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Research article

A class of explicit implicit alternating difference schemes for generalized time fractional Fisher equation

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Abstract: The generalized time fractional Fisher equation is one of the significant models to describe the dynamics of the system. The study of effective numerical techniques for the equation has important scientific significance and application value. Based on the alternating technique, this article combines the classical explicit difference scheme and the implicit difference scheme to construct a class of explicit implicit alternating difference schemes for the generalized time fractional Fisher equation. The unconditional stability and convergence with order $O\left(\tau^{2-\alpha}+h^2\right)$ of the proposed schemes are analyzed. Numerical examples are performed to verify the theoretical analysis. Compared with the classical implicit difference scheme, the calculation cost of the explicit implicit alternating difference schemes is reduced by almost 60%. Numerical experiments show that the explicit implicit alternating difference schemes are also suitable for solving the time fractional Fisher equation with initial weak singularity and have an accuracy of order $O\left(\tau^{\alpha}+h^2\right)$, which verify that the methods proposed in this paper are efficient for solving the generalized time fractional Fisher equation.

Keywords: generalized time fractional Fisher equation; explicit implicit alternating difference scheme; stability; convergence; numerical experiments
Mathematics Subject Classification: 65M06, 65M12

1. Introduction

Fractional derivatives provide an excellent tool for describing some phenomena with hereditary and memory properties and make up for the defects of integer order derivatives such as many parameters and unclear meanings. Fractional partial differential equations (FPDEs) are important mathematical models that describe many natural phenomena such as physics, chemistry, and biology [1–4]. It is to be noted that the research of FPDEs has gained much interest in the last few decades [5–8].

The generalized fractional Fisher equation is the most classical and simplest nonlinear reaction-diffusion equation and plays an important role in describing the dynamics of the system. In this paper, we consider the following generalized time fractional Fisher equation [2, 9]:

$$\begin{cases} D_t^{\alpha} u(x,t) = \frac{\partial^2 u}{\partial x^2} + f(x,t,u(x,t)), \\ u(x,0) = \mu(x), x \in [0,L], \\ u(0,t) = v_1(t), u(L,t) = v_2(t), t \in (0,T], \end{cases} \tag{1.1}$$

where $\mu(x), v_1(t), v_2(t)$ are the given functions with proper smoothness. The nonlinear source term $f(x,t,u(x,t))$ satisfies the Lipschitz condition with respect to u, that is

$$|f(x,t,u) - f(x,t,v)| \le l|u - v|,$$

where $0 < l < 1$ is called the Lipschitz constant for function $f(\cdot,\cdot,\cdot)$. The Caputo fractional derivative D_t^{α} is defined by

$$D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{\mathrm{d}\xi}{(t-\xi)^{\alpha}}, 0 < \alpha < 1,$$

where $\Gamma(\cdot)$ is the Gamma function.

For $\alpha = 1$, $f(u) = u(1-u)$, Eq (1.1) becomes the integer order Fisher equation which was initially proposed by Fisher to investigate the dynamics of the temporal spatial propagation of a virile gene in an infinite domain [10]. The equation represents the evolution of the population due to the two competing processes and changes in the interaction of diffusion and nonlinear reaction can be observed. This equation arises in heat and mass transfer, biology, and ecology. This equation and its modified form are also widely used in chemical kinetics [11], branching Brownian motion [12], epidemics [13] and other fields. The benefit of using fractional Fisher equation in physical processes is its nonlocal property. It indicates that the upcoming system state is also reliant on its past states. So, fractional models are more precise. Because most FPDEs do not have exact solutions, studying the numerical solutions of the generalized fractional Fisher equation has important scientific significance and engineering application value [14–16].

Many scholars have adopted various techniques, such as Haar wavelet method [17], residual power series method [18], homotopy analysis transform method [19], Jacobi wavelet collocation method [20], B-spline collocation method [21] to obtain the numerical solutions of the fractional Fisher equation. Zhang et al. [22] constructed a fully discrete scheme for the time fractional Fisher equation based on the finite difference method and local discontinuous Galerkin finite element method. The stability and error estimation of the scheme were discussed. Alquran et al. [23] analyzed the time fractional Fisher equation both analytically and numerically. A technique combining Sinc-collocation and finite difference method was used to solve the equation numerically. Atangana [24] presented further useful properties of the Caputo-Fabrizio fractional derivative and applied it to modify the Fisher equation. Then, he used the iterative method to solve the modified equation. The works above are interesting and instructive, but the computational efficiency of the numerical scheme is still a key issue that needs to be considered.

Finite difference method is one of the dominant numerical methods for solving FPDEs. Since its relatively simple programming and high computational efficiency, the finite difference method has been widely used in various fields of natural science and engineering technology. Several finite difference schemes have been proposed to solve FPDEs [25–28]. Based on the $L1$ approximation of the modified fractional derivative and the idea of Du Fort-Frankel scheme, Liao et al. [29] proposed unconditionally stable explicit difference schemes for time fractional diffusion equations. Liu et al. [30] constructed a numerical scheme for the time fractional diffusion equation with a nonlinear source term based on the finite element method and finite difference approximation. Khader and Saad [31] used the properties of Chebyshev polynomials to reduce the fractional Fisher equation to a system of ordinary differential equations, which was solved by the finite difference method. Zhang and Yang [32] constructed a class of explicit-implicit difference scheme and implicit-explicit difference scheme for time fractional reaction-diffusion equation. The numerical experiments verified the calculation efficiency of the schemes was improved by nearly 41% compared with the implicit difference scheme.

The aim of this work is to provide effective numerical methods to solve the generalized time fractional Fisher equation. We construct explicit implicit alternating difference schemes and the schemes can improve calculation efficiency on the basis of ensuring good accuracy. We combine the explicit scheme and implicit scheme into two-step schemes, in which half of the step length is calculated by the explicit scheme and the other half of the step length is calculated by the implicit scheme. In this way, only one tridiagonal matrix needs to be solved in each double step, hence the calculation efficiency is improved. The present methods also can be applied to solve the time fractional Fisher equation with initial weak singularity.

2. Explicit implicit alternating scheme for the generalized time fractional Fisher equation

2.1. The construction of explicit implicit alternating scheme

To derive the proposed scheme, we first divide the solution area $\Omega = \{(x,t)|0 \le x \le L, 0 \le t < T\}$ into equidistant rectangular grids. Take two positive integers M and N, denote $h = \frac{L}{M}$, $\tau = \frac{T}{N}$, $x_i = ih(0 \le i \le M)$, $t_k = k\tau(0 \le k \le N)$. Let u_i^k represent the numerical approximation of $u(x_i, t_k)$. Second-order central difference scheme is adopted to approximate $\frac{\partial^2 u}{\partial x^2}$ in the Eq (1.1).

$$\frac{\partial^2 u(x_i, t_n)}{\partial x^2} = \frac{u(x_{i-1}, t_n) - 2u(x_i, t_n) + u(x_{i+1}, t_n)}{h^2} + O(h^2).$$

The time fractional derivative can be approximated by $L1$ formula [33, 34],

$$\begin{aligned} &D_t^\alpha u(x, t_k) \\ =&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \left[u\left(x, t_{k-j}\right) - u\left(x, t_{k-j-1}\right)\right]\left[(j+1)^{1-\alpha} - j^{1-\alpha}\right] + O(\tau^{2-\alpha}) \\ =&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[c_0 u(x, t_k) - c_{k-1} u(x, t_0) - \sum_{j=1}^{k-1} (c_{j-1} - c_j) u(x, t_{k-j})\right] + O(\tau^{2-\alpha}), \end{aligned} \tag{2.1}$$

where $c_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j = 0, 1, ..., k-1$.

In order to establish the explicit implicit alternating difference scheme for Eq (1.1), we give the classical explicit difference scheme and classical implicit difference scheme of Eq (1.1) first.

The explicit difference scheme is:

$$D_t^\alpha u(x_i, t_n) = \frac{1}{h^2}\left(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}\right) + f\left(x_i, t_{n-1}, u_i^{n-1}\right). \tag{2.2}$$

The implicit difference scheme is:

$$D_t^\alpha u(x_i,t_n)=\frac{1}{h^2}\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)+f\left(x_i,t_n,u_i^n\right). \tag{2.3}$$

Let $f(x_i,t_n,u_i^n)=f_i^n, p=\tau^\alpha\Gamma(2-\alpha), r=\frac{p}{h^2}$.
The above schemes can be rewritten as:
when $n=1$, we have

$$u_i^1=ru_{i-1}^0+(1-2r)u_i^0+ru_{i+1}^0+pf_i^0, \tag{2.4}$$

$$-ru_{i-1}^1+(1+2r)u_i^1-ru_{i+1}^1=u_i^0+pf_i^1, \tag{2.5}$$

when $n>1$, we have

$$u_i^{n+1}=ru_{i-1}^n+(1-2r-c_1)u_i^n+ru_{i+1}^n+\sum_{j=1}^{n-1}(c_j-c_{j+1})u_i^{n-j}+c_nu_i^0+pf_i^n, \tag{2.6}$$

$$-ru_{i-1}^{n+1}+(1+2r)u_i^{n+1}-ru_{i+1}^{n+1}=(1-c_1)u_i^n+\sum_{j=1}^{n-1}(c_j-c_{j+1})u_i^{n-j}+c_nu_i^0+pf_i^{n+1}. \tag{2.7}$$

Divide the grid points into two groups according to the parity of the time index n. Based on the alternating method, we use the explicit scheme in the odd-numbered time layer and the implicit scheme in the even-numbered time layer:

$$u_i^{2n+1}=ru_{i-1}^{2n}+(1-2r-c_1)u_i^{2n}+ru_{i+1}^{2n}+\sum_{j=1}^{2n-1}(c_j-c_{j+1})u_i^{2n-j}+c_{2n}u_i^0+pf_i^{2n}, \tag{2.8}$$

$$-ru_{i-1}^{2n+2}+(1+2r)u_i^{2n+2}-ru_{i+1}^{2n+2}=(1-c_1)u_i^{2n+1}+\sum_{j=1}^{2n}(c_j-c_{j+1})u_i^{2n+1-j}+c_{2n+1}u_i^0+pf_i^{2n+2}. \tag{2.9}$$

Therefore, we can give the explicit implicit alternating scheme for the generalized time fractional Fisher Eq (1.1) as follows.

$$\begin{cases}u^{2n+1}=\left(v_1I-rG_1\right)u^{2n}+\sum_{j=1}^{2n-1}v_{j+1}u^{2n-j}+c_{2n}u^0+pf^{2n}+g^{2n},\\ \left(I+rG_1\right)u^{2n+2}=v_1u^{2n+1}+\sum_{j=1}^{2n}v_{j+1}u^{2n+1-j}+c_{2n+1}u^0+pf^{2n+2}+g^{2n+2},\end{cases} \tag{2.10}$$

where $v_j=c_{j-1}-c_j(j=1,2,...,N), u^n=(u_1^n,u_2^n,...,u_{M-1}^n)^T(n=0,1,...,N), f^n=(f_1^n,f_2^n,...,f_{M-1}^n)^T(n=0,1,...,N), g^n=(-ru_0^n,0,...,0,-ru_M^n)^T(n=0,1,...,N)$, and

$$G_1=\begin{pmatrix}2&-1&&&&\\-1&2&-1&&&\\&-1&2&-1&&\\&&\ddots&\ddots&\ddots&\\&&&-1&2&-1\\&&&&-1&2\end{pmatrix}_{(M-1)\times(M-1)}.$$

2.2. The existence and uniqueness of the explicit implicit alternating scheme solution

Obviously, the matrix $(I + rG_1)$ is strictly diagonally dominant, thus.

Theorem 2.1. *The explicit implicit alternating scheme* (2.10) *for the generalized time fractional Fisher Eq* (1.1) *has a unique solution.*

2.3. Stability of the explicit implicit alternating scheme

In order to facilitate numerical analysis, we substitute the Eq (2.8) into the Eq (2.9) to eliminate u_i^{2n+1} and obtain

$$\begin{aligned}
& - ru_{i-1}^{2n+2} + (1 + 2r)u_i^{2n+2} - ru_{i+1}^{2n+2} \\
=&(1 - c_1)\left[ru_{i-1}^{2n} + (1 - 2r - c_1)u_i^{2n} + ru_{i+1}^{2n} + \sum_{j=1}^{2n-1}(c_j - c_{j+1})u_i^{2n-j} + c_{2n}u_i^0 + pf_i^{2n}\right] \\
& + \sum_{j=1}^{2n}(c_j - c_{j+1})u_i^{2n+1-j} + c_{2n+1}u_i^0 + pf_i^{2n+2} \qquad (2.11)\\
=&rv_1u_{i-1}^{2n} + [v_1(v_1 - 2r) + v_2]u_i^{2n} + rv_1u_{i+1}^{2n} + \sum_{j=1}^{2n-1}(v_1v_{j+1} + v_{j+2})u_i^{2n-j} \\
& + (v_1c_{2n} + c_{2n+1})u_i^0 + v_1pf_i^{2n} + pf_i^{2n+2}.
\end{aligned}$$

Suppose that $\tilde{u}_i^n$ is the approximate solution of difference scheme (2.11), the round-off error ε_i^n is defined as $\varepsilon_i^n = \tilde{u}_i^n - u_i^n, n = 0, 1, ..., N, i = 0, 1, ..., M$, and ε_i^n satisfies the discretized Eq (2.11). Denote $\tilde{f}_i^n = f(x_i, t_n, \tilde{u}_i^n)$, we have

$$\begin{aligned}
& - r\varepsilon_{i-1}^{2n+2} + (1 + 2r)\varepsilon_i^{2n+2} - r\varepsilon_{i+1}^{2n+2} \\
=&rv_1\varepsilon_{i-1}^{2n} + [v_1(v_1 - 2r) + v_2]\varepsilon_i^{2n} + rv_1\varepsilon_{i+1}^{2n} + \sum_{j=1}^{2n-1}(v_1v_{j+1} + v_{j+2})\varepsilon_i^{2n-j} \qquad (2.12)\\
& + (v_1c_{2n} + c_{2n+1})\varepsilon_i^0 + v_1p(\tilde{f}_i^{2n} - f_i^{2n}) + p(\tilde{f}_i^{2n+2} - f_i^{2n+2}).
\end{aligned}$$

Since $f(x, t, u(x, t))$ satisfies the Lipschitz condition with respect to u, we have

$$\left|\tilde{f}_i^n - f_i^n\right| \le l\left|\tilde{u}_i^n - u_i^n\right| = l\left|\varepsilon_i^n\right|.$$

Supposing that $\|\varepsilon^n\|_\infty = \max_{0\le i\le M}|\varepsilon_i^n|$, we can get the following theorem.

Theorem 2.2. *The explicit implicit alternating scheme* (2.10) *for the generalized time fractional Fisher Eq* (1.1) *is unconditionally stable, and we have*

$$\|\varepsilon^n\|_\infty \le K\|\varepsilon^0\|_\infty, n = 0, 1, ..., N, \qquad (2.13)$$

where K is a positive number independent of n, h and τ.

Proof. Suppose that $\|\varepsilon^n\|_\infty = \max_{0\le i\le M}|\varepsilon_i^n| = |\varepsilon_m^n|$,
when $n = 1$, obviously we have

$$\begin{aligned}\left|\varepsilon_m^1\right| &\le r\left|\varepsilon_{m-1}^0\right| + (1-2r)\left|\varepsilon_m^0\right| + r\left|\varepsilon_{m+1}^0\right| + p\left|\tilde{f}_m^0 - f_m^0\right| \\ &\le r\left|\varepsilon_{m-1}^0\right| + (1-2r)\left|\varepsilon_m^0\right| + r\left|\varepsilon_{m+1}^0\right| + pl\left|\varepsilon_m^0\right| \le (1+pl)\left|\varepsilon_m^0\right| \le K\left|\varepsilon_m^0\right|.\end{aligned} \tag{2.14}$$

That is $\|\varepsilon^1\|_\infty \le K\|\varepsilon^0\|_\infty$.

Assume that $\|\varepsilon^n\|_\infty \le K\|\varepsilon^0\|_\infty$ holds for $n = 1, 2, ..., 2s+1$. Now we prove that the inequality (2.13) also holds for $n = 2s+2$.

Using (2.11), we have

$$\begin{aligned}\left|\varepsilon_m^{2s+2}\right| &\le -r\left|\varepsilon_{m-1}^{2s+2}\right| + (1+2r)\left|\varepsilon_m^{2s+2}\right| - r\left|\varepsilon_{m+1}^{2s+2}\right| \\ &\le \left|-r\varepsilon_{m-1}^{2s+2} + (1+2r)\varepsilon_m^{2s+2} - r\varepsilon_{m+1}^{2s+2}\right| \\ &\le \left|rv_1\varepsilon_{m-1}^{2s} + [v_1(v_1-2r)+v_2]\varepsilon_m^{2s} + rv_1\varepsilon_{m+1}^{2s} + \sum_{j=1}^{2s-1}(v_1v_{j+1}+v_{j+2})\varepsilon_m^{2s-j}\right| \\ &\quad + \left|(v_1c_{2s}+c_{2s+1})\varepsilon_m^0 + v_1p(\tilde{f}_m^{2s} - f_m^{2s}) + p(\tilde{f}_m^{2s+2} - f_m^{2s+2})\right| \\ &\le \left|rv_1\varepsilon_{m-1}^{2s} + [v_1(v_1-2r)+v_2]\varepsilon_m^{2s} + rv_1\varepsilon_{m+1}^{2s} + \sum_{j=1}^{2s-1}(v_1v_{j+1}+v_{j+2})\varepsilon_m^{2s-j}\right| \\ &\quad + (v_1c_{2s}+c_{2s+1})\left|\varepsilon_m^0\right| + v_1l\left|\varepsilon_m^{2s}\right| + l\left|\varepsilon_m^{2s+2}\right|.\end{aligned} \tag{2.15}$$

Then

$$\begin{aligned}\left|\varepsilon_m^{2s+2}\right| &\le \left|\frac{rv_1}{1-l}\varepsilon_{m-1}^{2s} + \frac{[v_1(v_1-2r)+v_2]}{1-l}\varepsilon_m^{2s} + \frac{rv_1}{1-l}\varepsilon_{m+1}^{2s} + \sum_{j=1}^{2s-1}\frac{(v_1v_{j+1}+v_{j+2})}{1-l}\varepsilon_m^{2s-j}\right| \\ &\quad + \frac{(v_1c_{2s}+c_{2s+1})}{1-l}\left|\varepsilon_m^0\right| + \frac{v_1l}{1-l}\left|\varepsilon_m^{2s}\right| \\ &\le \frac{(v_1^2+v_2)+v_1l}{1-l}\left|\varepsilon_m^{2s}\right| + \sum_{j=1}^{2s-1}\frac{(v_1v_{j+1}+v_{j+2})}{1-l}\left|\varepsilon_m^{2s-j}\right| + \frac{(v_1c_{2s}+c_{2s+1})}{1-l}\left|\varepsilon_m^0\right|.\end{aligned} \tag{2.16}$$

Since $\sum_{j=1}^{2s-1} v_{j+1} < 1$, we obtain

$$\left|\varepsilon_m^{2s+2}\right| \le \frac{(v_1^2+v_2)+v_1l}{1-l}\left|\varepsilon_m^{2s}\right| + K_1\left|\varepsilon_m^0\right| \le (K_1+K_2)\left|\varepsilon_m^0\right| \le K\left|\varepsilon_m^0\right|.$$

That is $\|\varepsilon^{2s+2}\|_\infty \le K\|\varepsilon^0\|_\infty$.
Therefore, there exists a positive K independent of n, h and τ, such that $\|\varepsilon^n\|_\infty \le K\|\varepsilon^0\|_\infty, n = 0, 1, ..., N$.
The proof is completed. □

2.4. Convergence of the explicit implicit alternating scheme

First, we analyze the accuracy of the explicit implicit alternating scheme. The explicit scheme and implicit scheme are expanded by Taylor series at the point u_i^{n+1}, respectively. Let R_E and R_I represent the local truncation errors of the explicit scheme and implicit scheme, respectively. According to

Stynes et al. [35], we assume that u_{tt}, u_{xxxx} are bounded over the intervals $[0, T]$ and $[0, L]$. That is, there exists a positive constant F, such that

$$|u_{tt}| \leq F, |u_{xxxx}| \leq F. \tag{2.17}$$

The local truncation error of the explicit scheme is:

$$R_E = -u_{xx} + \tau u_{xxt} - \frac{\tau^2}{2}u_{xxtt} - \frac{h^2}{12}u_{xxxx} - \tau\frac{\partial f(u)}{\partial t} + O\left(\tau^{2-\alpha} + h^2\right).$$

The local truncation error of the implicit scheme is:

$$R_I = -u_{xx} - \tau u_{xxt} - \frac{\tau^2}{2}u_{xxtt} - \frac{h^2}{12}u_{xxxx} + \tau\frac{\partial f(u)}{\partial t} + O\left(\tau^{2-\alpha} + h^2\right).$$

$\tau u_{xxt}, \tau\frac{\partial f(u)}{\partial t}$ in R_E and $\tau u_{xxt}, \tau\frac{\partial f(u)}{\partial t}$ in R_I have the same forms but opposite signs. Therefore, two error terms can be cancelled out by using the explicit scheme and implicit scheme alternately. Thus, the accuracy of the explicit implicit alternating scheme is second order in space and $2-\alpha$ order in time.

Suppose that $u(x_i, t_n)$ and u_i^n are the exact solution and numerical solution of the differential equation, respectively. Denote $e_i^n = u(x_i, t_n) - u_i^n, i = 0, 1, ..., M, n = 1, 2, ..., N$. Apparently, $e^0 = 0$. Substitute $e_i^n = u(x_i, t_n) - u_i^n$ into Eq (2.10), we have

$$\begin{cases} e^{2n+1} = (v_1 I - rG_1)\, e^{2n} + \sum_{j=1}^{2n-1} v_{j+1} e^{2n-j} + c_{2n}e^0 + p\left(f(u^{2n}) - f^{2n}\right) + pR^{2n+1}, \\ (I + rG_1)\, e^{2n+2} = v_1 e^{2n+1} + \sum_{j=1}^{2n} v_{j+1} e^{2n+1-j} + c_{2n+1}e^0 + p\left(f(u^{2n+2}) - f^{2n+2}\right) + pR^{2n+2}, \end{cases} \tag{2.18}$$

where $R^n = O(\tau^{2-\alpha} + h^2)$. Thus, there exists $K > 0$, such that $\|\tau^\alpha R^n\| \leq K\tau^\alpha(\tau^{2-\alpha} + h^2)$.

Theorem 2.3. *Suppose that $u(x, t)$ satisfies the smooth condition* (2.17)*, the explicit implicit alternating scheme* (2.10) *for the generalized time fractional Fisher Eq* (1.1) *is convergent, and we have*

$$\|u(x_i, t_n) - u_i^n\|_\infty \leq C(\tau^{2-\alpha} + h^2), i = 0, 1, ..., M, n = 0, 1, ..., N, \tag{2.19}$$

where C is a positive constant.

Proof. Suppose that $\|\varepsilon^n\|_\infty = \max_{0\leq i\leq M}|\varepsilon_i^n| = |\varepsilon_m^n|$,
when $n = 1$, obviously we have

$$\begin{aligned} \left|e_m^1\right| &\leq r\left|e_{m-1}^0\right| + (1-2r)\left|e_m^0\right| + r\left|e_{m+1}^0\right| + p\left|f(u_m^0) - f_m^0\right| + \left|pR^1\right| \\ &\leq \tau^\alpha R^1 \leq K\tau^\alpha(\tau^{2-\alpha} + h^2) = c_0^{-1}K\tau^\alpha(\tau^{2-\alpha} + h^2). \end{aligned} \tag{2.20}$$

That is $\|e^1\|_\infty \leq c_0^{-1}K\tau^\alpha(\tau^{2-\alpha} + h^2)$.

Assume that $\|e^n\|_\infty \leq c_{n-1}^{-1}K\tau^\alpha(\tau^{2-\alpha} + h^2)$ holds for $n = 1, 2, ..., 2s + 1$. Now we prove that the inequality (2.19) also holds for $n = 2s + 2$.

Using (2.11), we have

$$
\begin{aligned}
\left|e_m^{2s+2}\right| \le & -r\left|e_{m-1}^{2s+2}\right| + (1+2r)\left|e_m^{2s+2}\right| - r\left|e_{m+1}^{2s+2}\right| \\
\le & \left|-re_{m-1}^{2s+2} + (1+2r)e_m^{2s+2} - re_{m+1}^{2s+2}\right| \\
\le & \left|rv_1 e_{m-1}^{2s} + [v_1(v_1-2r)+v_2]e_m^{2s} + rv_1 e_{m+1}^{2s} + \sum_{j=1}^{2s-1}(v_1 v_{j+1} + v_{j+2})e_m^{2s-j}\right| \\
& + \left|(v_1 c_{2s} + c_{2s+1})e_m^0 + v_1 p(f(u_m^{2s}) - f_m^{2s}) + p(f(u_m^{2s+2}) - f_m^{2s+2})\right| + \left|pR^{2s+2}\right| \\
\le & \left|rv_1 e_{m-1}^{2s} + [v_1(v_1-2r)+v_2]e_m^{2s} + rv_1 e_{m+1}^{2s} + \sum_{j=1}^{2s-1}(v_1 v_{j+1} + v_{j+2})e_m^{2s-j}\right| \\
& + v_1 l\left|e_m^{2s}\right| + l\left|e_m^{2s+2}\right| + \left|pR^{2s+2}\right|.
\end{aligned}
\tag{2.21}
$$

Then

$$
\begin{aligned}
\left|e_m^{2s+2}\right| \le & \left|\frac{rv_1}{1-l}e_{m-1}^{2s} + \frac{[v_1(v_1-2r)+v_2]}{1-l}e_m^{2s} + \frac{rv_1}{1-l}e_{m+1}^{2s} + \sum_{j=1}^{2s-1}\frac{(v_1 v_{j+1} + v_{j+2})}{1-l}e_m^{2s-j}\right| \\
& + \frac{v_1 l}{1-l}\left|e_m^{2s}\right| + \frac{|pR^{2s+2}|}{1-l} \\
\le & \frac{(v_1^2+v_2)+v_1 l}{1-l}\left|e_m^{2s}\right| + \sum_{j=1}^{2s-1}\frac{(v_1 v_{j+1} + v_{j+2})}{1-l}\left|e_m^{2s-j}\right| + \frac{|pR^{2s+2}|}{1-l}.
\end{aligned}
\tag{2.22}
$$

Since $\sum_{j=1}^{2s-1} v_{j+1} < 1$, we have

$$
\left|e_m^{2s+2}\right| \le c_{2s-1}^{-1}K_1\tau^\alpha(\tau^{2-\alpha}+h^2) + c_{2s+1}^{-1}K_2\tau^\alpha(\tau^{2-\alpha}+h^2) \le c_{2s+1}^{-1}K\tau^\alpha(\tau^{2-\alpha}+h^2).
$$

That is $\|e^{2s+2}\|_\infty \le c_{2s+1}^{-1}K\tau^\alpha(\tau^{2-\alpha}+h^2)$.

From

$$
\lim_{n\to\infty}\frac{c_n^{-1}}{n^\alpha} = \lim_{n\to\infty}\frac{n^{-\alpha}}{(n+1)^{1-\alpha} - n^{1-\alpha}} = \lim_{n\to\infty}\frac{n^{-1}}{(1+\frac{1}{n})^{1-\alpha} - 1} = \frac{1}{1-\alpha}.
$$

Then, there exists $C_1 > 0$, such that

$$
\|e^n\|_\infty \le c_{n-1}^{-1}K\tau^\alpha(\tau^{2-\alpha}+h^2) \le K\frac{c_{n-1}^{-1}}{n^\alpha}n^\alpha\tau^\alpha(\tau^{2-\alpha}+h^2) \le \frac{K}{1-\alpha}n^\alpha\tau^\alpha(\tau^{2-\alpha}+h^2) \le C_1 T^\alpha(\tau^{2-\alpha}+h^2).
$$

Therefore

$$
\|u(x_i,t_n) - u_i^n\|_\infty \le C(\tau^{2-\alpha}+h^2), i = 0,1,...,M, n = 0,1,...,N.
$$

The proof is completed. □

3. Implicit explicit alternating scheme for the generalized time fractional Fisher equation

Similar to the construction of the explicit implicit alternating scheme for the generalized time fractional Fisher equation, the implicit explicit alternating scheme is established. In the odd-numbered layer, using the implicit scheme to calculate; and in the even-numbered layer, using the explicit scheme to calculate. The implicit explicit alternating scheme for the generalized time fractional Fisher equation is as follows:

$$\begin{cases}(I+rG_1)\,u^{2n+1}=v_1u^{2n}+\sum_{j=1}^{2n-1}v_{j+1}u^{2n-j}+c_{2n}u^0+pf^{2n+1}+g^{2n+1},\\ u^{2n+2}=(v_1I-rG_1)\,u^{2n+1}+\sum_{j=1}^{2n}v_{j+1}u^{2n+1-j}+c_{2n+1}u^0+pf^{2n+1}+g^{2n+1}.\end{cases} \tag{3.1}$$

The numerical analysis of the implicit explicit alternating scheme can be proved similarly.

Theorem 3.1. *Suppose that $u(x,t)$ satisfies the smooth condition* (2.17)*, the implicit explicit alternating scheme* (3.1) *for the generalized time fractional Fisher Eq* (1.1) *is uniquely solvable, unconditionally stable and convergent, and we have*

$$\|u(x_i,t_n)-u_i^n\|_\infty\le C(\tau^{2-\alpha}+h^2),\, i=0,1,...,M, n=0,1,...,N,$$

where C is a positive constant.

A class of schemes proposed in this paper retain the absolute stability of the implicit scheme, and reduce the number of solving the implicit solutions about 50%. So the calculation efficiency can be improved. The explicit implicit alternating scheme (2.10) and implicit explicit alternating scheme (3.1) are two-step schemes and the difference is just the order of using explicit scheme and implicit scheme. Therefore, the calculation efficiency of the explicit implicit alternating scheme is the same as that of the implicit explicit alternating scheme. In numerical experiments, we only consider the explicit implicit alternating scheme (2.10) for the generalized time fractional Fisher equation.

Remark 3.1. *Note that we get the convergence order of the explicit implicit alternating difference scheme under the condition given in* (2.17) *on the solution u of the Eq* (1.1)*. In general, the solution $u(x,t)$ and its derivatives have weak singularity near $t=0$. Therefore, the temporal convergence order of scheme* (2.10) *and scheme* (3.1) *are obviously lower than $2-\alpha$ order when the solution $u(x,t)$ is non-smooth. In fact, the convergence order of the explicit implicit alternating schemes of the generalized time fractional Fisher Eq* (1.1) *is $O(\tau^\alpha+h^2)$ in this case (see reference [35]). We also confirm this conclusion in the numerical experiment.*

4. Numerical experiments

All experiments are performed on Intel Core i5-8265 CPU. The results are coded by Matlab R2017b.
Example 1. The time fractional Fisher equation [36]:

$$\begin{cases}\dfrac{\partial^\alpha u\,(x,t)}{\partial t^\alpha}=\dfrac{\partial^2 u}{\partial x^2}+u(1-u)+g_1(x,t),\\ u(x,0)=0, 0\le x\le 1,\\ u(0,t)=u(1,t)=0, 0\le t\le 1,\end{cases} \tag{4.1}$$

where $g_1(x,t)=24t^{4-\alpha}sin(2\pi x)/\Gamma(5-\alpha)+4\pi^2t^4sin(2\pi x)-t^4sin(2\pi x)(1-t^4sin(2\pi x))$.
The exact solution of Eq (4.1) is $u(x,t)=t^4sin(2\pi x)$.

Compare two numerical scheme solutions with the exact solution at $t=0.6$. The computational results can be seen in Table 1. These data show that the accuracy of explicit implicit alternating scheme (2.10) and implicit scheme are similar, and the errors with the exact solution are both small. The absolute error graph of the explicit implicit alternating scheme is shown in Figure 1 for different values of fractional order derivative.

Take $\alpha = 0.4$, we compare the explicit implicit alternating scheme solution with the exact solution. It can be seen from Figure 2 that the numerical solution is in excellent agreement with the exact solution at different moments, indicating that the explicit implicit alternating scheme (2.10) is a high precision scheme.

Table 1. Exact solution and numerical solutions for different α with $M = N = 100$ at $t = 0.6$.

α		x			
		0.2	0.4	0.6	0.8
	Exact solution	1.2326E-01	7.6177E-02	-7.6177E-02	-1.2326E-01
0.4	Scheme (2.10) solution	1.2319E-01	7.6116E-02	-7.6188E-02	-1.2324E-01
	Implicit scheme solution	1.2331E-01	7.6211E-02	-7.6205E-02	-1.2331E-01
	Exact solution	1.2326E-01	7.6177E-02	-7.6177E-02	-1.2326E-01
0.6	Scheme (2.10) solution	1.2314E-01	7.6089E-02	-7.6160E-02	-1.2320E-01
	Implicit scheme solution	1.2334E-01	7.6229E-02	-7.6225E-02	-1.2334E-01
	Exact solution	1.2326E-01	7.6177E-02	-7.6177E-02	-1.2326E-01
0.95	Scheme (2.10) solution	1.2328E-01	7.6178E-02	-7.6233E-02	-1.2333E-01
	Implicit scheme solution	1.2365E-01	7.6422E-02	-7.6424E-02	-1.2366E-01

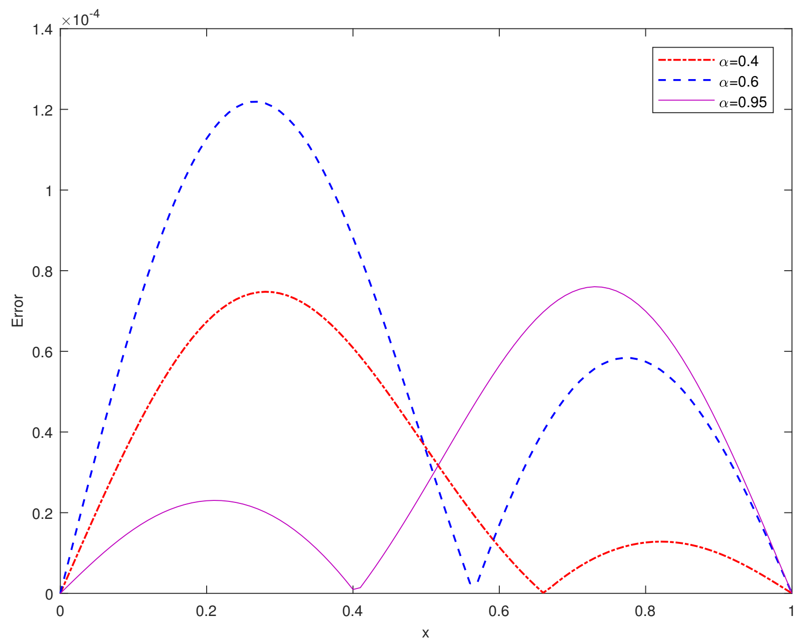

Figure 1. Absolute error of scheme (2.10) with $M = N = 100$ at $t = 0.6$.

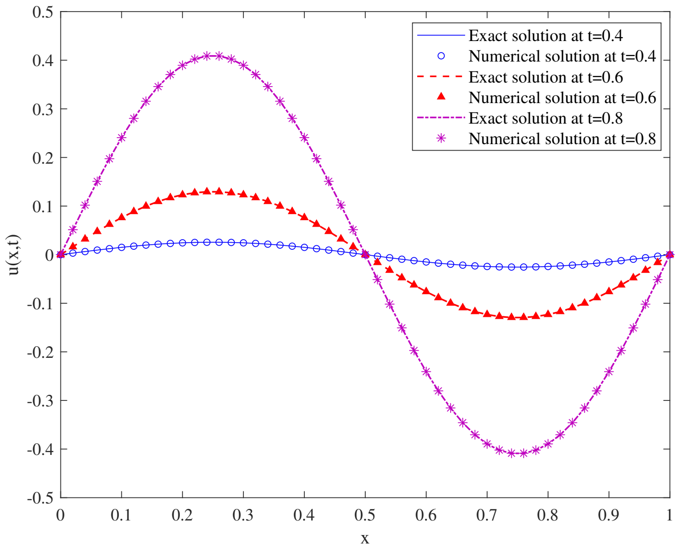

Figure 2. Exact solution and numerical solution of scheme (2.10) at different time with $\alpha = 0.4$.

For testing the temporal convergence order ($Order1$) and the spatial convergence order ($Order2$) of the explicit implicit alternating scheme (2.10), denote $E_\infty(h,\tau) = \max\limits_{0\le n\le N} \|u(x_i,t_n) - u_i^n\|_\infty$. Compute the $Order1$ and $Order2$ by the standard formulas [33, 37]:

$$Order_1 = \log_2\left(\frac{E_\infty(h,2\tau)}{E_\infty(h,\tau)}\right), Order_2 = \log_2\left(\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}\right).$$

Table 2. Error, temporal convergence orders and CPU time of two numerical schemes ($h = \frac{1}{200}$).

α	N	Scheme (2.10)			Implicit scheme		
		$E_\infty(h,\tau)$	$Order_1$	CPU time(s)	$E_\infty(h,\tau)$	$Order_1$	CPU time(s)
0.4	20	3.7238E-02	——	0.0126	2.1760E-03	——	0.0166
	40	1.3150E-02	1.50	0.0266	6.9341E-04	1.65	0.0390
	80	4.5280E-03	1.54	0.0606	2.4218E-04	1.52	0.0839
0.5	20	4.7618E-02	——	0.0126	2.3465E-03	——	0.0167
	40	1.7708E-02	1.43	0.0266	7.7885E-04	1.59	0.0388
	80	6.5023E-03	1.46	0.0605	2.7649E-04	1.49	0.0841
0.6	20	6.1869E-02	——	0.0125	2.6446E-03	——	0.0172
	40	2.4386E-02	1.34	0.0266	9.4330E-04	1.49	0.0388
	80	9.4376E-03	1.37	0.0608	3.5375E-04	1.41	0.0839

Here we choose $M = 200, N = 20, 40, 80$. The computation results are recorded in Table 2. These results demonstrate that temporal convergence orders of the explicit implicit alternating scheme (2.10) are close to $2 - \alpha$. This verifies the Theorem 2.3. Error results and convergence orders in spatial direction are listed in Table 3. From Table 3, we can see that the spatial convergence orders of explicit implicit alternating scheme (2.10) are almost second order, which are the same as the implicit scheme. Tables 2, 3 also show that compared with the implicit scheme, the explicit implicit alternating scheme can reduce the CPU time and storage.

Table 3. Error, spatial convergence orders and CPU time of two numerical schemes ($\tau = h^2$).

α	M	Scheme (2.10)			Implicit scheme		
		$E_\infty(h,\tau)$	$Order_2$	CPU time(s)	$E_\infty(h,\tau)$	$Order_2$	CPU time(s)
0.4	8	5.5983E-02	——	0.0109	5.7266E-02	——	0.0110
	16	1.3648E-02	2.04	0.1267	1.3966E-02	2.04	0.1310
	32	3.3879E-03	2.01	1.9428	3.4698E-03	2.01	2.0239
0.5	8	5.5189E-02	——	0.0108	5.6702E-02	——	0.0111
	16	1.3454E-02	2.04	0.1265	1.3825E-02	2.04	0.1311
	32	3.3399E-03	2.01	1.9423	3.4339E-03	2.01	2.0241
0.6	8	5.4380E-02	——	0.0109	5.6125E-02	——	0.0110
	16	1.3255E-02	2.04	0.1264	1.3677E-02	2.04	0.1307
	32	3.2899E-03	2.01	1.9421	3.3956E-03	2.01	2.0232

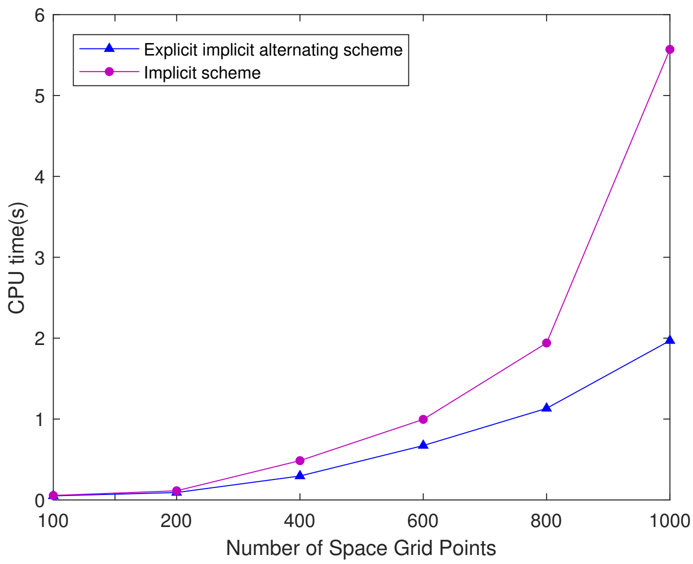

Figure 3. Comparison of CPU time among two schemes.

We will further compare the computational costs of two numerical methods and indicate the effectiveness of the explicit implicit alternating scheme (2.10).

Take $\alpha = 0.4, N = 200$ and $M = 100, 200, 400, 600, 800, 1000$. From Figure 3, we can see that the CPU time of two numerical schemes is almost the same when $M \leq 100$. This conclusion is consistent with the results in Table 3. But as the number of spatial grids increases, the advantages of the explicit implicit alternating scheme become more and more obvious. When $M = 1000$, the calculation cost of the explicit implicit alternating scheme is reduced by almost 60% compared with the implicit scheme. It shows that when the calculation accuracy is almost similar, the explicit implicit alternating scheme (2.10) is more efficient for solving the time fractional Fisher equation.

To verify the Remark 3.1 and show the explicit implicit alternating scheme (2.10) is effective for solving the time fractional Fisher equation with weak singularity at the initial time, we discuss the following equation.

Example 2. The time fractional Fisher equation with initial weak singularity [36]:

$$\begin{cases} \dfrac{\partial^\alpha u\,(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u}{\partial x^2} + u(1-u) + g_2(x,t), \\ u(x,0) = 0, 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, 0 < t \leq 1, \end{cases} \tag{4.2}$$

where $g_2(x,t) = \Gamma(1+\alpha)sin(x) + t^\alpha sin(x) - t^\alpha sin(x)(1 - t^\alpha sin(x))$.
The exact solution of Eq (4.2) is $u(x,t) = t^\alpha sin(x)$.

Compare two numerical scheme solutions with the exact solution at $t = 0.6$. We can see from Table 4 that the explicit implicit alternating scheme approximates the exact solution better than the implicit scheme. The absolute error graph is shown in Figure 4 for different values of fractional order derivative.

Table 4. Exact solution and numerical solutions for different α with $M = N = 100$ at $t = 0.6$.

α		x			
		0.2	0.4	0.6	0.8
0.4	Exact solution	4.7916E-01	7.7529E-01	7.7529E-01	4.7916E-01
	Scheme (2.10) solution	4.7881E-01	7.7462E-01	7.7462E-01	4.7881E-01
	Implicit scheme solution	4.7864E-01	7.7452E-01	7.7452E-01	4.7864E-01
0.6	Exact solution	4.3262E-01	7.0000E-01	7.0000E-01	4.3262E-01
	Scheme (2.10) solution	4.3207E-01	6.9898E-01	6.9898E-01	4.3207E-01
	Implicit scheme solution	4.3168E-01	6.9856E-01	6.9856E-01	4.3168E-01
0.95	Exact solution	3.6179E-01	5.8540E-01	5.8540E-01	3.6179E-01
	Scheme (2.10) solution	3.6205E-01	5.8560E-01	5.8560E-01	3.6205E-01
	Implicit scheme solution	3.6127E-01	5.8459E-01	5.8459E-01	3.6127E-01

Take $\alpha = 0.4$, we compare the explicit implicit alternating scheme solution with the exact solution. It can be seen from Figure 5 that the numerical solution is in a good agreement with the exact solution at different moments, indicating that the explicit implicit alternating scheme (2.10) is also suitable for solving the time fractional Fisher equation with initial weak singularity.

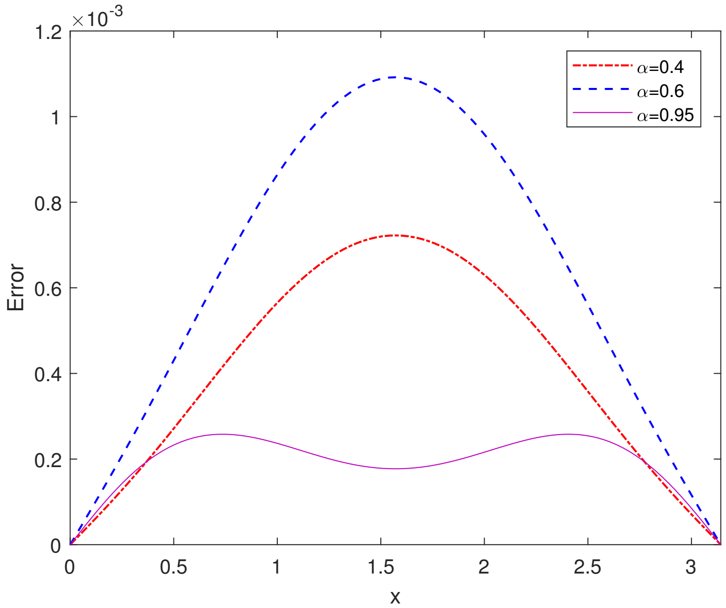

Figure 4. Absolute error of scheme (2.10) with $M = N = 100$ at $t = 0.6$.

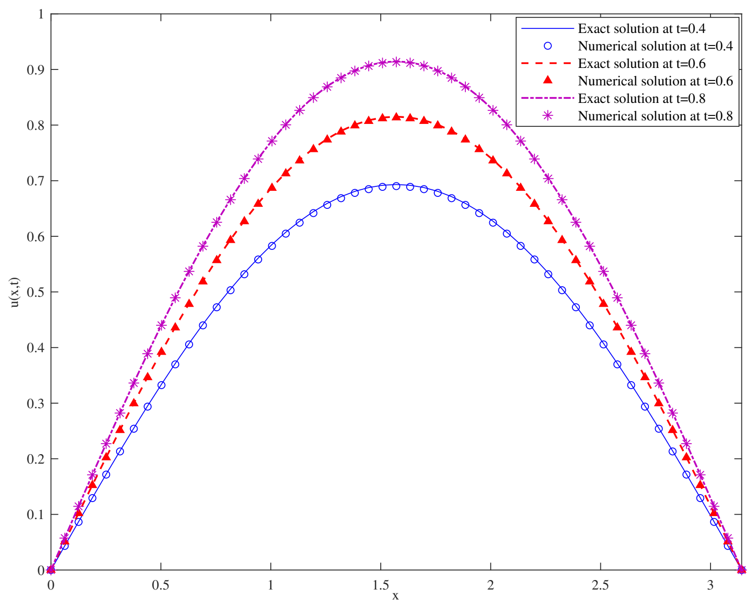

Figure 5. Exact solution and numerical solution of scheme (2.10) at different time with $\alpha = 0.4$.

We will discuss the temporal convergence order and the spatial convergence order of the explicit implicit alternating scheme (2.10).

Table 5 exhibits that the temporal convergence order of the explicit implicit alternating scheme (2.10) reaches α order, which matches Remark 3.1. And it is more accurate than the implicit scheme. Under the assumption of a nonsmooth solution, the time convergence order is lower than $2 - \alpha$ order.

We also can see that compared with the implicit scheme, the explicit implicit alternating scheme can reduce the computational costs. The discussion of convergence order in spatial direction and CPU time of two numerical schemes are given in Table 6. The calculation results show the spatial accuracy and the CPU time of two numerical schemes are almost the same and spatial convergence orders almost reach second order.

Table 5. Error, temporal convergence orders and CPU time of two numerical schemes ($h = \frac{1}{200}$).

α	N	Scheme (2.10)			Implicit scheme		
		$E_\infty(h,\tau)$	$Order_1$	CPU time(s)	$E_\infty(h,\tau)$	$Order_1$	CPU time(s)
0.4	48	4.4048E-02	——	0.0312	3.6214E-02	——	0.0468
	96	3.3382E-02	0.40	0.0748	2.9932E-02	0.27	0.1062
	192	2.5299E-02	0.40	0.2152	2.3780E-02	0.33	0.2760
0.5	48	3.0975E-02	——	0.0315	2.8502E-02	——	0.0486
	96	2.1903E-02	0.50	0.0757	2.1016E-02	0.44	0.1188
	192	1.5488E-02	0.50	0.2158	1.5170E-02	0.47	0.2817
0.6	48	2.0309E-02	——	0.0317	1.9521E-02	——	0.0508
	96	1.3399E-02	0.60	0.0792	1.3169E-02	0.57	0.1198
	192	8.8398E-03	0.60	0.1973	8.7730E-03	0.59	0.3031

Table 6. Error, spatial convergence orders and CPU time of two numerical schemes ($\tau = h^4$).

α	M	Scheme (2.10)			Implicit scheme		
		$E_\infty(h,\tau)$	$Order_2$	CPU time(s)	$E_\infty(h,\tau)$	$Order_2$	CPU time(s)
0.4	4	1.8920E-02	——	0.0459	1.8926E-02	——	0.0452
	8	5.1389E-03	1.88	7.2350	5.0223E-03	1.91	7.2377
	16	1.3470E-03	1.93	1857.23	1.3267E-03	1.92	1857.85
0.5	4	1.8877E-02	——	0.0468	1.8882E-02	——	0.0454
	8	4.8374E-03	1.96	7.2325	4.8374E-03	1.96	7.2439
	16	1.2160E-03	1.99	1856.02	1.2160E-03	1.99	1859.95
0.6	4	1.8684E-02	——	0.0471	1.8689E-02	——	0.0457
	8	4.7993E-03	1.96	7.2382	4.7993E-03	1.96	7.2450
	16	1.2072E-03	1.99	1848.37	1.2072E-03	1.99	1856.72

Based on the analysis of the data in Tables 1–6 and the description of Figures 1–5, we can see the correctness of the theories and the effectiveness of the numerical algorithms.

5. Conclusions

In order for the difference scheme to be widely used, its computational efficiency is very important. Adjusting the implementation details and reorganizing the operation process are typical strategies. We construct a class of explicit implicit alternating schemes for solving the generalized time fractional Fisher equation. It is proved theoretically and numerically that proposed schemes are unconditionally stable having an accuracy of order $O\left(\tau^{2-\alpha}+h^2\right)$. Numerical experiments show that explicit implicit alternating schemes are also suitable for solving the time fractional Fisher equation with initial weak singularity and have an accuracy of order $O\left(\tau^{\alpha}+h^2\right)$.

The explicit implicit alternating schemes show the numerical advantages of symmetrical discretization: the methods have good accuracy and can decrease the computation costs in solving the generalized time fractional Fisher equation. Based on the above reasons, explicit implicit alternating schemes have broad application prospects. In the future, we would like to extend the methods to solve other nonlinear time fractional partial differential equations.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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