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Research article

On the nonlinear system of fourth-order beam equations with integral boundary conditions

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Abstract: The purpose of this paper is to establish an existence theorem for a system of nonlinear fourth-order differential equations with two parameters

$$\begin{cases} u^{(4)} + A(x)u &= \lambda f(x,u,v,u'',v''),\ 0 < x < 1, \\ v^{(4)} + B(x)v &= \mu g(x,u,v,u'',v''),\ 0 < x < 1 \end{cases}$$

subject to the coupled integral boundary conditions:

$$\begin{cases} u(0) = u'(1) = u'''(1) = 0,\ u''(0) &= \int_0^1 p(x)v''(x)dx, \\ v(0) = v'(1) = v'''(1) = 0,\ v''(0) &= \int_0^1 q(x)u''(x)dx, \end{cases}$$

where $A,\ B \in C[0,1]$, $p,q \in L^1[0,1]$, $\lambda > 0, \mu > 0$ are two parameters and $f,g : [0,1] \times [0,\infty) \times [0,\infty) \times (-\infty,0) \times (-\infty,0) \to \mathbb{R}$ are two continuous functions satisfy the growth conditions.

Keywords: fourth-order differential equations; nonlocal boundary conditions; existence theorem; Schauder's fixed point theorem
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1. Introduction

In [1], the authors investigated the following system of fourth-order differential equations

$$\begin{cases} u^{(4)} &= f(x,u,v,u'',v''),\ 0 < x < 1, \\ v^{(4)} &= g(x,u,v,u'',v''),\ 0 < x < 1, \end{cases} \tag{1.1}$$

subject to the boundary conditions

$$\begin{cases} u(0) = u''(0) = 0,\ u(1) = u''(1) &= 0, \\ v(0) = v''(0) = 0,\ v(1) = v''(1) &= 0, \end{cases} \tag{1.2}$$

where $f, g \in C((0,1) \times (0,\infty) \times (0,\infty) \times (-\infty,0) \times (-\infty,0),\ \mathbb{R})$. The results were obtained by approximating the fourth-order system to a second-order singular one and using a fixed point index theorem on cones.

Recently, the authors [2] studied the following system

$$\begin{cases} u^{(4)} + \beta_1 u'' - \alpha_1 u &= f(x,u,v),\ 0 < x < 1, \\ v^{(4)} + \beta_2 v'' - \alpha_2 v &= g(x,u,v),\ 0 < x < 1, \end{cases} \tag{1.3}$$

subject to the above boundary conditions, where $\alpha_i, \beta_i \in \mathbb{R},\ i = 1,2$ satisfy $\beta_i < 2\pi^2,\ \alpha_i \geq -\frac{\beta_i^2}{4}$, $\frac{\alpha_i}{\pi^4} + \frac{\beta_i}{\pi^2} < 1$, and established the existence of positive solutions for this system with superlinear or sublinear nonlinearities.

In [3], the author considered the nonlinear fourth-order differential equation

$$\begin{cases} u^{(4)} &= \lambda f(x,u,v),\ 0 < x < 1, \\ v^{(4)} &= \mu g(x,u,v),\ 0 < x < 1, \end{cases} \tag{1.4}$$

subject to the coupled integral boundary conditions

$$\begin{cases} u(0) = u'(1) = u'''(1) = 0,\ u''(0) &= \int_0^1 p(x) v''(x) dx, \\ v(0) = v'(1) = v'''(1) = 0,\ v''(0) &= \int_0^1 q(x) u''(x) dx, \end{cases} \tag{1.5}$$

where p and q are continuous functions on $[0,1]$, λ and μ are two positive parameters and $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous functions, and established a sufficient condition $\int_0^1 p(x)dx \int_0^1 q(x)dx < 1$ with the extreme limits of f and g to guarantee a unique positive solution (u,v) in $C[0,1] \times C[0,1]$ for this problem by using the Guo-Krasnosel'skii fixed point theorem and the Green's functions.

In this paper, we establish an existence theorem for the following boundary value problem

$$\begin{cases} u^{(4)} + A(x)u &= \lambda f(x,u,v,u'',v''),\ 0 < x < 1, \\ v^{(4)} + B(x)v &= \mu g(x,u,v,u'',v''),\ 0 < x < 1, \end{cases} \tag{1.6}$$

subject to the integral boundary conditions (1.5), where $A, B \in C[0,1]$, $p, q \in L^1[0,1]$ and $f, g : [0,1] \times [0,\infty) \times [0,\infty) \times (-\infty,0) \times (-\infty,0) \to \mathbb{R}$ are continuous functions and satisfy the growth conditions with variable parameters:

$$|f(x,u,v,w,z)| \leq a_1(x)|u| + b_1(x)|v| + c_1(x)|w| + d_1(x)|z| + e_1(x), \tag{1.7}$$

$$|g(x,u,v,w,z)| \leq a_2(x)|u| + b_2(x)|v| + c_2(x)|w| + d_2(x)|z| + e_2(x), \tag{1.8}$$

where $a_i(x), b_i(x), c_i(x), d_i(x), e_i(x),\ i = 1,2$ are positive continuous functions on $[0,1]$. Moreover, we will assume that

$$\sup_{0\leq x\leq 1} A(x) = A_1 < \frac{\alpha}{\sqrt{C_1}},\ \sup_{0\leq x\leq 1} B(x) = B_1 < \frac{\beta}{\sqrt{C_1}},\ 0 < \lambda \leq \frac{\alpha - A_1\sqrt{C_1}}{7k_1 + 1}, 0 < \mu \leq \frac{\beta - B_1\sqrt{C_1}}{7k_2 + 1}, \tag{1.9}$$

$$\sigma = \int_0^1 p(x)dx \int_0^1 q(x)dx < 1, \ \frac{\alpha}{1-\sigma} \le \sqrt{C_1}, \ \frac{\beta}{1-\sigma} \le \sqrt{C_1}. \tag{1.10}$$

where $k_i = \max\{a_i, b_i, c_i, d_i\}$, $a_i = \max\limits_{0\le x\le 1} a_i(x)$, $b_i = \max\limits_{0\le x\le 1} b_i(x)$, $c_i = \max\limits_{0\le x\le 1} c_i(x)$, $d_i = \max\limits_{0\le x\le 1} d_i(x)$, $e_i = \max\limits_{0\le x\le 1} e_i(x)$, $C_1 = \dfrac{4}{\pi^2 - 4}$, $\alpha = 1 - \int_0^1 p^2(x) > 0$ and $\beta = 1 - \int_0^1 q^2(x) > 0$.

It is well-known that fourth-order differential equations play a major role in physics and elasticity theory, and lead to wide range of applications in mechanical engineering. Equations of (1.6) represent deflections of beams, where u and v denote the deflections and f, g denote the distributed loads on the beams, and each distributed load depends on the deflection. Here; u'' is the bending moment stiffness, and $u^{(4)}$ is the load density stiffness, and λ, μ are parameters that represent the reciprocal of the flexural rigidity of the material of each beam, which measures the resistance to bend. The simplest case is when the load $f = f(x)$ depends only on position x, but more general representations of f could also include other variables, such as the deflection u and the bending moment stiffness u'', which arise frequently in applications to mechanics. In [4], we established an existence and uniqueness theorem for the boundary value problem

$$u^{(4)} + A(x)u = \lambda f(x, u, u''), \ 0 < x < 1, \tag{1.11}$$

subject to the integral boundary conditions

$$u(0) = u(1) = \int_0^1 p(x)u(x)dx, \ \ u''(0) = u''(1) = \int_0^1 q(x)u''(x)dx, \tag{1.12}$$

where $A \in C[0,1]$, $p, q \in L^1[0,1]$ and f is continuous on $[0,1]\times\mathbb{R}\times\mathbb{R}$ and satisfies a growth condition with variable parameters:

$$|f(x, u, v)| \le a(x)\,|u| + b(x)\,|v| + c(x), \tag{1.13}$$

where a, b, c are positive continuous functions on $[0,1]$. This problem has attracted the attention of many researchers due to its important and various applications to mechanics and construction engineering (see [5–11] and references therein). With the exception of [9, 11] none of the authors discussed the problem with integral conditions. In [4], we investigated the problem in the case of small deflections, which usually occurs when the material of the beam has high flexural rigidity (i.e., λ is small), and so we imposed an upper bound for λ. This natural assumption of small deflections is essential to neglect shear distortion and effects of rotatory inertia and this will lead to Euler-Bernoulli Equation. Moreover, it is critical when it comes to industrial applications since large deflections of beams used in building towers, skyscrapers, bridges, and other constructions may cause cracks in the beams and this could lead to disastrous effects, such as the collapse of the construction for example.

In this paper, we wish to extend our study to the case of systems of beam equations rather than a single equation. Particularly speaking, investigating the existence of small deflections (i.e., solutions to the system (1.6) under some natural assumptions).

In system (1.6) the situation is a bit more complicated since it represents two different beams each of which exhibits a small deflection, and both deflections and their bending moment stiffness affect the distributed loads on both beams. This situation arises in heavy construction (towers, skyscrapers) where the deflection of one beam could affect the load on the next beam and the other surrounding beams. Studying these systems with a careful analysis of deflections could help reach the typical design for the structure to be built. In [1] the boundary conditions models a simply supported beam (provided that $p = q = 0$). In this study, since deflection of a beam is affected by loads on surrounding beams, it is more suitable to study boundary condition of the form (1.5) which represent beams that are simply supported at one end and sliding clamped at the other end, which is the same model studied by [3] . This important study is an extension of [4] to system of beams, and it generalizes the works of [1–3]. The system (1.6) generalizes the preceding systems in the following sense:

1). If $A = B = 0$ then (1.6) reduces to (1.4) and (1.1).
2). If f and g depend only on u and v then (1.6) reduces to (1.3) and (1.4).
3). If $p(x) = q(x) = 0$ then the condition (1.5) reduces to (1.2).

2. Existence theorem

The system (1.6) subject to the integral boundary conditions (1.5) can be converted into the following coupled system:

$$\begin{cases} u'' &= w, \ u(0) = u'(1) = 0, \\ v'' &= z, \ v(0) = v'(1) = 0, \\ w'' &= -A(x)u + \lambda f(x, \ u, \ v, \ w, \ z), \ w(0) = \int_0^1 p(x)z(x)dx, \ w'(1) = 0, \\ z'' &= -B(x)v + \mu g(x, \ u, \ v, \ w, \ z), \ z(0) = \int_0^1 q(x)w(x)dx, \ z'(1) = 0. \end{cases} \tag{2.1}$$

Thus, we shall prove the following statement

Proposition 2.1. *If the conditions (1.7)–(1.10) hold, then there exist two constants $M > 0$ and $M^* > 0$ such that for any $x \in [0, 1]$ and any solution $(u, \ v)$ to the system (2.1), we have*

$$\| u \|_{3,\rho} \leq M \text{ and } \ \| v \|_{3,\rho} \leq M^*, \tag{2.2}$$

where

$$\| u \|_{3,\rho} = \max_{0\leq x\leq 1} \left(|u(x)| + |u'(x)| + |\rho(x)u''(x)| + |u'''(x)|\right), \tag{2.3}$$

$$\| v \|_{3,\rho} = \max_{0\leq x\leq 1} \left(|v(x)| + |v'(x)| + |\rho(x)v''(x)| + |v'''(x)|\right), \tag{2.4}$$

and $\rho(x) = x(1 - \frac{x}{2}), \ x \in [0, 1]$.

Proof. Multiplying both sides of the first equation of (2.1) by u and integrating the resulting equation from 0 to 1, then employing integration by parts, we obtain

$$u'(1)u(1) - u'(0)u(0) - \int_0^1 u'^2(x)dx = \int_0^1 u(x)w(x)dx. \tag{2.5}$$

Taking into account $u(0) = u'(1) = 0$, we have

$$\int_0^1 u'^2(x)dx = -\int_0^1 u(x)w(x)dx. \tag{2.6}$$

The integral $\int_0^1 u(x)w(x)dx$ can be estimated by means of the Cauchy-Schwarz inequality

$$\left|\int_0^1 u(x)w(x)dx\right| \le \left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 w^2(x)dx\right)^{\frac{1}{2}}. \tag{2.7}$$

Using the Wirtinger's inequality [12]:

$$\int_a^b h^2(x)dx \le \frac{4(b-a)^2}{\pi^2}\int_a^b (h'(x))^2dx, \tag{2.8}$$

provided $h \in C^1[a, b]$ and $h(a) = 0$, we obtain

$$\int_0^1 u^2(x)dx \le \frac{4}{\pi^2}\int_0^1 u'^2(x)dx. \tag{2.9}$$

Thus

$$\int_0^1 u'^2(x)dx \le \frac{4}{\pi^2}\int_0^1 w^2(x)dx. \tag{2.10}$$

Adding (2.9) and (2.10), we obtain

$$\int_0^1 u^2(x)dx + \left(1 - \frac{4}{\pi^2}\right)\int_0^1 u'^2(x)dx \le \frac{4}{\pi^2}\int_0^1 w^2(x)dx. \tag{2.11}$$

Consequently,

$$\int_0^1 u^2(x)dx + \int_0^1 u'^2(x)dx \le C_1\int_0^1 w^2(x)dx, \tag{2.12}$$

where $C_1 = \dfrac{4}{\pi^2 - 4}$.

Similarly, for the second equation of (2.1), we obtain

$$\int_0^1 v^2(x)dx + \int_0^1 v'^2(x)dx \le C_1\int_0^1 z^2(x)dx. \tag{2.13}$$

Proceeding as before, multiplying both sides of the third equation of (2.1) by $\rho(x)v$ and integrating the resulting equation from 0 to 1, then employing integration by parts, taking into account the nonlocal boundary conditions $w(0) = \int_0^1 p(x)z(x)dx$ and $w'(1) = 0$, we obtain

$$\begin{aligned}\int_0^1 w^2(x)dx + 2\int_0^1 \rho(x)(w'(x))^2dx = \left[\int_0^1 p(x)z(x)dx\right]^2 + 2\int_0^1 A(x)\rho(x)u(x)w(x)dx\\ -2\lambda\int_0^1 f(x,u,v,w,z)\rho(x)w(x)dx.\end{aligned} \tag{2.14}$$

Note that, since $\sup\limits_{0\le x\le 1}\rho(x) = \frac{1}{2}$,

$$2\left|\int_0^1 A(x)\rho(x)u(x)w(x)dx\right| \le A_1\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 w^2(x)dx\right)^{\frac{1}{2}}, \tag{2.15}$$

and

$$\left[\int_0^1 p(x)z(x)dx\right]^2 \le \left(\int_0^1 p^2(x)dx\right)\left(\int_0^1 z^2(x)dx\right). \tag{2.16}$$

Applying (1.7) to $f(x,\ u,\ v,\ w,\ z)$ to obtain

$$\left|\int_0^1 f(x,u,v,w,z)\rho(x)w(x)dx\right| \le \frac{a_1}{2}\int_0^1 |u(x)w(x)|\,dx + \frac{b_1}{2}\int_0^1 |v(x)w(x)|\,dx + \frac{c_1}{2}\int_0^1 w^2(x)dx$$
$$+\frac{d_1}{2}\int_0^1 |w(x)z(x)|\,dx + \frac{1}{2}\int_0^1 |e_1(x)w(x)|\,dx. \tag{2.17}$$

The integrals on the right hand side of the above inequality can be estimated by means of the $\epsilon-$ inequality:

$$\int_0^1 |F(x)G(x)|\,dx \le \frac{1}{\epsilon}\int_0^1 F^2(x)dx + \epsilon\int_0^1 G^2(x)dx,\ \epsilon > 0. \tag{2.18}$$

Thus

$$\left|\int_0^1 f(x,u,v,w,z)\rho(x)w(x)dx\right| \le \frac{a_1}{2\epsilon_1}\int_0^1 u^2(x)dx + \frac{a_1\epsilon_1}{2}\int_0^1 w^2(x)dx + \frac{b_1}{2\epsilon_2}\int_0^1 v^2(x)dx$$
$$+\frac{b_1\epsilon_2}{2}\int_0^1 w^2(x)dx + \frac{c_1}{2}\int_0^1 w^2(x)dx + \frac{d_1\epsilon_3}{2}\int_0^1 w^2(x)dx$$
$$+\frac{d_1}{2\epsilon_3}\int_0^1 z^2(x)dx + \frac{e_1^2}{2\epsilon_4} + \frac{\epsilon_4}{2}\int_0^1 w^2(x)dx,\ \epsilon_i > 0,\ i = 1, ..., 4. \tag{2.19}$$

But

$$\int_0^1 u^2(x)dx \le \int_0^1 u^2(x)dx + \int_0^1 u'^2(x)dx \le C_1\int_0^1 w^2(x)dx, \tag{2.20}$$

and

$$\int_0^1 v^2(x)dx \le \int_0^1 v^2(x)dx + \int_0^1 v'^2(x)dx \le C_1\int_0^1 z^2(x)dx. \tag{2.21}$$

Substituting (2.20) and (2.21) into (2.15) and (2.16), we obtain

$$2\left|\int_0^1 A(x)\rho(x)u(x)w(x)dx\right| \le A_1\sqrt{C_1}\int_0^1 w^2(x)dx, \tag{2.22}$$

and

$$\left|\int_0^1 f(x,u,v,w,z)\rho(x)w(x)dx\right| \le \left(\frac{a_1C_1}{2\epsilon_1} + \frac{a_1\epsilon_1}{2} + \frac{c_1}{2} + \frac{b_1\epsilon_2}{2} + \frac{d_1\epsilon_3}{2} + \frac{\epsilon_4}{2}\right)\int_0^1 w^2(x)dx$$
$$+\left(\frac{b_1C_1}{2\epsilon_2} + \frac{d_1}{2\epsilon_3}\right)\int_0^1 z^2(x)dx + \frac{e_1^2}{2\epsilon_4}. \tag{2.23}$$

Now using (2.14), (2.16), (2.22) and (2.23), we obtain

$$\left[1 - A_1\sqrt{C_1} - \lambda\left(\frac{a_1C_1}{\epsilon_1} + a_1\epsilon_1 + c_1 + b_1\epsilon_2 + d_1\epsilon_3 + \epsilon_4\right)\right]\int_0^1 w^2(x)dx + 2\int_0^1 \rho(x)w'^2(x)dx$$
$$\leq \left[\lambda\left(\frac{b_1C_1}{\epsilon_1} + \frac{d_1}{\epsilon_2}\right) + \int_0^1 p^2(x)dx\right]\int_0^1 z^2(x)dx + \frac{\lambda e_1^2}{\epsilon_4}. \tag{2.24}$$

If we choose $\epsilon_i = 1,\ i = 1, ..., 4$, then

$$\left[1 - A_1\sqrt{C_1} - \lambda\left(\frac{a_1C_1}{2} + a_1 + b_1 + c_1 + d_1 + 1\right)\right]\int_0^1 w^2(x)dx + 2\int_0^1 \rho(x)w'^2(x)dx$$
$$\leq \left[\frac{\lambda}{2}(b_1C_1 + d_1) + \int_0^1 p^2(x)dx\right]\int_0^1 z^2(x)dx + \frac{\lambda e_1^2}{2}. \tag{2.25}$$

Since $\frac{a_1C_1}{2} + a_1 + b_1 + c_1 + d_1 + 1 < K_1 = 5k_1 + 1$, where $k_1 = \max\{a_1, b_1, c_1, d_1\}$,

$$\left[1 - A_1\sqrt{C_1} - \lambda K_1\right]\int_0^1 w^2(x)dx + 2\int_0^1 \rho(x)w'^2(x)dx$$
$$\leq \left[\frac{\lambda}{2}(b_1C_1 + d_1) + \int_0^1 p^2(x)dx\right]\int_0^1 z^2(x)dx + \frac{\lambda e_1^2}{2}. \tag{2.26}$$

Similarly, for the fourth equation of (2.1), we have

$$\left[1 - B_1\sqrt{C_1} - \mu K_2\right]\int_0^1 z^2(x)dx + 2\int_0^1 \rho(x)z'^2(x)dx$$
$$\leq \left[\frac{\mu}{2}(b_2C_1 + d_2) + \int_0^1 q^2(x)dx\right]\int_0^1 w^2(x)dx + \frac{\mu e_2^2}{2}, \tag{2.27}$$

where $K_2 = 5k_2 + 1$, and $k_2 = \max\{a_2, b_2, c_2, d_2\}$. Since $A_1 < \frac{\alpha}{\sqrt{C_1}}$, $B_1 < \frac{\beta}{\sqrt{C_1}}$, and in view of $\lambda < \frac{\alpha - A_1\sqrt{C_1}}{K_1}$ and $\mu < \frac{\beta - B_1\sqrt{C_1}}{K_2}$, we have

$$\gamma_1\int_0^1 w^2(x)dx + 2\int_0^1 \rho(x)w'^2(x)dx \leq \delta_1\int_0^1 z^2(x)dx + \frac{\lambda e_1^2}{2}, \tag{2.28}$$

and

$$\gamma_2\int_0^1 z^2(x)dx + 2\int_0^1 \rho(x)z'^2(x)dx \leq \delta_2\int_0^1 w^2(x)dx + \frac{\mu e_2^2}{2}, \tag{2.29}$$

where

$$\gamma_1 = 1 - A_1\sqrt{C_1} - \lambda K_1 > 0,\ \gamma_2 = 1 - B_1\sqrt{C_1} - \mu K_2 > 0, \tag{2.30}$$

and

$$\delta_1 = \frac{\lambda}{2}(b_1C_1 + d_1) + \int_0^1 p^2(x)dx,\ \delta_2 = \frac{\mu}{2}(b_2C_1 + d_2) + \int_0^1 q^2(x)dx. \tag{2.31}$$

Hence

$$\int_0^1 w^2(x)dx < \int_0^1 w^2(x)dx + \int_0^1 \rho(x)w'^2(x)dx \le \frac{\delta_1}{\gamma_1}\int_0^1 z^2(x)dx + \frac{\lambda e_1^2}{2\gamma_1}, \tag{2.32}$$

and

$$\int_0^1 z^2(x)dx < \int_0^1 z^2(x)dx + \int_0^1 \rho(x)z'^2(x)dx \le \frac{\delta_2}{\gamma_2}\int_0^1 w^2(x)dx + \frac{\mu e_2^2}{2\gamma_2}. \tag{2.33}$$

Substituting (2.33) into (2.32), we obtain

$$\int_0^1 w^2(x)dx \le M_1, \tag{2.34}$$

where $M_1 = \dfrac{\frac{\delta_1\mu e_2^2}{2\gamma_1\gamma_2} + \frac{\lambda e_1^2}{2\gamma_1}}{1 - \frac{\delta_1\delta_2}{\gamma_1\gamma_2}}$. It is easy to see from the upper bound of λ in (1.9) that $\lambda < \dfrac{\alpha - \sqrt{C_1}A_1}{K_1 + \frac{b_1C_1}{2} + \frac{d_1}{2}}$, hence $\delta_1 < \gamma_1$. Similarly, $\mu < \dfrac{\beta - \sqrt{C_1}B_1}{K_2 + \frac{b_2C_1}{2} + \frac{d_2}{2}}$, hence $\delta_2 < \gamma_2$ and therefore $\delta_1\delta_1 < \gamma_1\gamma_2$. Hence

$$\int_0^1 z^2(x)dx \le M_2, \tag{2.35}$$

where $M_2 = \frac{\delta_2}{\gamma_2}M_1 + \frac{\mu e_2^2}{2\gamma_2}$. Combining (2.34) and (2.35) with (2.12) and (2.13), respectively, we have

$$\int_0^1 u^2(x)dx + \int_0^1 u'^2(x)dx \le C_1M_1, \tag{2.36}$$

and

$$\int_0^1 v^2(x)dx + \int_0^1 v'^2(x)dx \le C_1M_2. \tag{2.37}$$

From (2.32) and (2.33), we obtain

$$\int_0^1 w^2(x)dx + \int_0^1 \rho(x)w'^2(x)dx \le M_3, \tag{2.38}$$

where $M_3 = \frac{\delta_1}{\gamma_1}M_2 + \frac{\lambda e_1^2}{2\gamma_1}$ and

$$\int_0^1 z^2(x)dx + \int_0^1 \rho(x)z'^2(x)dx \le M_4, \tag{2.39}$$

where $M_4 = \frac{\delta_2}{\gamma_2}M_1 + \frac{\mu e_2^2}{2\gamma_2}$.

From the third equation of (2.1), we have

$$\int_0^1 (w'')^2(x)dx = \int_0^1 \left[-A(x)u + \lambda f(x, u(x), v(x), w(x), z(x))\right]^2 dx. \tag{2.40}$$

Applying the growth condition (1.7) and using the sum of square inequality $(\sum_{i=1}^{n} h_i)^2 \leq n \sum_{i=1} h_i^2$, we obtain

$$\int_0^1 (w'')^2(x)dx \leq 2\left[A_1^2 \int_0^1 u^2(x)dx + \lambda^2 \int_0^1 f^2(x, u(x), v(x), w(x), z(x))dx\right], \tag{2.41}$$

and

$$\int_0^1 f^2 dx \leq 5\left[a_1^2 \int_0^1 u^2(x)dx + b_1^2 \int_0^1 v^2(x)dx + c_1^2 \int_0^1 w^2(x)dx + d_1^2 \int_0^1 z^2(x)dx + e_1^2\right]. \tag{2.42}$$

Hence

$$\int_0^1 (w''(x))^2 dx \leq M_5, \tag{2.43}$$

where $M_5 = 2A_1^2 C_1 M_1 + 10\lambda^2 \left(a_1^2 C_1 M_1 + b_1^2 C_1 M_2 + c_1^2 M_1 + d_1^2 M_2 + e_1^2\right)$.

Similarly, from the fourth equation of (2.1), we have

$$\int_0^1 (z''(x))^2 dx \leq M_6. \tag{2.44}$$

where $M_6 = 2B_1^2 C_1 M_2 + 10\mu^2 \left(a_2^2 C_1 M_1 + b_2^2 C_1 M_2 + c_2^2 M_1 + d_2^2 M_2 + e_2^2\right)$.

On the other hand, we have

$$u(x) = \int_0^x u'(x)dx,\ u(0) = 0 \text{ and } u'(x) = -\int_x^1 u''(x)dx,\ u'(1) = 0. \tag{2.45}$$

Thus

$$|u(x)| \leq \left(\int_0^1 (u'(x))^2 dx\right)^{\frac{1}{2}} \leq \sqrt{C_1 M_1}, \tag{2.46}$$

and

$$|u'(x)| \leq \left(\int_0^1 (u''(x))^2 dx\right)^{\frac{1}{2}} \leq \sqrt{M_1}. \tag{2.47}$$

Also, from

$$\rho(x)w(x) = \int_0^x (\rho(x)w(x))'\, dx,\ \rho(0) = 0, \tag{2.48}$$

we obtain

$$|\rho(x)w(x)| \leq \int_0^1 \left|(\rho(x)w(x))'\right| dx \leq \int_0^1 |\rho'(x)w(x) + \rho(x)w'(x)|\, dx. \tag{2.49}$$

Using $\sup_{0\leq x\leq 1} |\rho'(x)| = 1,\ \sup_{0\leq x\leq 1} \rho(x) \,|= \frac{1}{2}$ and applying Hölder's inequality, we obtain

$$|\, \rho(x)w(x) \,| \leq \sqrt{2}\left[\int_0^1 (w^2(x) + \rho(x)w'(x)^2)dx\right]^{\frac{1}{2}} \leq \sqrt{2M_3}. \tag{2.50}$$

We also have

$$w'(x) = \int_x^1 w''(x)dx,\ w'(1) = 0. \tag{2.51}$$

Hence

$$|w'(x)| \le \left(\int_0^1 (w''(x))^2 dx\right)^{\frac{1}{2}} \le \sqrt{M_5}. \tag{2.52}$$

Similarly, we obtain

$$\max_{0\le x\le 1} \left(|v(x)| + |v'(x)| + |\rho(x)z(x)| + |z'(x)|\right) \le \sqrt{C_1 M_2} + \sqrt{M_2} + \sqrt{2M_4} + \sqrt{M_6}. \tag{2.53}$$

Thus, the resulting inequalities imply the required result, and complete the proof of the proposition. □

The fundamental theorem used in proving the existence of the solution is Schauder's theorem. In order to make use of this theorem, it is sufficient to present the following lemmas.

Lemma 2.2. *Let $h : [0, 1] \to \mathbb{R}$ be a continuous function. The unique solution u of the boundary value problem*

$$u'' = h(x),\ 0 < x < 1, \tag{2.54}$$

subject to the boundary conditions $u(0) = u'(1) = 0$ is given by

$$u(x) = \int_0^1 \widehat{g}(x, y)h(y)dy, \tag{2.55}$$

where

$$\widehat{g}(x, y) = \begin{cases} -x, & 0 \le x \le y \le 1, \\ -y, & 0 \le y \le x \le 1. \end{cases} \tag{2.56}$$

Proof. Integrating this equation twice, we obtain

$$u(x) = \int_0^x \left[\int_1^y h(s)ds\right] dy + \delta_1 x + \delta_2, \tag{2.57}$$

where $\delta_i,\ i = 1, 2$ are constants of integration. Integrations by parts of the integral with respect to y in this equation gives

$$u(x) = -x \int_x^1 h(y)dy - \int_0^x yh(y)dy + \delta_1 x + \delta_2. \tag{2.58}$$

We determine $\delta_1 = \delta_2 = 0$ from $u(0) = u'(1) = 0$. It follows that

$$u(x) = -x \int_x^1 h(y)dy - \int_0^x yh(y)dy. \tag{2.59}$$

The proof is complete. □

Lemma 2.3. *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be continuous functions. The unique solution (u, v) of the following system*

$$\begin{cases} w'' &= h_1(x),\ 0 < x < 1, \\ z'' &= h_2(x),\ 0 < x < 1, \end{cases} \tag{2.60}$$

subject to the integral boundary conditions

$$\begin{cases} w(0) &= \int_0^1 p(x)z(x)dx,\ w'(1) = 0, \\ z(0) &= \int_0^1 p(x)w(x)dx,\ z'(1) = 0, \end{cases} \tag{2.61}$$

is given by

$$w(x) = \int_0^1 G_1(x,y)h_1(y)dy + \int_0^1 G_2(x,y)h_2(y)dy,\ z(x) = \int_0^1 G_3(x,y)h_2(y)dy + \int_0^1 G_4(x,y)h_1(y)dy, \tag{2.62}$$

where $G_i(x;y),\ i = 1,2,3,4$ are the Green functions of this boundary value problem and given by

$$G_1(x,y) = \widehat{g}(x,y) + \frac{1}{1-\sigma}\int_0^1 p(x)dx \int_0^1 q(x)\widehat{g}(x,y)dx,\ G_2(x,y) = \frac{1}{1-\sigma}\int_0^1 p(x)\widehat{g}(x,y)dx, \tag{2.63}$$

and

$$G_3(x,y) = \widehat{g}(x,y) + \frac{1}{1-\sigma}\int_0^1 q(x)dx \int_0^1 p(x)\widehat{g}(x,y)dx,\ G_4(x,y) = \frac{1}{1-\sigma}\int_0^1 q(x)\widehat{g}(x,y)dx, \tag{2.64}$$

where $\sigma = \int_0^1 p(x)dx \int_0^1 q(x)dx$.

Proof. Proceeding as in the previous proof of lemma, we obtain

$$\begin{cases} w &= \int_0^1 \widehat{g}(x,y)h_1(y)dy + \delta_1 x + \delta_2,\ 0 < x < 1, \\ z &= \int_0^1 \widehat{g}(x,y)h_2(y)dy + \delta_3 x + \delta_4,\ 0 < x < 1. \end{cases} \tag{2.65}$$

We determine $\delta_i,\ i = 1,2,3,4$ from the given boundary conditions, which imply that $\delta_1 = \delta_3 = 0,\ \delta_2 = \int_0^1 p(x)z(x)dx,\ \delta_4 = \int_0^1 q(x)w(x)dx$. It follows that

$$\begin{cases} w &= \int_0^1 \widehat{g}(x,y)h_1(y)dy + \int_0^1 p(x)z(x)dx,\ 0 < x < 1, \\ z &= \int_0^1 \widehat{g}(x,y)h_2(y)dy + \int_0^1 q(x)w(x)dx,\ 0 < x < 1. \end{cases} \tag{2.66}$$

Solving this system for $\int_0^1 p(x)z(x)dx$ and $\int_0^1 q(x)w(x)dx$, we obtain

$$\begin{cases} \int_0^1 q(x)w(x)dx &= \frac{1}{1-\sigma}\left[\int_0^1 q(x)(\int_0^1 \widehat{g}(x,y)h_1(y)dy)dx + \int_0^1 q(x)dx \int_0^1 p(x)(\int_0^1 \widehat{g}(x,y)h_2(y)dy)dx\right], \\ \int_0^1 p(x)z(x)dx &= \frac{1}{1-\sigma}\left[\int_0^1 p(x)(\int_0^1 \widehat{g}(x,y)h_2(y)dy)dx + \int_0^1 p(x)dx \int_0^1 q(x)(\int_0^1 \widehat{g}(x,y)h_1(y)dy)dx\right]. \end{cases} \tag{2.67}$$

A simple computation leads to

$$\begin{cases} w(x) &= \int_0^1 \widehat{g}(x,y)h_1(y)dy + \frac{1}{1-\alpha}\left[\int_0^1 q(x)(\int_0^1 \widehat{g}(x,y)h_1(y)dy)dx + \int_0^1 q(x)dx \int_0^1 p(x)(\int_0^1 \widehat{g}(x,y)h_2(y)dy)dx\right], \\ z(x) &= \int_0^1 \widehat{g}(x,y)h_2(y)dy + \frac{1}{1-\alpha}\left[\int_0^1 p(x)(\int_0^1 \widehat{g}(x,y)h_2(y)dy)dx + \int_0^1 p(x)dx \int_0^1 q(x)(\int_0^1 \widehat{g}(x,y)h_1(y)dy)dx\right], \end{cases} \tag{2.68}$$

which are what we had to prove. □

Thus, in view of these two lemmas and from (2.1), we obtain an equivalent integral system

$$u = \int_0^1 \widehat{g}(x,s)w(s)ds, \tag{2.69}$$

$$v = \int_0^1 \widehat{g}(x,s)z(s)ds, \tag{2.70}$$

$$\begin{aligned} w = &-\int_0^1 G_1(x,s)A(s)u(s)ds + \lambda \int_0^1 G_1(x,s)f(s,u(s),v(s),w(s),z(s))ds \\ &- \int_0^1 G_2(x,s)B(s)v(s)ds + \mu \int_0^1 G_2(x,s)g(s,u(s),v(s),w(s),z(s))ds, \end{aligned} \tag{2.71}$$

and

$$\begin{aligned} z = &-\int_0^1 G_3(x,s)B(s)v(s)ds + \mu \int_0^1 G_3(x,s)g(s,u(s),v(s),w(s),z(s))ds \\ &- \int_0^1 G_4(x,s)A(s)w(s)ds + \lambda \int_0^1 G_4(x,s)f(s,u(s),v(s),w(s),z(s))ds. \end{aligned} \tag{2.72}$$

Inserting (2.71) and (2.72) into (2.69) and (2.70), we obtain

$$\begin{aligned} u = &-\int_0^1 \widehat{G}_1(x,t)A(t)u(t)dt + \lambda \int_0^1 \widehat{G}_1(x,t)f(t,u(t),v(t),w(t),z(t))dt \\ &- \int_0^1 \widehat{G}_2(x,t)B(t)v(t)dt + \mu \int_0^1 \widehat{G}_2(x,t)g(t,u(t),v(t),w(t),z(t))dt, \end{aligned} \tag{2.73}$$

and

$$\begin{aligned} v = &-\int_0^1 \widehat{G}_3(x,t)B_(t)v(t)dt + \mu \int_0^1 \widehat{G}_3(x,t)g(t,u(t),v(t),w(t),z(t))dt \\ &- \int_0^1 \widehat{G}_4(x,t)A(t)w(t)dt + \lambda \int_0^1 \widehat{G}_4(x,t)f(t,u(t),v(t),w(t),z(t))dt, \end{aligned} \tag{2.74}$$

respectively, where $\widehat{G}_i(x,t) = \int_0^1 \widehat{g}(x,s)G_i(s,t)ds,\ i = 1, ..., 4.$
Consider the Banach space $\mathbb{Y}_\rho = C^3_\rho[0,1] \subset C^3[0,1]$ with norm $\| u \|_{3,\rho}$, where $\rho(x) = x(1-\frac{x}{2})$, and define the operator $T : \mathbb{X} \longrightarrow \mathbb{X}$ by $T(u,v) = (T_1(u,v), T_2(u,v))$, where $\mathbb{X} = \mathbb{Y}_\rho \times \mathbb{Y}_\rho$ with norm $\| (u,v) \|_{3,\rho} = \| u \|_{3,\rho} + \| v \|_{3,\rho}$, and

$$\begin{aligned} T_1(u,v) = &-\int_0^1 \widehat{G}_1(x,t)A(t)u(t)dt + \lambda \int_0^1 \widehat{G}_1(x,t)f(t,u(t),v(t),w(t),z(t))dt \\ &- \int_0^1 \widehat{G}_2(x,t)B(t)v(t)dt + \mu \int_0^1 \widehat{G}_2(x,t)g(t,u(t),v(t),w(t),z(t))dt, \end{aligned} \tag{2.75}$$

and

$$T_2(u,v) = -\int_0^1 \widehat{G}_3(x,t)B(t)v(t)dt + \mu \int_0^1 \widehat{G}_3(x,t)g(t,u(t),v(t),w(t),z(t))dt$$

$$-\int_0^1 \widehat{G}_4(x,t)A(t)w(t)dt + \lambda\int_0^1 \widehat{G}_4(x,t)f(t,u(t),v(t),w(t),z(t))dt. \quad (2.76)$$

Consider the closed and convex set

$$\mathbb{S} = \left\{(u,v) \in \mathbb{X} : \|(u,v)\|_{3,\rho} \le 6(M+M^*)\right\}. \quad (2.77)$$

Lemma 2.4. *For any $(u,v) \in \mathbb{S}$, $T(u,v)$ is contained in $\mathbb{S}$.*

Proof. Note first that the differentiability of $\widehat{G}_i,\ i = 1,...,4$ allows differentiation under integral sign. From the definition of $T(u,v)$, we have

$$\begin{aligned} T_1'(u,v) = &-\int_0^1 \widehat{G}_{1,x}(x,t)A(t)u(t)dt + \lambda\int_0^1 \widehat{G}_{1,x}(x,t)f(t,u(t),v(t),w(t),z(t))dt \\ &-\int_0^1 \widehat{G}_{2,x}(x,t)B(t)v(t)dt + \mu\int_0^1 \widehat{G}_{2,x}(x,t)g(t,u(t),v(t),w(t),z(t))dt, \end{aligned} \quad (2.78)$$

$$\begin{aligned} T_1''(u,v) = &-\int_0^1 \widehat{G}_{1,xx}(x,t)A(t)u(t)dt + \lambda\int_0^1 \widehat{G}_{1,xx}(x,t)f(t,u(t),v(t),w(t),z(t))dt \\ &-\int_0^1 \widehat{G}_{2,xx}(x,t)B(t)v(t)dt + \mu\int_0^1 \widehat{G}_{2,xx}(x,t)g(t,u(t),v(t),w(t),z(t))dt, \end{aligned} \quad (2.79)$$

and

$$\begin{aligned} T_1'''(u,v) = &-\int_0^1 \widehat{G}_{1,xxx}(x,t)A(t)u(t)dt + \lambda\int_0^1 \widehat{G}_{1,xxx}(x,t)f(t,u(t),v(t),w(t),z(t))dt \\ &-\int_0^1 \widehat{G}_{2,xxx}(x,t)B(t)v(t)dt + \mu\int_0^1 \widehat{G}_{2,xxx}(x,t)g(t,u(t),v(t),w(t),z(t))dt. \end{aligned} \quad (2.80)$$

Thus,

$$\begin{aligned} |T_1(u,v)| \le &\int_0^1 \left|\widehat{G}_1(x,t)\right| |A(t)|\,|u(t)|\,dt + \lambda\int_0^1 \left|\widehat{G}_1(x,t)\right| |f(t,u(t),v(t),w(t),z(t))|\,dt \\ &+\int_0^1 \left|\widehat{G}_2(x,t)\right| |B(t)|\,|v(t)|\,dt + \mu\int_0^1 \left|\widehat{G}_2(x,t)\right| |g(t,u(t),v(t),w(t),z(t))|\,dt. \end{aligned} \quad (2.81)$$

Using $\left|\widehat{g}(x,s)\right| \le 1$, $\left|\widehat{G}_1(s,t)\right| \le \frac{1}{1-\sigma}$, $\sigma = \int_0^1 p(x)dx\int_0^1 q(x)dx$, $\left|\widehat{G}_2(s,t)\right| \le \frac{1}{1-\sigma}$, $\left|\widehat{G}_3(s,t)\right| \le \frac{1}{1-\sigma}\int_0^1 p(x)dx$, $\left|\widehat{G}_4(s,t)\right| \le \frac{1}{1-\sigma}\int_0^1 q(x)dx$, $|A(t)| \le A_1$, $|B(t)| \le B_1$ and the growth conditions on f and g with the above estimates of u,v,w,z, (Proposition 2.1) and with sufficiently small values of λ and μ, thus there exist constants $D_i > 0, i = 1,2,3,4$ such that

$$\max_{0\le x\le 1} |T_1(u,v)| \le D_1 = \max\left(\frac{A_1}{1-\sigma}, \frac{B_1}{1-\sigma}\right)(M+M^*). \quad (2.82)$$

Since $A_1 < \frac{\alpha}{\sqrt{C_1}}$, $B_1 < \frac{\beta}{\sqrt{C_1}}$, $\frac{\alpha}{1-\sigma} \le \sqrt{C_1}$ and $\frac{\beta}{1-\sigma} \le \sqrt{C_1}$, we have

$$\max_{0\le x\le 1} |T_1(u,v)| \le M + M^*. \quad (2.83)$$

Proceeding in this way for $T_1'(u,v)$, $T_1''(u,v)$ and $T_1'''(u,v)$, we obtain

$$\max_{0\le x\le 1} \mid T_1'(u,v) \mid\le D_2 \le M+M^*,\ \max_{0\le x\le 1} \mid \rho(x)T_1''(u,v) \mid\le D_3 \le M+M^*,\ \max_{0\le x\le 1} \mid T_1'''(u,v) \mid\le D_4 \le M+M^*. \tag{2.84}$$

Thus

$$\| T_1(u,v) \|_{3,\rho}\le 3(M+M^*). \tag{2.85}$$

A similar argument of the above gives

$$\| T_2(u,v) \|_{3,\rho}\le 3(M+M^*). \tag{2.86}$$

It follows that $\| T(u,v) \|_{3,\rho}\le 6(M+M^*)$. Taking into account the continuity of $f(x,u,v,u'',v'')$, $g(x,u,v,u'',v'')$, u,v,u'' and v'', it follows that $T(u,v)$ is continuous. This shows that $T(u,v)$ is also contained in $\mathbb{S}$. □

To prove that $T(u,v)$ is compact we use the Arzela-Ascoli Lemma, that is; $T(\mathbb{S})$ must be closed, bounded and equicontinuous. In order to prove that $T(\mathbb{S})$ is equicontinuous, it is sufficient to prove that if $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in [0,1]$ the inequality

$$|T\,(u(x),v(x)) - T(u(y),v(y))| \le \epsilon, \tag{2.87}$$

is satisfied for any x and y in the interval $[0,1]$ with $\mid x-y \mid< \delta$. Indeed, by the definition of $T_1(u,v)$, there exists a constant $K_1 > 0$ such that

$$\begin{aligned}|T_1\,(u(x),v(x)) - T_1(u(y),v(y))| &\le \left|\int_y^x \widehat{G_1}(x,t)A(t)u(t)dt + \lambda\int_y^x \widehat{G_1}(x,t)f(t,u(t),v(t),w(t),z(t))dt\right| \\ &+ \left|\int_y^x \widehat{G_2}(x,t)B(t)v(t)dt + \mu\int_y^x \widehat{G_2}(x,t)g(t,u(t),v(t),w(t),z(t))dt\right| \\ &\le K_1\,|x-y|\,.\end{aligned} \tag{2.88}$$

Similarly, for $T_2\,(u(x),v(x))$, we have

$$|T_2\,(u(x),v(x)) - T_2(u(y),v(y))| \le K_2\,|x-y| \quad \text{for any} \quad x,y \in [0,1], \tag{2.89}$$

which proves the equicontinuous of $T(u,v)$.

Consequently, $T(u,v)$ has a fixed point by the Schauder's fixed point theorem.

Thus, we have the following theorem

Theorem 2.5. *Under the hypothesis of Proposition 1, there exists a continuous solution* (u,v) *in* $C_\rho^3[0,1] \times C_\rho^3[0,1]$ *which satisfies system (1.6) with the boundary conditions (1.5).*

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Conflict of interest

The authors declare no conflict of interest.

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