+}f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{a}^{\chi} \left(\frac{\chi^{\rho} - \tau^{\rho}}{\rho} \right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \quad (\rho > 0) \tag{1.5}$$

and

$$^{\rho}\mathcal{K}^{\alpha}_{b-}f(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{b} \left(\frac{\tau^{\rho} - \chi^{\rho}}{\rho} \right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \quad (\rho > 0), \tag{1.6}$$

respectively.

The above fractional operators are known as Erdélyi-Kober fractional integrals in [18], or Katugampola fractional integrals in [16] or ρ–Riemann-Liouville fractional integrals in [6], which generalized fractional integrals of Riemann-Liouville and Hadamard, respectively [17]:

$$\begin{aligned} \lim_{\rho \to 1} [^{\rho}\mathcal{K}^{\alpha}_{a+}f(\chi)] &= \lim_{\rho \to 1} \frac{1}{\Gamma(\alpha)} \int_{a}^{\chi} \left(\frac{\chi^{\rho} - \tau^{\rho}}{\rho} \right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ &= \frac{1}{\Gamma(\alpha)} \int_{a}^{\chi} (\chi - \tau)^{\alpha-1} f(\tau)\mathrm{d}\tau = \mathcal{R}^{\alpha}_{a+}f(\chi) \end{aligned} \tag{1.7}$$

and

$$\lim_{\rho\to 0}[{}^{\rho}\mathcal{K}^{\alpha}_{a+}f(\chi)] = \lim_{\rho\to 0}\frac{1}{\Gamma(\alpha)}\int_a^{\chi}\Big(\frac{\chi^{\rho}-\tau^{\rho}}{\rho}\Big)^{\alpha-1}f(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}$$
$$= \frac{1}{\Gamma(\alpha)}\int_a^{\chi}(\ln\chi-\ln\tau)^{\alpha-1}f(\tau)\frac{\mathrm{d}\tau}{\tau} = \mathcal{H}^{\alpha}_{a+}f(\chi). \tag{1.8}$$

The similar results for right-sided fractionals integral also hold.

For more results on the fractional integrals please see [1, 3, 4, 8, 13–15, 21, 24] and the references therein.

For any convex function $f : [a, b] \to \mathbb{R}$, the following double inequalities

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(\tau)\mathrm{d}\tau \le \frac{f(a)+f(b)}{2} \tag{1.9}$$

are known as Hermite-Hadamard inequality [12, 19].

In [5], Chen et al. established the following Hermite-Hadamard type inequalities based on the Katugampola fractional integrals.

Theorem 1.1. *Suppose that $f : [a^{\rho}, b^{\rho}] \to \mathbb{R}$ is a positive function with $\rho > 0$ and $0 \le a < b$, and $f \in \mathfrak{X}^p_c(a^{\rho}, b^{\rho})$. If f is a convex function on $[a, b]$, then for any $\alpha > 0$*

$$f\Big(\frac{a^{\rho}+b^{\rho}}{2}\Big) \le \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}\mathcal{K}^{\alpha}_{a+}f(b^{\rho}) +^{\rho}\mathcal{K}^{\alpha}_{b-}f(a^{\rho})] \le \frac{f(a^{\rho})+f(b^{\rho})}{2}, \tag{1.10}$$

where the fractional integrals are considered for the function $f(\chi^{\rho})$ and evaluated at a and b, respectively.

Furthermore, Chen et al. also gave some right estimations of the Hermite-Hadamard type inequalities for the Katugampola fractional integrals in [5].

Theorem 1.2. *Suppose that $f : [a^{\rho}, b^{\rho}] \to \mathbb{R}$ is a differentiable function on (a^{ρ}, b^{ρ}) with $\rho > 0$ and $0 \le a < b$, and $f \in \mathfrak{X}^p_c(a^{\rho}, b^{\rho})$. If $|f'|$ is a convex function on $[a^{\rho}, b^{\rho}]$, then for any $\alpha > 0$*

$$\Big|\frac{f(a^{\rho})+f(b^{\rho})}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}\mathcal{K}^{\alpha}_{a+}f(b^{\rho}) +^{\rho}\mathcal{K}^{\alpha}_{b-}f(a^{\rho})]\Big|$$
$$\le \frac{b^{\rho}-a^{\rho}}{2(\alpha+1)}[|f'(a^{\rho})| + |f'(b^{\rho})|], \tag{1.11}$$

where the fractional integrals are considered for the function $f(\chi^{\rho})$ and evaluated at a and b, respectively.

Theorem 1.3. *Suppose that $f : [a^{\rho}, b^{\rho}] \to \mathbb{R}$ is a differentiable function on (a^{ρ}, b^{ρ}) with $\rho > 0$ and $0 \le a < b$, and $f \in \mathfrak{X}^p_c(a^{\rho}, b^{\rho})$. If $|f'|$ is convex on $[a^{\rho}, b^{\rho}]$, then for any $\alpha > 0$*

$$\Big|\frac{f(a^{\rho})+f(b^{\rho})}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}\mathcal{K}^{\alpha}_{a+}f(b^{\rho}) +^{\rho}\mathcal{K}^{\alpha}_{b-}f(a^{\rho})]\Big|$$
$$\le \frac{b^{\rho}-a^{\rho}}{2(\alpha+1)}\Big(1-\frac{1}{2^{\alpha}}\Big)[|f'(a^{\rho})| + |f'(b^{\rho})|], \tag{1.12}$$

where the fractional integrals are considered for the function $f(\chi^{\rho})$ and evaluated at a and b, respectively.

For more results on the convexity and Hermite-Hadamard type inequalities please see [2,7,9–11,23] and the references therein.

In the paper, based on Erdélyi-Kober fractional integrals ${}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f(b^{\rho})$ and ${}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f(a^{\rho})$ for any $\chi \in [a,b]$ with $f \in \mathfrak{X}^{p}_{c}(a,b)$, authors establish some new Hermite-Hadamard type inequalities for convex function. The obtained inequalities generalize the corresponding results for Riemann-Liouville fractional integrals by taking limits when a parameter $\rho \to 1$. As applications, the error estimations of Hermite-Hadamard type inequality are also provided.

2. Main results

Firstly, we establish Hermite-Hadamard type inequality for the Erdélyi-Kober fractional integrals ${}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f(b^{\rho})$ and ${}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f(a^{\rho})$ for any $\chi \in [a,b]$ with $f \in \mathfrak{X}^{p}_{c}(a,b)$.

Theorem 2.1. *Suppose that $f : [a^{\rho}, b^{\rho}] \to \mathbb{R}$ is a function with $\rho > 0$ and $0 \le a < b$, and $f \in \mathfrak{X}^{p}_{c}(a^{\rho}, b^{\rho})$. If f is a convex function on $[a^{\rho}, b^{\rho}]$, then for any $\alpha > 0$ and any $\chi \in [a,b]$,*

$$
\begin{aligned}
f\Big(\frac{1}{\alpha+1}\frac{a^{\rho}+b^{\rho}}{2}+\frac{\alpha}{\alpha+1}\chi^{\rho}\Big) &\le \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2}\Big[\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f(b^{\rho})}{(b^{\rho}-\chi^{\rho})^{\alpha}}+\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f(a^{\rho})}{(\chi^{\rho}-a^{\rho})^{\alpha}}\Big] \\
&\le \frac{1}{\alpha+1}\Big[\frac{f(a^{\rho})+f(b^{\rho})}{2}+\alpha f(\chi^{\rho})\Big].
\end{aligned} \tag{2.1}
$$

Proof. It easy to follow that

$$
\begin{aligned}
&\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2}\Big[\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f(b^{\rho})}{(b^{\rho}-\chi^{\rho})^{\alpha}}+\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f(a^{\rho})}{(\chi^{\rho}-a^{\rho})^{\alpha}}\Big] \\
=&\frac{\rho\alpha}{2}\Big[\int_{\chi}^{b}\Big(\frac{b^{\rho}-t^{\rho}}{b^{\rho}-\chi^{\rho}}\Big)^{\alpha-1}\frac{t^{\rho-1}f(t^{\rho})}{b^{\rho}-\chi^{\rho}}\mathrm{d}t+\int_{a}^{\chi}\Big(\frac{t^{\rho}-a^{\rho}}{\chi^{\rho}-a^{\rho}}\Big)^{\alpha-1}\frac{t^{\rho-1}f(t^{\rho})}{\chi^{\rho}-a^{\rho}}\mathrm{d}t\Big].
\end{aligned} \tag{2.2}
$$

Making the integral transformations $t^{\rho} = \tau^{\rho}\chi^{\rho} + (1-\tau^{\rho})b^{\rho}$ and $t^{\rho} = (1-\tau^{\rho})a^{\rho} + \tau^{\rho}\chi^{\rho}$ respectively in (2.2), it is obtained that

$$
\begin{aligned}
&\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2}\Big[\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f(b^{\rho})}{(b^{\rho}-\chi^{\rho})^{\alpha}}+\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f(a^{\rho})}{(\chi^{\rho}-a^{\rho})^{\alpha}}\Big] \\
=&\frac{\rho\alpha}{2}\int_{0}^{1}\tau^{\rho\alpha-1}[f(\tau^{\rho}\chi^{\rho}+(1-\tau^{\rho})b^{\rho})+f((1-\tau^{\rho})a^{\rho}+\tau^{\rho}\chi^{\rho})]\mathrm{d}\tau.
\end{aligned} \tag{2.3}
$$

Then by the convexity of f and Jensen's inequality, we obtain

$$
\begin{aligned}
&\frac{\rho\alpha}{2}\int_{0}^{1}\tau^{\rho\alpha-1}[f(\tau^{\rho}\chi^{\rho}+(1-\tau^{\rho})b^{\rho})+f((1-\tau^{\rho})a^{\rho}+\tau^{\rho}\chi^{\rho})]\mathrm{d}\tau \\
\ge&\rho\alpha\int_{0}^{1}\tau^{\rho\alpha-1}f\Big(\tau^{\rho}\chi^{\rho}+(1-\tau^{\rho})\frac{a^{\rho}+b^{\rho}}{2}\Big)\mathrm{d}\tau \\
\ge&\rho\alpha f\Bigg(\frac{\int_{0}^{1}\tau^{\rho\alpha-1}\Big[\tau^{\rho}\chi^{\rho}+(1-\tau^{\rho})\frac{a^{\rho}+b^{\rho}}{2}\Big]\mathrm{d}\tau}{\int_{0}^{1}\tau^{\rho\alpha-1}\mathrm{d}\tau}\Bigg)\int_{0}^{1}\tau^{\rho\alpha-1}\mathrm{d}\tau
\end{aligned}
$$

$$=f\Big(\frac{1}{\alpha+1}\frac{a^\rho+b^\rho}{2}+\frac{\alpha}{\alpha+1}\chi^\rho\Big), \tag{2.4}$$

which completes the left inequality of Theorem 2.1.

On the another hand, by the convexity of f again, we have

$$\begin{aligned}
&\frac{\rho\alpha}{2}\int_0^1 \tau^{\rho\alpha-1}[f(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)+f((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)]\mathrm{d}\tau\\
\le&\frac{\rho\alpha}{2}\int_0^1 \tau^{\rho\alpha-1}[\tau^\rho f(\chi^\rho)+(1-\tau^\rho)f(b^\rho)+(1-\tau^\rho)f(a^\rho)+\tau^\rho f(\chi^\rho)]\mathrm{d}\tau\\
=&\rho\alpha\int_0^1 \tau^{\rho\alpha-1}\Big[\tau^\rho f(\chi^\rho)+(1-\tau^\rho)\frac{f(a^\rho)+f(b^\rho)}{2}\Big])\mathrm{d}\tau\\
=&\frac{1}{\alpha+1}\Big[\frac{f(a^\rho)+f(b^\rho)}{2}+\alpha f(\chi^\rho)\Big],
\end{aligned} \tag{2.5}$$

which completes the right inequality of Theorem 2.1. □

Corollary 2.1.1. *With the assumptions of Theorem 2.1 and taking $\chi^\rho=\frac{a^\rho+b^\rho}{2}$, it reduces that*

$$\begin{aligned}
f\Big(\frac{a^\rho+b^\rho}{2}\Big)\le&\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\Big[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)\Big]\\
\le&\frac{1}{\alpha+1}\Big[\frac{f(a^\rho)+f(b^\rho)}{2}+\alpha f\Big(\frac{a^\rho+b^\rho}{2}\Big)\Big]\le\frac{f(a^\rho)+f(b^\rho)}{2}.
\end{aligned} \tag{2.6}$$

In particular, taking limits when $\chi\to a$ and $\chi\to b$ respectively in the inequality (2.1), and using the L'Hospital rule, we have the following result.

Corollary 2.1.2. *With the assumptions of Theorem 2.1 and taking limits when $\chi\to a$ and $\chi\to b$ respectively, it reduces that*

$$\begin{aligned}
f\Big(\frac{a^\rho+b^\rho}{2}\Big)\le&\frac{1}{2}\Big[f\Big(\frac{(2\alpha+1)a^\rho+b^\rho}{2(\alpha+1)}\Big)+f\Big(\frac{a^\rho+(2\alpha+1)b^\rho}{2(\alpha+1)}\Big)\Big]\\
\le\frac{f(a^\rho)+f(b^\rho)}{4}+&\frac{\rho^\alpha\Gamma(\alpha+1)}{4(b^\rho-a^\rho)^\alpha}\Big[{}^\rho\mathcal{K}^\alpha_{a+}f(b^\rho)+{}^\rho\mathcal{K}^\alpha_{b-}f(a^\rho)\Big]\le\frac{f(a^\rho)+f(b^\rho)}{2}.
\end{aligned} \tag{2.7}$$

3. Error estimations of Hermite-Hadamard type inequality

Now we give a interest equality.

Lemma 3.1. *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable on (a^ρ,b^ρ) with $\rho>0$ and $0\le a<b$, and $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$. If the generalized fractional integrals exist, then for any $\alpha>0$ and any $\chi\in[a,b]$, the equality*

$$\begin{aligned}
&\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\Big]\\
=&\frac{\rho}{2}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}[(b^\rho-\chi^\rho)f'(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)-(\chi^\rho-a^\rho)f'((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)]\mathrm{d}\tau
\end{aligned} \tag{3.1}$$

holds, where the fractional integrals are considered for the function $f(\chi^\rho)$ and evaluated at a and b, respectively.

Proof. Using integration by parts, it easy to follow that

$$
\begin{aligned}
&\frac{\rho}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}[(b^\rho-\chi^\rho)f'(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)-(\chi^\rho-a^\rho)f'((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)]\mathrm{d}\tau\\
=&\frac{f(b^\rho)}{2}-\frac{\rho\alpha}{2}\int_0^1 \tau^{\rho\alpha-1}f(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)\mathrm{d}\tau\\
&+\frac{f(a^\rho)}{2}-\frac{\rho\alpha}{2}\int_0^1 \tau^{\rho\alpha-1}f((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)\mathrm{d}\tau\\
=&\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\Big], \qquad (3.2)
\end{aligned}
$$

which completes the proof of Lemma 3.1. □

In particular, taking limits when $\chi \to a$ and $\chi \to b$ respectively in the identity (3.1), and summing the obtained identities, we obtain

$$
\begin{aligned}
&\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}[{}^\rho\mathcal{K}^\alpha_{a+}f(b^\rho)+{}^\rho\,\mathcal{K}^\alpha_{b-}f(a^\rho)]\\
=&\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1 [(1-\tau^\rho)^\alpha-\tau^{\rho\alpha}]\tau^{\rho-1}f'(\tau^\rho a^\rho+(1-\tau^\rho)b^\rho)\mathrm{d}\tau, \qquad (3.3)
\end{aligned}
$$

which is Lemma 2.4 in [5].

Next, we establish some integral inequalities by the differentiability, the convexity and Lemma 3.1.

If the function $|f'|$ is convex, then the following integral inequality holds.

Theorem 3.1. *Suppose that* $f:[a^\rho,b^\rho]\to\mathbb{R}$ *is differentiable on* (a^ρ,b^ρ) *with* $\rho>0$ *and* $0\le a<b$*, and* $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$*. If* $|f'|$ *is convex on* $[a^\rho,b^\rho]$*, then the inequality*

$$
\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\Big]\right|\\
\le&\frac{\alpha}{4(\alpha+1)(\alpha+2)}[(\alpha+3)(\chi^\rho-a^\rho)|f'(a^\rho)|+(\alpha+1)(b^\rho-a^\rho)|f'(\chi^\rho)|+(\alpha+3)(b^\rho-\chi^\rho)|f'(b^\rho)|] \qquad (3.4)
\end{aligned}
$$

holds for any $\alpha>0$ *and any* $\chi\in[a,b]$*.*

Proof. Using Lemma 3.1 and the convexity of the function $|f'|$, we have

$$
\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\Big]\right|\\
\le&\frac{\rho}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}\Big[(b^\rho-\chi^\rho)\Big|f'(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)\Big|+(\chi^\rho-a^\rho)\Big|f'((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)\Big|\Big]\mathrm{d}\tau\\
\le&\frac{\rho(b^\rho-\chi^\rho)}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}[\tau^\rho|f'(\chi^\rho)|+(1-\tau^\rho)|f'(b^\rho)|]\mathrm{d}\tau\\
&+\frac{\rho(\chi^\rho-a^\rho)}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}[(1-\tau^\rho)|f'(a^\rho)|+\tau^\rho|f'(\chi^\rho)|]\mathrm{d}\tau. \qquad (3.5)
\end{aligned}
$$

By simple computation, the inequality (3.4) is obtained which completes the proof of Theorem 3.1. □

If we take $\chi^\rho = \frac{a^\rho+b^\rho}{2}$ in the inequality (3.4), then we get a integral inequality.

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}-}f(a^\rho)+{}^\rho\mathcal{K}^\alpha_{\sqrt[\rho]{\frac{a^\rho+b^\rho}{2}}+}f(b^\rho)\right]\right|\\&\le\frac{\alpha(b^\rho-a^\rho)}{8(\alpha+1)(\alpha+2)}\left[(\alpha+3)|f'(a^\rho)|+2(\alpha+1)\left|f'\left(\frac{a^\rho+b^\rho}{2}\right)\right|+(\alpha+3)|f'(b^\rho)|\right].\end{aligned}\tag{3.6}$$

Also, making limits when $\rho\to1$ in the inequality (3.6), we immediately get the integral inequalities for Riemann-Liouville fractional integrals:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathcal{R}^\alpha_{\frac{a+b}{2}-}f(a)+\mathcal{R}^\alpha_{\frac{a+b}{2}+}f(b)\right]\right|\\&\le\frac{\alpha(b-a)}{8(\alpha+1)(\alpha+2)}\left[(\alpha+3)|f'(a)|+2(\alpha+1)\left|f'\left(\frac{a+b}{2}\right)\right|+(\alpha+3)|f'(b)|\right].\end{aligned}\tag{3.7}$$

If the function $|f'|^q(q>1)$ is convex, then the below inequality holds.

Theorem 3.2. *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable on (a^ρ,b^ρ) with $\rho>0$ and $0\le a<b$, and $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$. If $|f'|^q(q>1)$ is convex on $[a^\rho,b^\rho]$, then the inequality*

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\&\le\frac{1}{2\alpha}\left[\mathbf{B}\left(\frac{2q-r-1}{q-1},\frac{\rho q-r-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\left\{(\chi^\rho-a^\rho)\left[\mathbf{B}\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)|f'(\chi^\rho)|^q\right.\right.\\&\quad\left.+\left(\mathbf{B}\left(r+1,\frac{r+1}{\rho\alpha}\right)-\mathbf{B}\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)\right)|f'(a^\rho)|^q\right]^{1/q}\\&\quad+(b^\rho-\chi^\rho)\left[\mathbf{B}\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)|f'(\chi^\rho)|^q\right.\\&\quad\left.\left.+\left(\mathbf{B}\left(r+1,\frac{r+1}{\rho\alpha}\right)-\mathbf{B}\left(r+1,\frac{\rho+r+1}{\rho\alpha}\right)\right)|f'(b^\rho)|^q\right]^{1/q}\right\}\end{aligned}\tag{3.8}$$

holds for any $\alpha>0$, $\chi\in[a,b]$ and $0\le r\le q$, where $\mathbf{B}(\mu,\nu)$ is classical beta function defined by

$$\mathbf{B}(\mu,\nu)=\int_0^1 s^{\mu-1}(1-s)^{\nu-1}ds\qquad(\mu>0,\nu>0).\tag{3.9}$$

Proof. Using the identity (3.1), Hölder's inequality and the convexity of the function $|f'|^q(q>1)$, we have

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\&\le\frac{\rho}{2}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}\left[(b^\rho-\chi^\rho)\left|f'(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)\right|+(\chi^\rho-a^\rho)\left|f'((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)\right|\right]d\tau\\&\le\frac{\rho(b^\rho-\chi^\rho)}{2}\left(\int_0^1(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{q(\rho-1)-r}{q-1}}d\tau\right)^{1-1/q}\end{aligned}$$

$$\times\Big[\int_0^1 (1-\tau^{\rho\alpha})^r\tau^r[\tau^\rho|f'(\chi^\rho)|^q+(1-\tau^\rho)|f'(b^\rho)|^q]\mathrm{d}\tau\Big]^{1/q}$$
$$+\frac{\rho(\chi^\rho-a^\rho)}{2}\Big(\int_0^1 (1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{q(\rho-1)-r}{q-1}}\mathrm{d}\tau\Big)^{1-1/q}$$
$$\times\Big[\int_0^1 (1-\tau^{\rho\alpha})^r\tau^r[\tau^\rho|f'(a^\rho)|^q+(1-\tau^\rho)|f'(\chi^\rho)|^q]\mathrm{d}\tau\Big]^{1/q}. \tag{3.10}$$

By direct calculation, we obtain the inequality (3.8) which completes the proof of Theorem 3.2. □

In particular, making $r=0$, $r=1$ and $r=q$, respectively, then

$$\Big|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\Big]\Big|$$
$$\leq\frac{\rho^{1/q}}{2\alpha^{1-1/q}(\rho+1)^{1/q}}\Big[\mathbf{B}\Big(\frac{2q-1}{q-1},\frac{\rho q-1}{\rho\alpha(q-1)}\Big)\Big]^{1-1/q}$$
$$\times\Big\{(\chi^\rho-a^\rho)[|f'(\chi^\rho)|^q+\rho|f'(a^\rho)|^q]^{1/q}+(b^\rho-\chi^\rho)[|f'(\chi^\rho)|^q+\rho|f'(b^\rho)|^q]^{1/q}\Big\} \tag{3.11}$$

and

$$\Big|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\Big]\Big|$$
$$\leq\frac{\rho^2\alpha(q-1)^{2-2/q}}{2^{1+1/q}[(\rho+2)(\rho+2+\rho\alpha)(\rho\alpha+2)]}\Big(\frac{1}{(\rho q-2)[\rho q-2+\rho\alpha(q-1)]}\Big)^{1-1/q}$$
$$\times\Big\{(\chi^\rho-a^\rho)[2(2+\rho\alpha)|f'(\chi^\rho)|^q+\rho(\rho+2+\rho\alpha)|f'(a^\rho)|^q]^{1/q}$$
$$+(b^\rho-\chi^\rho)[2(2+\rho\alpha)|f'(\chi^\rho)|^q+\rho(\rho+2+\rho\alpha)|f'(b^\rho)|^q]^{1/q}\Big\} \tag{3.12}$$

and

$$\Big|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\Big]\Big|$$
$$\leq\frac{\rho^{1-1/q}}{2\alpha^{1/q}}\Big(\frac{q-1}{\rho q-q-1}\Big)^{1-1/q}\Big\{(\chi^\rho-a^\rho)\Big[\mathbf{B}\Big(q+1,\frac{\rho+q+1}{\rho\alpha}\Big)|f'(\chi^\rho)|^q$$
$$+\Big(\mathbf{B}\Big(q+1,\frac{q+1}{\rho\alpha}\Big)-\mathbf{B}\Big(q+1,\frac{\rho+q+1}{\rho\alpha}\Big)\Big)|f'(a^\rho)|^q\Big]^{1/q}$$
$$+(b^\rho-\chi^\rho)\Big[\mathbf{B}\Big(q+1,\frac{\rho+q+1}{\rho\alpha}\Big)|f'(\chi^\rho)|^q$$
$$+\Big(\mathbf{B}\Big(q+1,\frac{q+1}{\rho\alpha}\Big)-\mathbf{B}\Big(q+1,\frac{\rho+q+1}{\rho\alpha}\Big)\Big)|f'(b^\rho)|^q\Big]^{1/q}\Big\}. \tag{3.13}$$

Furthermore, utilizing Lemma 3.1 reduces to the below inequalities.

Theorem 3.3. *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable on (a^ρ,b^ρ) with $\rho>0$ and $0\leq a<b$, and $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$. If $|f'|^q (q>1)$ is convex on $[a^\rho,b^\rho]$, then the inequality*

$$\Big|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\Big]\Big|$$

$$
\begin{aligned}
\le & \frac{1}{2\alpha}\Big[\mathbf{B}\Big(\frac{2q-r-1}{q-1},\frac{\rho(q-r)+r-1}{\rho\alpha(q-1)}\Big)\Big]^{1-1/q}\Big\{(\chi^{\rho}-a^{\rho})\Big[\mathbf{B}\Big(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\Big)|f'(\chi^{\rho})|^{q}\\
&+\Big(\mathbf{B}\Big(r+1,\frac{\rho r-r+1}{\rho\alpha}\Big)-\mathbf{B}\Big(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\Big)\Big)|f'(a^{\rho})|^{q}\Big]^{1/q}\\
&+(b^{\rho}-\chi^{\rho})\Big[\mathbf{B}\Big(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\Big)|f'(\chi^{\rho})|^{q}\\
&+\Big(\mathbf{B}\Big(r+1,\frac{\rho r-r+1}{\rho\alpha}\Big)-\mathbf{B}\Big(r+1,\frac{\rho(r+1)-r+1}{\rho\alpha}\Big)\Big)|f'(b^{\rho})|^{q}\Big]^{1/q}\Big\} \qquad (3.14)
\end{aligned}
$$

holds for any $\alpha>0$, $\chi\in[a,b]$ *and* $0\le r\le q$, *where* $\mathbf{B}(\mu,\nu)$ *is classical beta function defined in* (3.9).

Proof. Using the identity (3.1), Hölder's inequality and the convexity of the function $|f'|^{q}(q>1)$, we have

$$
\begin{aligned}
&\Big|\frac{f(a^{\rho})+f(b^{\rho})}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2}\Big[\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f(a^{\rho})}{(\chi^{\rho}-a^{\rho})^{\alpha}}+\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f(b^{\rho})}{(b^{\rho}-\chi^{\rho})^{\alpha}}\Big]\Big|\\
\le&\frac{\rho}{2}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}\big[(b^{\rho}-\chi^{\rho})\big|f'(\tau^{\rho}\chi^{\rho}+(1-\tau^{\rho})b^{\rho})\big|+(\chi^{\rho}-a^{\rho})\big|f'((1-\tau^{\rho})a^{\rho}+\tau^{\rho}\chi^{\rho})\big|\big]\mathrm{d}\tau\\
\le&\frac{\rho(b^{\rho}-\chi^{\rho})}{2}\Big(\int_0^1(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{(\rho-1)(q-r)}{q-1}}\mathrm{d}\tau\Big)^{1-1/q}\\
&\times\Big[\int_0^1(1-\tau^{\rho\alpha})^{r}\tau^{(\rho-1)r}[\tau^{\rho}|f'(\chi^{\rho})|^{q}+(1-\tau^{\rho})|f'(b^{\rho})|^{q}]\mathrm{d}\tau\Big]^{1/q}\\
&+\frac{\rho(\chi^{\rho}-a^{\rho})}{2}\Big(\int_0^1(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{(\rho-1)(q-r)}{q-1}}\mathrm{d}\tau\Big)^{1-1/q}\\
&\times\Big[\int_0^1(1-\tau^{\rho\alpha})^{r}\tau^{(\rho-1)r}[\tau^{\rho}|f'(a^{\rho})|^{q}+(1-\tau^{\rho})|f'(\chi^{\rho})|^{q}]\mathrm{d}\tau\Big]^{1/q}. \qquad (3.15)
\end{aligned}
$$

By direct calculation, we obtain the inequality (3.14) which completes the proof of Theorem 3.3. □

In particular, making $r=1$ and $r=q$, respectively, then

$$
\begin{aligned}
&\Big|\frac{f(a^{\rho})+f(b^{\rho})}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2}\Big[\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f(a^{\rho})}{(\chi^{\rho}-a^{\rho})^{\alpha}}+\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f(b^{\rho})}{(b^{\rho}-\chi^{\rho})^{\alpha}}\Big]\Big|\\
\le&\frac{\alpha}{2^{1-1/q}(\alpha+1)(\alpha+2)^{1/q}}\Big\{(\chi^{\rho}-a^{\rho})[(\alpha+1)|f'(\chi^{\rho})|^{q}+(\alpha+3)|f'(a^{\rho})|^{q}]^{1/q}\\
&+(b^{\rho}-\chi^{\rho})[(\alpha+1)|f'(\chi^{\rho})|^{q}+(\alpha+3)|f'(b^{\rho})|^{q}]^{1/q}\Big\} \qquad (3.16)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\frac{f(a^{\rho})+f(b^{\rho})}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2}\Big[\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi-}f(a^{\rho})}{(\chi^{\rho}-a^{\rho})^{\alpha}}+\frac{{}^{\rho}\mathcal{K}^{\alpha}_{\chi+}f(b^{\rho})}{(b^{\rho}-\chi^{\rho})^{\alpha}}\Big]\Big|\\
\le&\frac{\rho^{1-1/q}}{2\alpha^{1/q}}\Big\{(\chi^{\rho}-a^{\rho})\Big[\mathbf{B}\Big(q+1,\frac{\rho(q+1)-q+1}{\rho\alpha}\Big)|f'(\chi^{\rho})|^{q}\\
&+\Big(\mathbf{B}\Big(q+1,\frac{\rho q-q+1}{\rho\alpha}\Big)-\mathbf{B}\Big(q+1,\frac{\rho(q+1)-q+1}{\rho\alpha}\Big)\Big)|f'(a^{\rho})|^{q}\Big]^{1/q}
\end{aligned}
$$

$$+ (b^\rho - \chi^\rho)\Big[\mathbf{B}\Big(q+1, \frac{\rho(q+1)-q+1}{\rho\alpha}\Big)|f'(\chi^\rho)|^q$$
$$+ \Big(\mathbf{B}\Big(q+1, \frac{\rho q - q+1}{\rho\alpha}\Big) - \mathbf{B}\Big(q+1, \frac{\rho(q+1)-q+1}{\rho\alpha}\Big)\Big)|f'(b^\rho)|^q\Big]^{1/q}\Big\}. \tag{3.17}$$

Theorem 3.4. *Suppose that $f : [a^\rho, b^\rho] \to \mathbb{R}$ is differentiable on (a^ρ, b^ρ) with $\rho > 0$ and $0 \le a < b$, and $f \in \mathfrak{X}_c^p(a^\rho, b^\rho)$. If $|f'|^q (q > 1)$ is convex on $[a^\rho, b^\rho]$, then the inequality*

$$\Big|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho - a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho - \chi^\rho)^\alpha}\Big]\Big|$$
$$\le \frac{\rho^{1/q}}{2\alpha^{1-1/q}[(\rho r - r+1)(\rho(r+1)-r+1)]^{1/q}}\Big[\mathbf{B}\Big(\frac{2q-1}{q-1}, \frac{(\rho-1)(q-r)+q-1}{\rho\alpha(q-1)}\Big)\Big]^{1-1/q}$$
$$\times \Big\{(\chi^\rho - a^\rho)[(\rho r - r+1)|f'(\chi^\rho)|^q + \rho|f'(a^\rho)|^q]^{1/q}$$
$$+ (b^\rho - \chi^\rho)[(\rho r - r+1)|f'(\chi^\rho)|^q + \rho|f'(b^\rho)|^q]^{1/q}\Big\} \tag{3.18}$$

holds for any $\alpha > 0$, $\chi \in [a, b]$ and $0 \le r \le q$, where $\mathbf{B}(\mu, \nu)$ is classical beta function defined in (3.9).

Proof. Using the identity (3.1), Hölder's inequality and the convexity of the function $|f'|^q (q > 1)$, we have

$$\Big|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho - a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho - \chi^\rho)^\alpha}\Big]\Big|$$
$$\le \frac{\rho}{2}\int_0^1 (1-\tau^{\rho\alpha})\tau^{\rho-1}[(b^\rho - \chi^\rho)\big|f'(\tau^\rho\chi^\rho + (1-\tau^\rho)b^\rho)\big| + (\chi^\rho - a^\rho)\big|f'((1-\tau^\rho)a^\rho + \tau^\rho\chi^\rho)\big|]\mathrm{d}\tau$$
$$\le \frac{\rho(b^\rho - \chi^\rho)}{2}\Big(\int_0^1 (1-\tau^{\rho\alpha})^{\frac{q}{q-1}}\tau^{\frac{(\rho-1)(q-r)}{q-1}}\mathrm{d}\tau\Big)^{1-1/q}$$
$$\times [\int_0^1 \tau^{(\rho-1)r}[\tau^\rho|f'(\chi^\rho)|^q + (1-\tau^\rho)|f'(b^\rho)|^q]\mathrm{d}\tau]^{1/q}$$
$$+ \frac{\rho(\chi^\rho - a^\rho)}{2}\Big(\int_0^1 (1-\tau^{\rho\alpha})^{\frac{q}{q-1}}\tau^{\frac{(\rho-1)(q-r)}{q-1}}\mathrm{d}\tau\Big)^{1-1/q}$$
$$\times [\int_0^1 \tau^{(\rho-1)r}[\tau^\rho|f'(a^\rho)|^q + (1-\tau^\rho)|f'(\chi^\rho)|^q]\mathrm{d}\tau]^{1/q}. \tag{3.19}$$

By direct calculation, we obtain the inequality (3.18) which completes the proof of Theorem 3.4. □

In particular, making $r = 1$ and $r = q$, respectively, then

$$\Big|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2}\Big[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho - a^\rho)^\alpha} + \frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho - \chi^\rho)^\alpha}\Big]\Big|$$
$$\le \frac{1}{2^{1+1/q}\alpha^{1-1/q}}\Big[\mathbf{B}\Big(\frac{2q-1}{q-1}, \frac{1}{\alpha}\Big)\Big]^{1-1/q}\Big\{(\chi^\rho - a^\rho)[|f'(\chi^\rho)|^q + |f'(a^\rho)|^q]^{1/q}$$
$$+ (b^\rho - \chi^\rho)[|f'(\chi^\rho)|^q + |f'(b^\rho)|^q]^{1/q}\Big\} \tag{3.20}$$

and

$$
\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\
\leq&\frac{\rho^{1/q}}{2\alpha^{1-1/q}[(\rho q-q+1)(\rho(q+1)-q+1)]}\left[\mathbf{B}\left(\frac{2q-1}{q-1},\frac{1}{\rho\alpha}\right)\right]^{1-1/q}\\
&\times\Big\{(\chi^\rho-a^\rho)[(\rho q-q+1)|f'(\chi^\rho)|^q+\rho|f'(a^\rho)|^q]^{1/q}\\
&+(b^\rho-\chi^\rho)[(\rho q-q+1)|f'(\chi^\rho)|^q+\rho|f'(b^\rho)|^q]^{1/q}\Big\}. \qquad (3.21)
\end{aligned}
$$

Theorem 3.5. *Suppose that $f:[a^\rho,b^\rho]\to\mathbb{R}$ is differentiable on (a^ρ,b^ρ) with $\rho>0$ and $0\leq a<b$, and $f\in\mathfrak{X}^p_c(a^\rho,b^\rho)$. If $|f'|^q(q>1)$ is convex on $[a^\rho,b^\rho]$, then the inequality*

$$
\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\
\leq&\frac{1}{2\alpha}\left[\mathbf{B}\left(\frac{2q-r-1}{q-1},\frac{\rho q-1}{\rho\alpha(q-1)}\right)\right]^{1-1/q}\Bigg\{(\chi^\rho-a^\rho)\Bigg[\mathbf{B}\left(r+1,\frac{\rho+1}{\rho\alpha}\right)|f'(\chi^\rho)|^q\\
&+\left(\mathbf{B}\left(r+1,\frac{1}{\rho\alpha}\right)-\mathbf{B}\left(r+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(a^\rho)|^q\Bigg]^{1/q}+(b^\rho-\chi^\rho)\Bigg[\mathbf{B}\left(r+1,\frac{\rho+1}{\rho\alpha}\right)|f'(\chi^\rho)|^q\\
&+\left(\mathbf{B}\left(r+1,\frac{1}{\rho\alpha}\right)-\mathbf{B}\left(r+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(b^\rho)|^q\Bigg]^{1/q}\Bigg\} \qquad (3.22)
\end{aligned}
$$

holds for any $\alpha>0$, $\chi\in[a,b]$ and $0\leq r\leq q$, where $\mathbf{B}(\mu,\nu)$ is classical beta function defined in (3.9).

Proof. Using the identity (3.1), Hölder's inequality and the convexity of the function $|f'|^q(q>1)$, we have

$$
\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|\\
\leq&\frac{\rho}{2}\int_0^1(1-\tau^{\rho\alpha})\tau^{\rho-1}[(b^\rho-\chi^\rho)\left|f'(\tau^\rho\chi^\rho+(1-\tau^\rho)b^\rho)\right|+(\chi^\rho-a^\rho)\left|f'((1-\tau^\rho)a^\rho+\tau^\rho\chi^\rho)\right|]\mathrm{d}\tau\\
\leq&\frac{\rho(b^\rho-\chi^\rho)}{2}\left(\int_0^1(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{q(\rho-1)}{q-1}}\mathrm{d}\tau\right)^{1-1/q}\\
&\times[\int_0^1(1-\tau^{\rho\alpha})^r[\tau^\rho|f'(\chi^\rho)|^q+(1-\tau^\rho)|f'(b^\rho)|^q]\mathrm{d}\tau]^{1/q}\\
&+\frac{\rho(\chi^\rho-a^\rho)}{2}\left(\int_0^1(1-\tau^{\rho\alpha})^{\frac{q-r}{q-1}}\tau^{\frac{q(\rho-1)}{q-1}}\mathrm{d}\tau\right)^{1-1/q}\\
&\times[\int_0^1(1-\tau^{\rho\alpha})^r[\tau^\rho|f'(a^\rho)|^q+(1-\tau^\rho)|f'(\chi^\rho)|^q]\mathrm{d}\tau]^{1/q}. \qquad (3.23)
\end{aligned}
$$

By direct calculation, we obtain the inequality (3.22) which completes the proof of Theorem 3.5. □

In particular, making $r=1$ and $r=q$, respectively, then

$$
\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|
$$

$$\leq\frac{\rho^2\alpha(q-1)^{2-2/q}}{2[(\rho+1)(\rho\alpha+1)\rho\alpha+\rho+1)]^{1/q}}\left[\frac{1}{(\rho q-1)[\rho q-1+\rho\alpha(q-1)]}\right]^{1-1/q}$$
$$\times\left\{(\chi^\rho-a^\rho)[(1+\rho\alpha)|f'(\chi^\rho)|^q+\rho(\rho\alpha+\rho+2)|f'(a^\rho)|^q]^{1/q}\right.$$
$$\left.+(b^\rho-\chi^\rho)[(1+\rho\alpha)|f'(\chi^\rho)|^q+\rho(\rho\alpha+\rho+2)|f'(b^\rho)|^q]^{1/q}\right\} \tag{3.24}$$

and

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2}\left[\frac{{}^\rho\mathcal{K}^\alpha_{\chi-}f(a^\rho)}{(\chi^\rho-a^\rho)^\alpha}+\frac{{}^\rho\mathcal{K}^\alpha_{\chi+}f(b^\rho)}{(b^\rho-\chi^\rho)^\alpha}\right]\right|$$
$$\leq\frac{\rho^{1-1/q}}{2\alpha^{1/q}}\left(\frac{q-1}{\rho q-1}\right)^{1-1/q}\left\{(\chi^\rho-a^\rho)\left[\mathbf{B}\left(q+1,\frac{\rho+1}{\rho\alpha}\right)|f'(\chi^\rho)|^q\right.\right.$$
$$\left.+\left(\mathbf{B}\left(q+1,\frac{1}{\rho\alpha}\right)-\mathbf{B}\left(q+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(a^\rho)|^q\right]^{1/q}+(b^\rho-\chi^\rho)\left[\mathbf{B}\left(q+1,\frac{\rho+1}{\rho\alpha}\right)|f'(\chi^\rho)|^q\right.$$
$$\left.\left.+\left(\mathbf{B}\left(q+1,\frac{1}{\rho\alpha}\right)-\mathbf{B}\left(q+1,\frac{\rho+1}{\rho\alpha}\right)\right)|f'(b^\rho)|^q\right]^{1/q}\right\}. \tag{3.25}$$

4. Conclusions

In this article, we firstly establish Hermite-Hadamard type inequality for the Erdélyi-Kober fractional integrals (2.1). Taking limits when $\chi\to a$ and $\chi\to b$ respectively in the inequality (2.1), it is the generalization and the refinement of Theorem 2.1 in the reference [5]. In addition, as an application, the error estimations of Hermite-Hadamard type inequality have been investigated. The derived inequalities can be seen as a generalization for Riemann-Liouville fractional integrals when $\rho\to 1$ and the Corollaries presented in the paper show that the results of this paper generalize and extend many existing results.

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Conflict of interest

The authors declare that they have no conflict of interest.

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