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Research article

Two fractional regularization methods for identifying the radiogenic source of the Helium production-diffusion equation

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Abstract: The predication of the helium diffusion concentration as a function of a source term in diffusion equation is an ill-posed problem. This is called inverse radiogenic source problem. Although some classical regularization methods have been considered for this problem, we propose two new fractional regularization methods for the purpose of reducing the over-smoothing of the classical regularized solution. The corresponding error estimates are proved under the a-priori and the a-posteriori regularization parameter choice rules. Some numerical examples are shown to display the necessarity of the methods.

Keywords: inverse radiogenic source problem; fractional regularization methods; regularized solution; regularization parameter choice rules; error estimates
Mathematics Subject Classification: 65M32, 35Q80

1. Introduction

The study of helium diffusion dynamics has attracted the attention of many scholars in recent years. Compared with other isotope systems, helium has a high yield. In addition, high-precision and high-sensitivity helium analysis is relatively easy, and the (U-Th)/He isotopes dating has reached a relatively low-temperature condition, which is very important for the study of thermochronology. In [1], the author gives the helium diffusion and low-temperature thermochronometry of apatite. The effects of long alpha-stopping distance on (U-Th)/He ages are studied in literature [2]. For the application of helium isotope as thermochronometer in terrestrial and extrater restrial materials, refer to the paper [3]. For more important applications of helium isotopes in physics, please refer to the literature [4–6].

This article will consider the prediction of helium concentration as a function of the spatial variable source term, which is closely related to helium isotope dating and is a low-temperature thermochronometry method. The helium production-diffusion model is as follows:

$$\begin{cases} \frac{\partial u(r,t)}{\partial t} = a(t)[\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial u(r,t)}{\partial r}] + f(r), & t \in (0,T), 0 < r < R, \\ u(r,0) = 0, & 0 < r < R, \\ u(R,t) = 0, & t \in (0,T), \\ \lim\limits_{r\to 0} u(r,t) \text{ bounded}, & t \in (0,T), \end{cases} \tag{1.1}$$

where $0 < a_0 \le a(t) \le a_1 (a_0$ and a_1 are constants), r is the dimensional radial variable, R is the radius of the spherical diffusion domain, given the final observation of the helium concentration as follows

$$u(r,T) = g(r), 0 < r < R. \tag{1.2}$$

The object is to reconstruct the source term $f(r)$.

Inverse source problem is a typical ill-posed problem, which has been studied by many scholars. For the discussion of the existence, uniqueness and stability of solutions, please refer to reference [7–11]. In reference [12, 13], Isakov has given some theoretical studies on the inverse source problem, and in [14], Isakov explains the general inverse problem of partial differential equations. In [15], Bao et al. has used the Tikhonov method and the truncation method with a-priori parameter choice rule to study the problem of the inverse radiogenic source identification problem. In [16], Zhang and Yan applied a-posteriori truncation method to the radiogenic source identification for the helium production-diffusion equation, which is closely connected to the helium isotopes dating as one method of the low-temperature thermochronometry. For more literature on numerical methods, see [17–25].

Regarding the research on the regularization theory of inverse radiogenic source problem, most of the results are discussed in the context of a-priori parameter selection, the numerical results of posterior parameters are more less. The a-priori method is based on the smoothness conditions of the solution, which is convenient for theoretical analysis, but it is difficult to verify. Therefore, in actual calculations, the regularization method of the posterior parameter selection rule is more widely used. In order to better identify the radiation source problem of helium production-diffusion Eq (1.1), we will give two fractional regularization methods, namely weighted fractional Tikhonov regularization method (WTRM) and fractional Landweber regularization method (FLRM). For related research on fractional regularization methods, please refer to the literatures [26–32]. Also, for the application of more fractional regularization methods please refer to [33–36].

The outline of the paper is as follows. In Section 2, we give some preliminary results; In order to overcome the difficulty, a novel a-priori bound is introduced. In Section 3, the ill-posedness of inverse radiogenic source promblem (1.1) is also given. In Section 4, we constructed the regularization solution by a WTRM. Moreover, we have performed a detailed convergence analysis and given the a-priori and a-posterior regularization parameters choice rule. In Section 5, we use the FLRM to give the regularization solution of the inverse source problem, and give the regularization parameter choice rule of the provided method and the corresponding error estimate. In Section 6, Numerical examples are given, and the numerical results show that the proposed method is accurate and effective. This article summarizes some general conclusions in Section 7.

2. Preliminaries

The following lemmas will be used.

Lemma 2.1. *Given* $0 < \alpha \le 1$, *for constants* $x > 0$, $\underline{c} = \frac{R^2}{2a_1\pi^2}$ *and* $\overline{c} = \frac{R^2}{a_0\pi^2}$, a_0 *and* a_1 *are the* *diffusivity [15], we hold the following inequality*

$$\frac{\overline{c}^{\alpha}x^2}{\underline{c}^{\alpha+1}+\mu x^{2\alpha+2}} \le c_1\mu^{-\frac{1}{\alpha+1}}. \tag{2.1}$$

where $c_1 = \frac{\underline{c}^{\alpha}\alpha^{\frac{\alpha}{\alpha+1}}}{\overline{c}^{\alpha}(\alpha+1)} > 0$ *are independent of* $\alpha, \overline{c}, \underline{c}$.

Proof. For $0 < \alpha \le 1$, we define the following function:

$$h_1(x) = \frac{\overline{c}^{\alpha}x^2}{\underline{c}^{\alpha+1}+\mu x^{2\alpha+2}},$$

where $x > 0$, $h_1(x)$ has a unique point $x_0 = \left(\frac{\overline{c}^{\alpha+1}}{\alpha\mu}\right)^{\frac{1}{\alpha+1}} \ge 0$ such that $h_1'(x_0) = 0$.
Clearly we have

$$h_1(x) \le h_1(x_0) = \frac{\underline{c}^{\alpha}\alpha^{\frac{\alpha}{\alpha+1}}}{\overline{c}^{\alpha}(\alpha+1)}\mu^{-\frac{1}{\alpha+1}} := c_1\mu^{-\frac{1}{\alpha+1}}.$$

Lemma 2.2. *For constants* $x > 0$, $0 < \alpha \le 1$ *and* $p > 0$, *we have*

$$\frac{\mu x^{2\alpha+2-p}}{\underline{c}^{\alpha+1}+\mu x^{2\alpha+2}} \le \begin{cases} c_2\mu^{\frac{p}{2\alpha+2}}, & 0 < p < 2\alpha+2, \\ c_3\mu, & p \ge 2\alpha+2. \end{cases} \tag{2.2}$$

where $c_2 = \overline{c}^{-\frac{p}{2}}\left(1-\frac{p}{2\alpha+2}\right)\left(\frac{p}{2\alpha+2-p}\right)^{\frac{p}{2\alpha+2}}$, $c_3 = \underline{c}^{-(\alpha+1)}$.

Proof. Given the following function:

$$h_2(x) = \frac{\mu x^{2\alpha+2-p}}{\underline{c}^{\alpha+1}+\mu x^{2\alpha+2}},$$

where $x > 0$.
If $0 < p < 2\alpha+2$, then $\lim\limits_{x\to 0} h_2(x) = \lim\limits_{x\to\infty} h_2(x) = 0$. Thus there exists a $x_0 = \overline{c}^{\frac{1}{2}}\left(\frac{2\alpha+2-p}{\mu p}\right)^{\frac{1}{2\alpha+2}} \ge 0$ which is a global maximizer such that $h_2'(x_0) = 0$, we have

$$h_2(x) \le \sup_{x\in(0,\infty)} h_2(x) \le h_2(x_0),$$

Thus, we have

$$h_2(x) \le h_2(x_0) = \overline{c}^{-\frac{p}{2}}\left(1-\frac{p}{2\alpha+2}\right)\left(\frac{p}{2\alpha+2-p}\right)^{\frac{p}{2\alpha+2}}\mu^{\frac{p}{2\alpha+2}} := c_2\mu^{\frac{p}{2\alpha+2}}.$$

If $p \ge 2\alpha+2$, for $x \ge 1$ then we have

$$h_2(x) = \frac{1}{(\overline{c}^{\alpha+1}+\mu x^{2\alpha+2})x^{p-(2\alpha+2)}}\mu \le \frac{1}{\overline{c}^{\alpha+1}+\mu x^{2\alpha+2}}\mu \le \underline{c}^{-(\alpha+1)}\mu := c_3\mu.$$

Lemma 2.3. *For constants* $x > 0$ *and* $0 < \alpha \le 1$, *we have*

$$\frac{\overline{c}\mu x^{2\alpha-p}}{\underline{c}^{\alpha+1}+\mu x^{2\alpha+2}} \le \begin{cases} c_4\mu^{\frac{p+2}{2\alpha+2}}, & 0 < p < 2\alpha, \\ c_5\mu, & p \ge 2\alpha. \end{cases} \tag{2.3}$$

where $c_4 = \frac{\overline{c}(p+2)^{2+p}(2\alpha-p)^{\frac{2\alpha-p}{2\alpha+2}}}{\underline{c}^{1+\frac{p}{2}}(2\alpha+2)}, c_5 = \frac{\overline{c}}{\underline{c}^{\alpha+1}} > 0.$

Proof. Define the following function:

$$h_3(x) = \frac{\overline{c}\mu x^{2\alpha-p}}{\underline{c}^{\alpha+1} + \mu x^{2\alpha+2}}.$$

If $0 < p < 2\alpha$, then $\lim_{x\to 0} h_3(x) = \lim_{x\to\infty} h_3(x) = 0$. Thus there exists a unique $x_0 = \underline{c}^{\frac{1}{2}}\left(\frac{2\alpha-p}{\mu(p+2)}\right)^{\frac{1}{2\alpha+2}} \geq 0$ which is a global maximizer such that $h_3'(x_0) = 0$, we have

$$h_3(x) \leq \sup_{x\in(0,\infty)} h_3(x) \leq h_3(x_0),$$

Therefore, we have

$$h_3(x) \leq h_3(x_0) = \frac{\overline{c}(p+2)^{2+p}(2\alpha-p)^{\frac{2\alpha-p}{2\alpha+2}}}{\underline{c}^{1+\frac{p}{2}}(2\alpha+2)}\mu^{\frac{p+2}{2\alpha+2}} := c_4\mu^{\frac{p+2}{2\alpha+2}}.$$

If $p \geq 2\alpha$, $x \geq 1$, we have

$$h_3(x) = \frac{\overline{c}\mu}{(\underline{c}^{\alpha+1} + \mu x^{2\alpha+2})x^{p-2\alpha}} \leq \frac{\overline{c}}{\underline{c}^{\alpha+1} + \mu x^{2\alpha+2}}\mu \leq \frac{\overline{c}}{\underline{c}^{\alpha+1}}\mu := c_5\mu.$$

Lemma 2.4. [37, 38] *For* $0 < \lambda < 1$, $\upsilon > 0$, $m \in \mathbb{N}$, *let* $r_m(\lambda) := (1-\lambda)^m$, *there holds:*

$$r_m(\lambda)\lambda^\upsilon \leq \theta_\upsilon(m+1)^{-\upsilon},$$

where

$$\theta_\upsilon = \begin{cases} 1, & 0 \leq \upsilon \leq 1, \\ \upsilon^\upsilon, & \upsilon > 1. \end{cases}$$

Lemma 2.5. *For* $k_n > 0$ *and* $\frac{1}{2} < \alpha < 1$, $0 < \beta k_n^2 < 1, m \geq 1$, *we have*

$$k_n^{-1}[1-(1-\beta k_n^2)^m]^\alpha \leq \beta^{\frac{1}{2}} m^{\frac{1}{2}}. \tag{2.4}$$

Proof. We define two functions with $\tau^2 := \beta k_n^2$:

$$\psi(\tau) = \beta\tau^{-2}[1-(1-\tau^2)^m]^{2\alpha}, \tag{2.5}$$

and

$$\phi(\tau) = \tau^{-2}[1-(1-\tau^2)^m]^{2\alpha}. \tag{2.6}$$

Obviously $\psi(\tau) = \beta\phi(\tau)$. These two functions are continuous in $\tau \in (0, 1)$.
For $\frac{1}{2} < \alpha < 1$ and $\tau \in (0, 1)$, using the Lemma 3.3 in [28], we have

$$\phi(\tau) \leq m, \quad \psi(\tau) \leq \beta m.$$

Therefore,

$$k_n^{-1}[1-(1-\beta k_n^2)^m]^\alpha \le \beta^{\frac{1}{2}} m^{\frac{1}{2}}.$$

Lemma 2.6. *For $m \ge 1, k_n > 0, 0 < \beta k_n^2 < 1$, we have*

$$k_n^{\frac{p}{2}}(1-\beta k_n^2)^m \le c(\beta,p) m^{-\frac{p}{4}} := c_6 m^{-\frac{p}{4}}, \tag{2.7}$$

where the constant $c_6 = \left(\frac{p}{4\beta}\right)^{\frac{p}{4}}$.

Proof. We introduce a new variable $x := k_n^2, x < 1/\beta$, and let

$$h_4(x) = (1-\beta x)^m x^{\frac{p}{4}}.$$

It is easy to see that there exists a unique $x_0 = \frac{z}{\beta(z+m)}$ with $z = \frac{p}{4}$ such that $h_4'(x_0) = 0$. We find that

$$\begin{aligned} h_4(x) \le h_4(x_0) \le \left(1-\frac{z}{z+m}\right)^m \left(\frac{z}{\beta(z+m)}\right)^z < \left(\frac{z}{\beta}\right)^z \left(\frac{1}{z+m}\right)^z \\ < \left(\frac{z}{\beta}\right)^z \left(\frac{1}{m}\right)^z := c_6 m^{-\frac{p}{4}}. \end{aligned}$$

3. Problem formulation and Ill-posedness

In this section, we derive an analytical solution for the inverse radiogenic source problem based on the eigenfunction expansion, and give an analysis on the ill-posedness of the inverse source problem (1.1).

Throughout this paper, the Hilbert space of square integrable functions on $[0,R]$ is denoted as $L^2([0,R])$. $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the inner product and norm on $L^2([0,R])$ respectively, introduced as follows

$$\langle f,g\rangle = \int_0^R f(r)g(r)dr, \text{ and } \|u\| = (\int_0^R |f(r)|^2 dr)^{\frac{1}{2}}.$$

In order to solve the problem (1.1), we introduce a new function $\omega(r,t)$ under the substitution

$$\omega(r,t) = ru(r,t).$$

It follows from the inverse radiogenic source problem (1.1), that ω satisfies:

$$\begin{cases} \frac{\partial\omega(r,t)}{\partial t} = a(t)\frac{\partial^2\omega(r,t)}{\partial r^2} + rf(r), & t\in(0,T), 0<r<R, \\ \omega(r,0) = 0, & 0<r<R, \\ \omega(R,t) = 0 & t\in(0,T), \\ \lim\limits_{r\to 0}\omega(r,t) \text{ bounded}, & t\in(0,T). \end{cases} \tag{3.1}$$

The corresponding final observation of the helium concentration becomes

$$\omega(r,T) = rg(r), \quad 0<r<R. \tag{3.2}$$

Applying the method of separation of variables, consider the solution of problem (3.1) of the form

$$\omega(r,t) = X(r)Y(t), \tag{3.3}$$

substitute it into the (3.1), we obtain the following Sturm-Liouville problem

$$\begin{aligned} & X''(r) + \lambda X(r) = 0. \qquad 0 < r < R, \\ & X(R) = 0, \\ & \lim_{r \to 0} r^{-1} X(r) \text{ bounded}, \end{aligned} \tag{3.4}$$

where λ is an unknown constant.

Through calculation, we can get the eigenvalues of (3.4) as follows

$$\lambda_n = \left(\frac{n\pi}{R}\right)^2, \ \ n = 1, 2, \cdots, \tag{3.5}$$

and the corresponding eigenfunctions are

$$X_n(r) = \sin\left(\frac{n\pi r}{R}\right). \tag{3.6}$$

From the orthogonality and completeness of the eigenfunction system $\{sin(\frac{n\pi r}{R})\}_{n=1}^{\infty}$ in $L^2([0, R])$, we get the solution $\omega(r, t)$ and the source term $r^1 f(r)$ can be represented as

$$\omega(r, t) = \sum_{n=1}^{\infty} X_n(r) Y_n(t), \tag{3.7}$$

$$rf(r) = \sum_{n=1}^{\infty} f_n X_n(r), \tag{3.8}$$

where

$$f_n = \frac{\int_0^R rf(r)\sin(\frac{n\pi r}{R})dr}{\int_0^R \sin^2(\frac{n\pi r}{R})dr} = \frac{2}{R}\int_0^R rf(r)\sin\left(\frac{n\pi r}{R}\right)dr, \ \ n = 1, 2, \ldots. \tag{3.9}$$

Substituting (3.7) and (3.8) into (3.1), we have

$$\begin{aligned} & Y_n'(t) + \lambda a(t) Y_n(t) = f_n, \\ & Y_n(0) = 0. \end{aligned}$$

Solving the above initial-value problem yields the solution

$$Y_n(t) = f_n \int_0^t e^{-\lambda_n \int_\tau^t a(s)ds} d\tau.$$

Therefore, the solution of (3.1) can be written as the infinite series

$$\omega(r, t) = \sum_{n=1}^{\infty} f_n X_n(r) \int_0^t e^{-\lambda_n \int_\tau^t a(s)ds} d\tau. \tag{3.10}$$

Evaluating (3.10) at t = T on both sides and using the final helium concentration (3.2) give

$$\omega(r, T) = ru(r, T) = rg(r) = \sum_{n=1}^{\infty} f_n X_n(r) \int_0^T e^{-\lambda_n \int_\tau^T a(s)ds} d\tau. \tag{3.11}$$

Define

$$\varphi_n(r) = \sqrt{\frac{2}{R}}X_n(r) = \sqrt{\frac{2}{R}}\sin\Big(\frac{n\pi r}{R}\Big). \tag{3.12}$$

It is easy to verify that the eigenfunctions $\{\varphi_n(r)\}_{n=1}^{\infty}$ form an orthonormal basis in $L^2([0, R])$. Using the eigenfunctions as a basis, formula (3.8) can be rewritten as:

$$rg(r) = \sum_{n=1}^{\infty} \int_0^T e^{-\lambda_n \int_\tau^T a(s)ds} d\tau \langle f_r(r), \varphi_n(r)\rangle \varphi_n(r). \tag{3.13}$$

For convenience, we denote

$$f_r(r) = rf(r) \text{ and } g_r(r) = rg(r). \tag{3.14}$$

To get f_r, define an operator $K : f_r \to g_r$, then the inverse source problem can be represented by the following linear operator equation:

$$Kf_r(r) = g_r(r). \tag{3.15}$$

Using (3.13), it holds

$$Kf_r(r) = \sum_{n=1}^{\infty} k_n \langle f_r(r), \varphi_n(r)\rangle \varphi_n(r) = g_r(r), \tag{3.16}$$

with

$$k_n = \int_0^T e^{-\lambda_n \int_\tau^T a(s)ds} d\tau, n = 1, 2, \ldots. \tag{3.17}$$

Therefore, the analytical solution of the inverse source problem is:

$$f_r(r) = \sum_{n=1}^{\infty} k_n^{-1} \langle g_r(r), \varphi_n(r)\rangle \varphi_n(r). \tag{3.18}$$

Because measurement errors exist in the data function g_r, the solution has to be reconstructed from noisy data g_r^δ which is assumed to satisfy

$$\|g_r^\delta - g_r\| \le \delta. \tag{3.19}$$

Here $\delta > 0$ represents the noise level, and both $g_r(r)$ and $g_r^\delta(r)$ are assumed to be functions in $L^2([0, R])$.

To study the ill-posed nature of the inverse problem, it is sufficient to investigate the decay property of the eigenvalues. From [15], we can see that the upper and lower bounds of k_n are as follows

$$\frac{\underline{c}}{n^2} \le k_n \le \frac{\overline{c}}{n^2}, as\ n \to \infty, \tag{3.20}$$

where the constant $\underline{c} = \frac{R^2}{2a_1\pi^2}$ and $\overline{c} = \frac{R^2}{a_0\pi^2}$.

From (3.20), we note that when $n \to \infty$, the eigenvalue $k_n \to 0$ [15, p7]. Fixed size data error can be amplified arbitrarily much by the factors k_n^{-1}. Therefore, the problem of identifying $f(r)$ is ill-posed. In the following, we will use two fractional regularization methods to solve the inverse radiogenic source promblem.

To obtain the error estimates, it is necessary to assume certain regularity of the exact source function. Here we assume that there exists an a priori estimate for the source function $f_r(r)$, i.e.,

$$\|f_r\|_p \leq E, \text{ for } p > 0, \tag{3.21}$$

where $E > 0$ is a constant. where the norm is defined in terms of the eigenfunctions

$$\|f_r\|_p = \Big\| \sum_{n=1}^{\infty} n^p \langle f_r, \varphi_n \rangle \varphi_n \Big\|. \tag{3.22}$$

It is easy to check $\|f_r\|_0 = \|f_r\|$.

4. Regularization solution of WTRM and convergence analysis

In this section, we propose a WTRM to solve the ill-posed problem (1.1) and give convergence estimate under the a-priori regularization parameter choice rule.

Then we consider WTRM to solve the ill-posed problem, the regularization solution is

$$f_r^{\delta,\mu}(r) = \sum_{n=1}^{\infty} \frac{k_n^{\alpha}}{k_n^{\alpha+1} + \mu} \langle g_r^{\delta}(r), \varphi_n(r) \rangle \varphi_n(r), \tag{4.1}$$

where $\mu > 0$ plays the role of regularization parameter, we define α as the fractional parameter. When $\alpha = 1$, it expresses the classic Tikhonov method. However, for $0 < \alpha < 1$ we can see it prevent the effect of oversmoothing and obtain a more accurate numerical results for the discontinuity of solution.

4.1. Convergence analysis of WTRM under an a-priori parameter choice rule

Theorem 4.1. *Suppose the a priori condition (3.21) and the noise assumption (3.19) hold,*
(1) If $0 < p < 2\alpha + 2$ *and choice* $\mu = \left(\frac{\delta}{E}\right)^{\frac{2\alpha+2}{p+2}}$, *we have a convergence estimate*

$$\|f_r^{\delta,\mu}(r) - f_r(r)\| \leq (c_1 + c_2) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{4.2}$$

(2) If $p \geq 2\alpha + 2$ *and choice* $\mu = \left(\frac{\delta}{E}\right)^{\frac{\alpha+1}{\alpha+2}}$, *we have a convergence estimate*

$$\|f_r^{\delta,\mu}(r) - f_r(r)\| \leq (c_1 + c_3) E^{\frac{1}{\alpha+2}} \delta^{\frac{\alpha+1}{\alpha+2}}. \tag{4.3}$$

Proof. By the triangle inequality, we know

$$\|f_r^{\delta,\mu}(r) - f_r(r)\| \leq \|f_r^{\delta,\mu}(r) - f_r^{\mu}(r)\| + \|f_r^{\mu}(r) - f_r(r)\| = I_1 + I_2. \tag{4.4}$$

We first give the estimate of I_1, with Lemma 2.1 and (3.19), we have

$$I_1 = \|f_r^{\delta,\mu}(r) - f_r^{\mu}(r)\| = \Big\| \sum_{n=1}^{\infty} \frac{k_n^{\alpha}}{k_n^{\alpha+1} + \mu} \langle g_r^{\delta} - g_r, \varphi_n \rangle_r \varphi_n \Big\|$$

$$
\begin{aligned}
&\leq \delta \sup_{n>0} \frac{(\frac{\bar{c}}{n^2})^{\alpha}}{(\frac{c}{n^2})^{\alpha+1}+\mu} \\
&\leq c_1 \mu^{-\frac{1}{\alpha+1}} \delta .
\end{aligned}
$$

Now we estimate I_2, by Lemma 2.2 and a priori bound condition (3.21), we can deduce that

$$
\begin{aligned}
I_2 = \|f_r^{\mu}(r) - f_r(r)\| &= \Big\| \sum_{n=1}^{\infty} \Big(1 - \frac{k_n^{\alpha+1}}{k_n^{\alpha+1}+\mu}\Big) k_n^{-1} \langle g_r, \varphi_n \rangle_r \varphi_n \Big\| \\
&\leq \Big\| \sum_{n=1}^{\infty} \frac{\mu n^{-p}}{k_n^{\alpha+1}+\mu} n^p \langle f_r, \varphi_n \rangle_r \varphi_n \Big\| \\
&\leq E \sup_{n>0} \frac{\mu n^{2\alpha+2-p}}{\underline{c}^{\alpha+1} + \mu n^{2\alpha+2}} \\
&\leq \begin{cases} c_2 \mu^{\frac{p}{2\alpha+2}} E, & 0 < p < 2\alpha+2, \\ c_3 \mu E, & p \geq 2\alpha+2. \end{cases}
\end{aligned}
$$

Combining the above two inequality and choose the regularization parameter μ, we obtain

$$
\|f_r^{\delta,\mu}(r) - f_r(r)\| \leq \begin{cases} (c_1 + c_2) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2\alpha+2, \\ (c_1 + c_3) E^{\frac{1}{\alpha+2}} \delta^{\frac{\alpha+1}{\alpha+2}}, & p \geq 2\alpha+2. \end{cases}
$$

4.2. Convergence analysis of WTRM under an a-posteriori parameter selection rule

In this section, we give the regularization parameter choice rule of the posterior fractional regularization method. We can also obtain a convergence rate for the regularized solution (4.1) under this parameter choice rule. The most general a posteriori rule is the Morozovs discrepancy principle [39].

We use the discrepancy principle in the following form:

$$
\left\| K f_r^{\delta,\mu}(r) - g_r^{\delta}(r) \right\| = \tau \delta, \tag{4.5}
$$

where $0 < \alpha \leq 1, \tau > 1$ is a constant, $\mu > 0$ is regularization parameter, K is defined by the operator Eq (3.15). According to the following lemma, we know there exists a unique solution for Eq (4.5) if $0 < \tau\delta < \|g_r^{\delta}\|$.

Lemma 4.2. *Let $d(\mu) = \|K f_r^{\delta,\mu}(r) - g_r^{\delta}(r)\|$, then we have the following conclusions: (1) $d(\mu)$ is a continuous function; (2) $\lim_{\mu\to 0} d(\mu) = \|g_r^{\delta}(r)\|$; (3) $\lim_{\mu\to\infty} d(\mu) = 0$; (4) $d(\mu)$ is a strictly decreasing function over* $(0, \infty)$.

Proof. The proofs are straightforward results by virtue of

$$
d(\mu) = \Big(\sum_{n=1}^{\infty} \Big(\frac{\mu}{k_n^{\alpha+1}+\mu} \Big)^2 (g_r^{\delta})^2 \Big)^{\frac{1}{2}}.
$$

Lemma 4.3. *If μ is the solution of Eq (4.5), we also obtain the following inequality:*

$$\mu^{-\frac{1}{\alpha+1}} \leq \begin{cases} \left(\frac{c_4}{\tau-1}\right)^{\frac{2}{p+2}} \left(\frac{E}{\delta}\right)^{\frac{2}{p+2}}, & 0 < p < 2\alpha, \\ \left(\frac{c_5}{\tau-1}\right)^{\frac{1}{\alpha+1}} \left(\frac{E}{\delta}\right)^{\frac{1}{\alpha+1}}, & p \geq 2\alpha. \end{cases} \tag{4.6}$$

Proof. From (4.5), and according to Lemma 4.2, we obtain

$$\begin{aligned} \tau\delta &= \left\| K f_r^{\delta,\mu}(r) - g_r^{\delta}(r) \right\| \\ &\leq \left\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1} + \mu} \langle g_r^{\delta} - g_r, \varphi_n \rangle_r \varphi_n \right\| + \left\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1} + \mu} \langle g_r^{\delta}, \varphi_n \rangle_r \varphi_n \right\| \\ &\leq \delta + \left\| \sum_{n=1}^{\infty} \frac{\mu k_n n^{-p}}{k_n^{\alpha+1} + \mu} n^p k_n^{-1} \langle g_r^{\delta}, \varphi_n \rangle_r \varphi_n \right\| \\ &\leq \delta + E \sup_{n>0} \frac{\overline{c} \mu n^{2\alpha-p}}{\underline{c}^{\alpha+1} + \mu n^{2\alpha+2}}. \end{aligned}$$

According to Lemma 2.3, we have

$$\tau\delta \leq \delta + E \begin{cases} c_4 \, \mu^{\frac{p+2}{2\alpha+2}}, & 0 < p < 2\alpha, \\ c_5 \, \mu, & p \geq 2\alpha. \end{cases}$$

This yields

$$\mu^{-\frac{1}{\alpha+1}} \leq \begin{cases} \left(\frac{c_4}{\tau-1}\right)^{\frac{2}{p+2}} \left(\frac{E}{\delta}\right)^{\frac{2}{p+2}}, & 0 < p < 2\alpha, \\ \left(\frac{c_5}{\tau-1}\right)^{\frac{1}{\alpha+1}} \left(\frac{E}{\delta}\right)^{\frac{1}{\alpha+1}}, & p \geq 2\alpha. \end{cases}$$

□

Theorem 4.4. *Suppose the a priori condition (3.21) and the noise assumption (3.19) hold, and the regularization parameter μ is chosen by discrepancy principle (4.5), then,*
(1) If $0 < p < 2\alpha$, we have a convergence estimate

$$\|f_r^{\delta,\mu}(r) - f_r(r)\| \leq \left(c_1 \left(\frac{c_4}{\tau - 1} \right)^{\frac{2}{p+2}} + \left(\frac{\tau + 1}{\underline{C}} \right)^{\frac{p}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{4.7}$$

(2) If $p \geq 2\alpha$, we have a convergence estimate

$$\|f_r^{\delta,\mu}(r) - f_r(r)\| \leq \left(\left(\frac{c_6}{\tau - 1} \right)^{\frac{1}{\alpha+1}} + \left(\frac{\tau + 1}{\underline{C}} \right)^{\frac{\alpha}{\alpha+1}} \lambda_1^{\frac{2\alpha-p}{2(\alpha+1)}} \right) E^{\frac{1}{\alpha+1}} \delta^{\frac{\alpha}{\alpha+1}}. \tag{4.8}$$

Proof. By the triangle inequality, we know

$$\|f_r^{\delta,\mu}(r) - f_r(r)\| \leq \|f_r^{\delta,\mu}(r) - f_r^{\mu}(r)\| + \|f_r^{\mu}(r) - f_r(r)\| = I_1 + I_2. \tag{4.9}$$

(1) For $0 < p < 2\alpha$. We first give the estimate of I_1, with Lemma 4.3 we have

$$I_1 = \|f_r^{\delta,\mu}(r) - f_r^{\mu}(r)\| \leq c_1 \mu^{-\frac{1}{\alpha+1}} \delta \leq c_1 \left(\frac{c_3}{\tau - 1} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{4.10}$$

Now we estimate I_2, by (3.19) and (3.21), we can deduce that

$$
\begin{aligned}
I_2 &= \|f_r^{\mu}(r) - f_r(r)\| = \Big\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1}+\mu} k_n^{-1} \langle g_r, \varphi_n \rangle \varphi_n \Big\| \\
&= \Big\| \sum_{n=1}^{\infty} \frac{\mu k_n}{k_n^{\alpha+1}+\mu} \frac{\langle f_r, \varphi_n \rangle \varphi_n}{k_n} \Big\| \\
&\le \Big\| \sum_{n=1}^{\infty} \frac{\mu k_n}{k_n^{\alpha+1}+\mu} \langle f_r, \varphi_n \rangle \varphi_n \Big\|^{\frac{p}{p+2}} \cdot \Big\| \frac{\mu k_n}{k_n^{\alpha+1}+\mu} \frac{\langle f_r, \varphi_n \rangle \varphi_n}{k_n^{\frac{p+2}{2}}} \Big\|^{\frac{2}{p+2}} \\
&\le \Big\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1}+\mu} \langle g_r, \varphi_n \rangle \varphi_n \Big\|^{\frac{p}{p+2}} \cdot \Big\| \sum_{n=1}^{\infty} \frac{\mu n^p}{k_n^{\frac{p}{2}} (k_n^{\alpha+1}+\mu)} n^{-p} \langle f_r, \varphi_n \rangle \varphi_n \Big\|^{\frac{2}{p+2}} \\
&\le \Big(\Big\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1}+\mu} \langle g_r - g_r^{\delta}, \varphi_n \rangle \varphi_n \Big\| \\
&\quad + \Big\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1}+\mu} \langle g_r^{\delta}, \varphi_n \rangle \varphi_n \Big\| \Big)^{\frac{p}{p+2}} \sup_{n>0} \left(\frac{n^{-p}}{(\frac{c}{n^2})^{\frac{p}{2}}} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \\
&\le \Big(\frac{\tau+1}{\underline{c}} \Big)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.
\end{aligned}
\tag{4.11}
$$

Combining (4.9), (4.10) and (4.11), we obtain

$$
\|f_r^{\delta,\mu}(r) - f_r(r)\| \le \left(c_1 \Big(\frac{c_4}{\tau-1} \Big)^{\frac{2}{p+2}} + \Big(\frac{\tau+1}{\underline{c}} \Big)^{\frac{p}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.
$$

(2) For $p \ge 2\alpha$. From (4.9), we first give the estimate of I_1, with Lemma 4.2 we have

$$
I_1 = \|f_r^{\delta,\mu}(r) - f_r^{\delta}(r)\| \le c_1 \delta \mu^{-\frac{1}{\alpha+1}} \le \Big(\frac{c_5}{\tau-1} \Big)^{\frac{1}{\alpha+1}} E^{\frac{1}{\alpha+1}} \delta^{\frac{\alpha}{\alpha+1}}. \tag{4.12}
$$

Then we estimate I_2, by Lemma 2.3 and (3.21), we known

$$
\begin{aligned}
I_2 &= \|f_r^{\mu}(r) - f_r(r)\| = \Big\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1}+\mu} k_n^{-1} \langle g_r, \varphi_n \rangle \varphi_n \Big\| = \Big\| \sum_{n=1}^{\infty} \frac{\mu k_n}{k_n^{\alpha+1}+\mu} \frac{\langle f_r, \varphi_n \rangle \varphi_n}{k_n} \Big\| \\
&\le \Big\| \sum_{n=1}^{\infty} \frac{\mu k_n}{k_n^{\alpha+1}+\mu} \langle f_r, \varphi_n \rangle \varphi_n \Big\|^{\frac{\alpha}{\alpha+1}} \cdot \Big\| \frac{\mu k_n}{k_n^{\alpha+1}+\mu} \frac{\langle f_r, \varphi_n \rangle \varphi_n}{k_n^{\alpha+1}} \Big\|^{\frac{1}{\alpha+1}} \\
&\le \Big\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1}+\mu} \langle g_r, \varphi_n \rangle \varphi_n \Big\|^{\frac{\alpha}{\alpha+1}} \cdot \Big\| \sum_{n=1}^{\infty} \frac{\mu n^{-p}}{k_n^{\alpha} (k_n^{\alpha+1}+\mu)} n^{p} \langle f_r, \varphi_n \rangle \varphi_n \Big\|^{\frac{1}{\alpha+1}} \\
&\le \Big(\Big\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1}+\mu} \langle g_r - g_r^{\delta}, \varphi_n \rangle \varphi_n \Big\| \\
&\quad + \Big\| \sum_{n=1}^{\infty} \frac{\mu}{k_n^{\alpha+1}+\mu} \langle g_r^{\delta}, \varphi_n \rangle \varphi_n \Big\| \Big)^{\frac{\alpha}{\alpha+1}} \Big\| \frac{n^{-p}}{(\frac{c}{n^2})^{\alpha}} n^{p} \langle f_r, \varphi_n \rangle \varphi_n \Big\|^{\frac{1}{\alpha+1}} \\
&\le \Big(\frac{\tau+1}{\underline{c}} \Big)^{\frac{\alpha}{\alpha+1}} E^{\frac{1}{\alpha+1}} \delta^{\frac{\alpha}{\alpha+1}}.
\end{aligned}
\tag{4.13}
$$

Combining (4.9), (4.12) and (4.13), we obtain

$$\|f_r^{\delta,\mu}(r) - f_r(r)\| \le \Big(\Big(\frac{c_5}{\tau - 1}\Big)^{\frac{1}{\alpha+1}} + \Big(\frac{\tau + 1}{\underline{c}}\Big)^{\frac{\alpha}{\alpha+1}}\Big)E^{\frac{1}{\alpha+1}}\delta^{\frac{\alpha}{\alpha+1}}.$$

This completes the proof.

5. Regularization solution of FLRM and convergence analysis

In this section, we propose a FLRM to solve the ill-posed problem (1.1) and give convergence estimate under the a-priori regularization parameter choice rule.

We denote regularization solution of FLRM with the noisy data as follows:

$$f_r^{\delta,m}(r) = \sum_{n=1}^{\infty}[1 - (1 - \beta k_n^2)^m]^{\alpha} k_n^{-1}\langle g_r^{\delta}(r), \varphi_n(r)\rangle\varphi_n(r),\ \frac{1}{2} < \alpha \le 1, \tag{5.1}$$

where $m \ge 1$ plays the role of regularization parameter, $0 < \beta < \frac{2}{k_n^2}$, α is called the fractional parameter. When $\alpha = 1$, it is the classic Landweber iterative method.

5.1. Convergence analysis of FLRM under an a-priori parameter selection rule

Now we give the main result of this section.

Theorem 5.1. *Suppose the a priori condition (3.21) and the noise assumption (3.19) hold, let* $m = \lfloor(\frac{E}{\delta})^{\frac{4}{p+2}}\rfloor$*, we have the convergence estimate*

$$\|f_r^{\delta,m}(r) - f_r(r)\| \le (\beta^{\frac{1}{2}} + c_6\underline{c}^{-\frac{p}{2}})E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, \tag{5.2}$$

where $\lfloor s\rfloor$ *denotes the largest integer smaller than or equal to s,* c_5 *are the positive constants depending on* β, p, α*, and* $\underline{c}$*.*

Proof. By the triangle inequality, we know

$$\|f_r^{\delta,m}(r) - f_r(r)\| \le \|f_r^{\delta,m}(r) - f_r^m(r)\| + \|f_r^m(r) - f_r(r)\| = J_1 + J_2. \tag{5.3}$$

As in the estimate of I_1, by Lemma 2.5 and (3.19), we have

$$\begin{aligned} J_1 &= \|f_r^{\delta,m}(r) - f_r^m(r)\| \\ &= \Big\|\sum_{n=1}^{\infty}[1 - (1 - \beta k_n^2)^m]^{\alpha} k_n^{-1}\langle g_r^{\delta} - g_r, \varphi_n\rangle\varphi_n\Big\| \\ &\le \delta\sup_{n>0} k_n^{-1}[1 - (1 - \beta k_n^2)^m]^{\alpha} \\ &\le \beta^{\frac{1}{2}} m^{\frac{1}{2}}\delta. \end{aligned}$$

Now we estimate J_2, by Lemma 2.6 and the a-priori bound condition (3.21), we can deduce that

$$J_2 = \|f_r^m(r) - f_r(r)\|$$

$$
\begin{aligned}
&= \Big\| \sum_{n=1}^{\infty} \Big[1 - [1-(1-\beta k_n^2)^m]^\alpha\Big] k_n^{-1} \langle g_r, \varphi_n\rangle \varphi_n \Big\| \\
&\le \Big\| \sum_{n=1}^{\infty} (1-\beta k_n^2)^m n^{-p} n^p k_n^{-1} \langle g_r, \varphi_n\rangle \varphi_n \Big\| \\
&\le E \sup_{n>0} (1-\beta k_n^2)^m n^{-p} \\
&\le c_6 \underline{c}^{-\frac{p}{2}} m^{-\frac{p}{4}} E.
\end{aligned}
$$

Combining the above two inequalities, we obtain

$$
\|f_m^\delta(r) - f(r)\| \le \beta^{\frac{1}{2}} m^{\frac{1}{2}} \delta + c_6 \underline{c}^{-\frac{p}{2}} m^{-\frac{p}{4}} E.
$$

Choose the regularization parameter m by

$$
m = \lfloor \Big(\frac{E}{\delta}\Big)^{\frac{4}{p+2}} \rfloor,
$$

then we have the following result

$$
\|f_r^{\delta,m}(r) - f_r(r)\| \le (\beta^{\frac{1}{2}} + c_6 \underline{c}^{-\frac{p}{2}}) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.
$$

5.2. Convergence analysis of FLRM under an a-posteriori parameter selection rule

Due to the semi-convergence property of iterative methods for ill-posed problems, we need a reliable stopping rule for detecting the moment from convergence to divergence. In this section, we give the a-posteriori parameter choice rule for the FLRM. We can obtain a convergence rate for the regularized solution (4.1) under this parameter choice rule. The most general a-posteriori rule is the Morozov's discrepancy principle [39].

We use the discrepancy principle in the following form:

$$
\Big\| K f_r^{\delta,m}(r) - g_r^\delta(r) \Big\| \le \tau\delta, \tag{5.4}
$$

where $\tau > 1$ is a user-supplied constant independent on δ, $m > 0$ is regularization parameter which makes (5.4) hold at the first time, K is defined by the operator Eq (3.15).

Lemma 5.2. *Let $d(m) = \|K f_r^{\delta,m}(r) - g_r^\delta(r)\Big\|$, then we have the following conclusions: (1) $d(m)$ is a continuous function; (2) $\lim_{m\to 0} d(m) = \|g_r^\delta(r)\|$; (3) $\lim_{m\to\infty} d(m) = 0$; (4) $d(m)$ is a strictly decreasing function over* $(0, \infty)$.

Proof. By (5.1) and (5.4), we have

$$
d(m) = \Big(\sum_{n=1}^{\infty} \Big[1 - [1-(1-\beta k_n^2)^m]^\alpha\Big]^2 (g_r^\delta)^2 \Big)^{\frac{1}{2}}.
$$

Obviously, $\lim_{m\to 0} d(m) = \Big(\sum_{n=1}^{\infty} (g_r^\delta)^2\Big)^{\frac{1}{2}} = \|g_r^\delta(r)\|$. Therefore the conclusions (1)–(4) are obvious.

Remark 5.3. *We assume that the noisy data $\|g_r^\delta\|$ is large enough such that $0 < \tau\delta < \|g_r^\delta\|$, thus* *according to Lemma 5.2, there exists a unique minimum solution for inequality (5.4).*

Lemma 5.4. *If m is the solution of Eq (5.4), we can obtain the following inequality:*

$$(m\beta)^{\frac{1}{2}} \leq \left(\frac{\theta_{\frac{p+2}{4}}}{\underline{c}^{\frac{p}{2}}(\tau-1)}\right)^{\frac{2}{p+2}} \left(\frac{E}{\delta}\right)^{\frac{2}{p+2}}, \tag{5.5}$$

where

$$\theta_{\frac{p+2}{4}} = \begin{cases} 1, & 0 \leq p \leq 2, \\ \left(\frac{p+2}{4}\right)^{\frac{p+2}{4}}, & p > 2. \end{cases}$$

Proof. From (5.4), and according to Lemma 2.4, we obtain

$$\begin{aligned} \tau\delta &\leq \left\| K f_r^{\delta,m}(r) - g_r^{\delta}(r) \right\| \\ &= \left\| \sum_{n=1}^{\infty} \left[1 - [1 - (1 - \beta k_n^2)^{m-1}]^{\alpha} \right] \langle g_r^{\delta}, \varphi_n \rangle \varphi_n \right\| \\ &\leq \left\| \sum_{n=1}^{\infty} (1 - \beta k_n^2)^{m-1} \langle g_r^{\delta} - g_r, \varphi_n \rangle \varphi_n \right\| + \left\| \sum_{n=1}^{\infty} (1 - \beta k_n^2)^{m-1} \langle g_r^{\delta}, \varphi_n \rangle \varphi_n \right\|, \end{aligned}$$

then

$$\begin{aligned} \tau\delta &\leq \delta + E \sup_{n>0} (1 - \beta k_n^2)^{m-1} k_n n^{-p} \\ &\leq \delta + \underline{c}^{-\frac{p}{2}} E \sup_{n>0} (1 - \beta k_n^2)^{m-1} (\beta k_n^2)^{\frac{p+2}{4}} \beta^{-\frac{p+2}{4}} \\ &\leq \delta + \underline{c}^{-\frac{p}{2}} \theta_{\frac{p+2}{4}} (m\beta)^{-\frac{p+2}{4}} E. \end{aligned}$$

This yields

$$(m\beta)^{\frac{1}{2}} \leq \left(\frac{\theta_{\frac{p+2}{4}}}{\underline{c}^{\frac{p}{2}}(\tau-1)}\right)^{\frac{2}{p+2}} \left(\frac{E}{\delta}\right)^{\frac{2}{p+2}}. \tag{5.6}$$

Theorem 5.5. *Suppose the a priori condition (3.21) and the noise assumption (3.19) hold, then we have the convergence estimate*

$$\|f_r^{\delta,m}(r) - f_r(r)\| \leq \left(\left(\frac{\theta_{\frac{p+2}{4}}}{\underline{c}^{\frac{p}{2}}(\tau-1)}\right)^{\frac{2}{p+2}} + \left(\frac{\tau+1}{\underline{c}}\right)^{\frac{p}{p+2}} \right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.$$

Proof. By the triangle inequality, we know

$$\|f_r^{\delta,m}(r) - f_r(r)\| \leq \|f_r^{\delta,m}(r) - f_r^m(r)\| + \|f_r^m(r) - f_r(r)\| = J_1 + J_2. \tag{5.7}$$

On the estimate of J_1, by Lemma 5.4, we have

$$J_1 = \|f_r^{\delta,m}(r) - f_r^m(r)\| \leq \beta^{\frac{1}{2}} m^{\frac{1}{2}} \delta \leq \left(\frac{\theta_{\frac{p+2}{4}}}{\underline{c}^{\frac{p}{2}}(\tau-1)}\right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{5.8}$$

Now we estimate J_2, by the a priori bound condition (3.21), we can deduce that

$$\begin{aligned} J_2 &= \|f_r^m(r) - f_r(r)\| \\ &= \Big\| \sum_{n=1}^{\infty} \Big[1 - [1 - (1 - \beta E_{\alpha,1}^2(-\lambda_n T^\alpha))^m]^\alpha\Big] f_n \varphi_n \Big\| \\ &\le \Big\| \sum_{n=1}^{\infty} (1 - \beta k_n^2)^m \langle f_r, \varphi_n\rangle \varphi_n \Big\| \\ &\le \Big\| \sum_{n=1}^{\infty} (1 - \beta k_n^2)^m \langle f_r, \varphi_n\rangle \varphi_n \Big\|^{\frac{p}{p+2}} \Big\| \sum_{n=1}^{\infty} (1 - \beta k_n^2)^m n^{-p} n^p \langle f_r, \varphi_n\rangle \varphi_n \Big\|^{\frac{2}{p+2}}, \end{aligned} \tag{5.9}$$

where we have used the Hölder inequality. Therefore, by the triangle inequality, we obtain

$$\begin{aligned} J_2 &\le \Bigg(\Big\| \sum_{n=1}^{\infty} (1 - \beta k_n^2)^m k_n^{-1} \langle (g_r - g_r^\delta), \varphi_n\rangle \varphi_n \Big\| \\ &+ \Big\| \sum_{n=1}^{\infty} (1 - \beta k_n^2)^m k_n^{-1} g_n^\delta \varphi_n \Big\| \Bigg)^{\frac{p}{p+2}} \Big\| \sum_{n=1}^{\infty} n^{-p} n^p \langle (f_r, \varphi_n\rangle \varphi_n \Big\|^{\frac{2}{p+2}} \\ &\le (\delta + \tau\delta)^{\frac{p}{p+2}} \sup_{n>0} (k_n n^2)^{-\frac{p}{p+2}} \\ &\le \Big(\frac{\tau + 1}{\underline{c}}\Big)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \end{aligned} \tag{5.10}$$

Combining (5.7), (5.8) and (5.10), we can get the conclusion.

6. Numerical examples

In this section, three simple numerical examples are presented to show the validity of the two fractional regularization methods. the simulated data aregenerated as:

$$g^\delta = g(1 + \delta \cdot randn(size(g))).$$

where g is the solution of the forward problem, and randn is the white noise, the magnitude δ indicates the noise level of the measured data. In the numerical experiment, we use the spatial discretization number $M = 401$, and we fix $a(t) = 1, T = 1, R = 1$. $f(r)$ is the exact solution, $f^{\delta,\mu}(r)$ and $f^{\delta,m}(r)$ are regularized solutions of the WTRM and FLRM, respectively. The relative error calculations of the WTRM and FLRM are as follows

$$RE(WTRM) = \sqrt{\sum_{i=1}^{M} r(i)(f(i) - f^{\delta,\mu}(i))^2} \Big/ \sqrt{\sum_{i=1}^{M} r(i)(f(i))^2},$$

$$RE(FLRM) = \sqrt{\sum_{i=1}^{M} r(i)(f(i) - f^{\delta,m}(i))^2} \Big/ \sqrt{\sum_{i=1}^{M} r(i)(f(i))^2},$$

where $\|\cdot\|$ is the $L^2([0, R])$ norm.

In order to obtain the artificial data g^δ, we need to solve the following forward problem:

$$\begin{cases} \frac{\partial u(r,t)}{\partial t} = a(t)[\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial u(r,t)}{\partial r}] + f(r), \ t \in (0,1), 0 < r < 1, \\ u(r,0) = 0, \\ u(1,t) = 0. \end{cases}$$

Since the diffusivity $a(t)$ is taken to be a constant, the specific representation of the eigenvalues, eigenfunctions and the regularization solution could be calculated as follows

$$k_n = \int_0^T e^{-\lambda_n \int_\tau^T a(t)dt} d\tau = \frac{1 - e^{-a\lambda_n T}}{a\lambda_n} = \frac{1 - e^{-a(n\pi)^2}}{a(n\pi)^2},$$

and the eigenfunctions

$$\varphi_n(r) = \sqrt{2}\sin(n\pi r).$$

Substituting the eigenfunctions gives a formula to compute the coefficients

$$\langle rg(r), \varphi_n(r)\rangle = \sqrt{2}\int_0^1 rg(r)\sin(n\pi r)dr.$$

Note that the right hand side is essentially the sine transform of the function $rg(r)$, which can be efficiently implemented by using a version of the fast Fourier transform for real functions [40]. Based on the explicit expressions for the eigensystems, the regularization solution of WTRM could be calculated as follows

$$f^{\delta,\mu}(r) = \frac{2}{r}\sum_{n=1}^{\infty} \frac{k_n^\alpha}{k_n^{\alpha+1} + \mu}\left[\int_0^1 rg^\delta(r)\sin(n\pi r)dr\right]\sin(n\pi r), \ \frac{1}{2} < \alpha \leq 1. \tag{6.1}$$

the regularization solution of FLRM could be calculated as follows

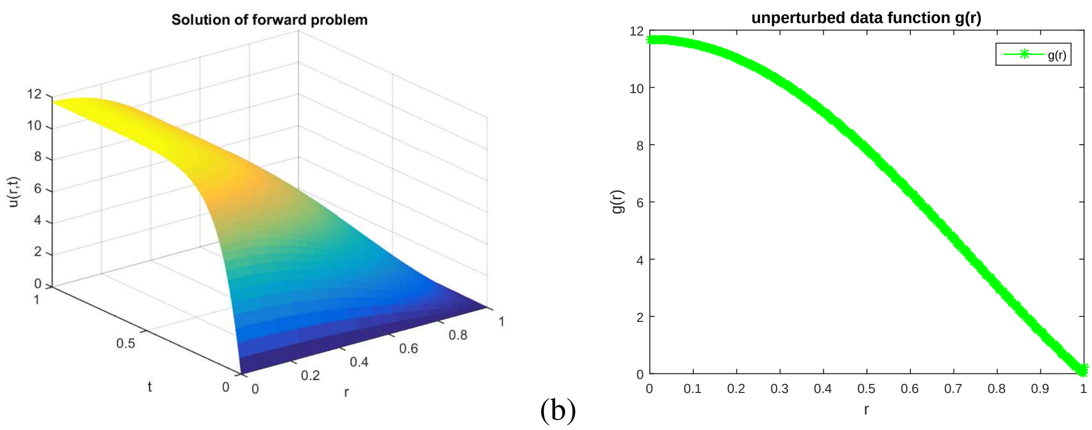
$$f^{\delta,m}(r) = \frac{2}{r}\sum_{n=1}^{\infty}[1 - (1 - \beta k_n^2)^m]^\alpha k_n^{-1}\left[\int_0^1 rg^\delta(r)\sin(n\pi r)dr\right]\sin(n\pi r), \ \frac{1}{2} < \alpha \leq 1. \tag{6.2}$$

In practical computation, we take the first $n = 40$ terms of the sum (6.1) and (6.2).

Example 6.1. Consider a smooth exact solution:

$$f_1(r) = 100(1 - r^2), \tag{6.3}$$

in interval $[0, 1]$.



(a) (b)

Figure 1. Example 6.1: (a) Solution of forward problem; (b) The unperturbed data function g(r).

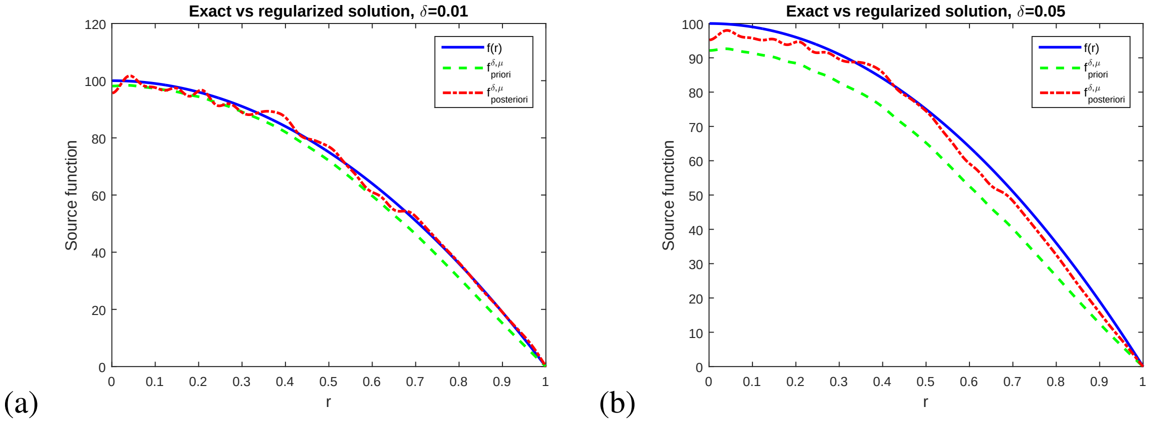


Figure 2. Example 6.1: Source function f(r) reconstructed by WTRM.
(a) $\delta = 0.01,\ \alpha = 0.65,\ \mu_{priori} = 0.0015,\ \mu_{posterior} = 1 * 10^{-5}$;
(b) $\delta = 0.05,\ \alpha = 0.65,\ \mu_{priori} = 0.0043,\ \mu_{posterior} = 1 * 10^{-4}$.

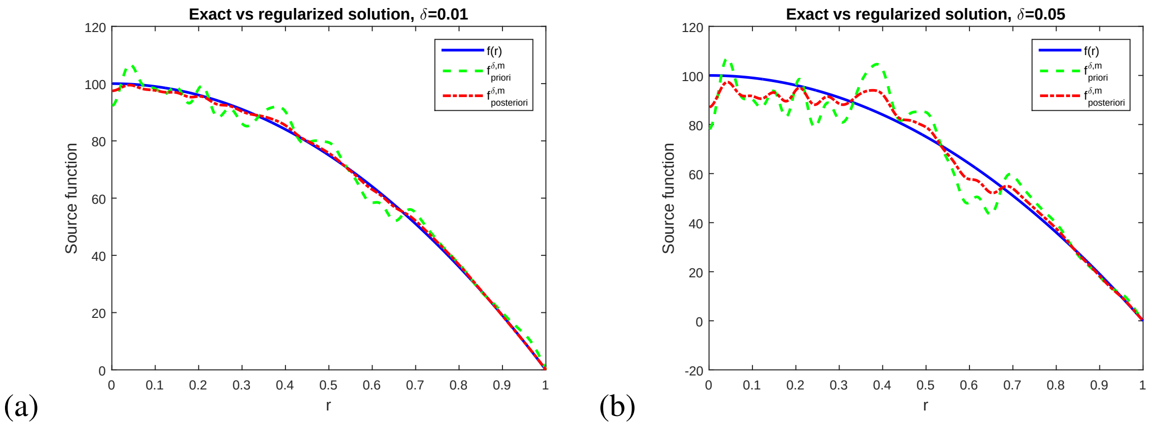


Figure 3. Example 6.1: Source function f(r) reconstructed by FLRM.
(a) $\delta = 0.01,\ \alpha = 0.8,\ \beta = 30,\ m_{priori} = 5407,\ m_{posterior} = 500$;
(b) $\delta = 0.05,\ \alpha = 0.8,\ \beta = 30,\ m_{priori} = 1849,\ m_{posterior} = 500$.

Table 1. Numerical results of Example 6.1.

(a)

δ	0.01	0.05
RE_{priori}	0.0444	0.1199
$RE_{posterior}$	0.0216	0.0386

Relative error of WTRM ($\alpha = 0.65$);

(b)

δ	0.01	0.05
RE_{priori}	0.0438	0.1178
$RE_{posterior}$	0.0117	0.0606

Relative error of FLRM ($\alpha = 0.8$).

Figures 1–3 show the numerical results of Example 6.1. Figure 1 shows the solution of forward problem and the unperturbed data function $g(r)$. Figure 2 shows the comparison for the regularization parameter chosen by both the a-priori WTRM and the a-posteriori WTRM with respect to different noise level. Figure 3 shows the comparison for the regularization parameter chosen by both the a-priori FLRM and the a-posteriori FLRM with respect to different noise level. Table 1(a) and 1(b) show the relative error between the exact solution and the approximate solution calculated by the WTRM and the FLRM, respectively.

From the numerical results, the approximate solution calculated by the a-posterior method is better than the a-prior method, and the relative error of the a-posterior calculation is smaller than that of the a-priori parameter choice. As the noise level increases, the numerical performance deteriorates.

Example 6.2. Consider the oscillating source function

$$f_2(r) = 2[1 + cos(3\pi r)], \tag{6.4}$$

in interval $[0, 1]$.

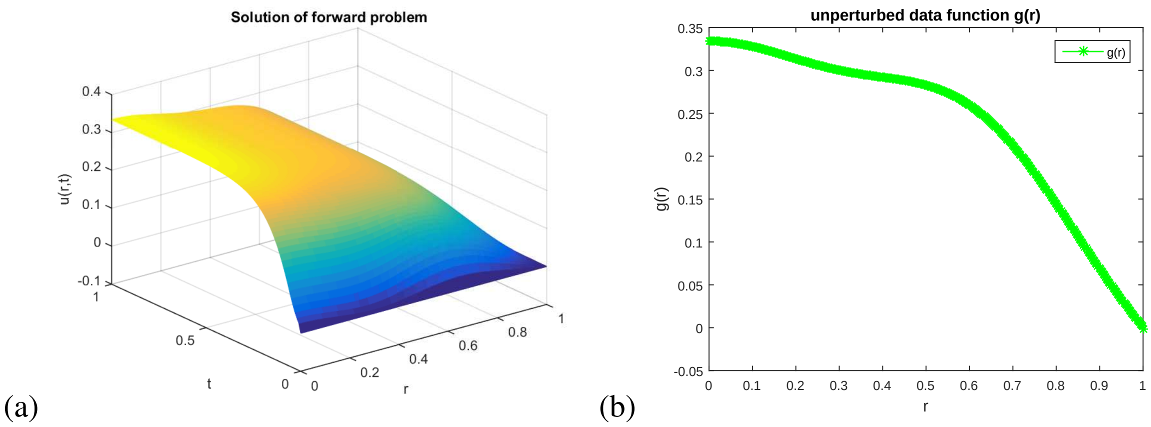


Figure 4. Example 6.2: (a) Solution of forward problem; (b) The unperturbed data function g(r).

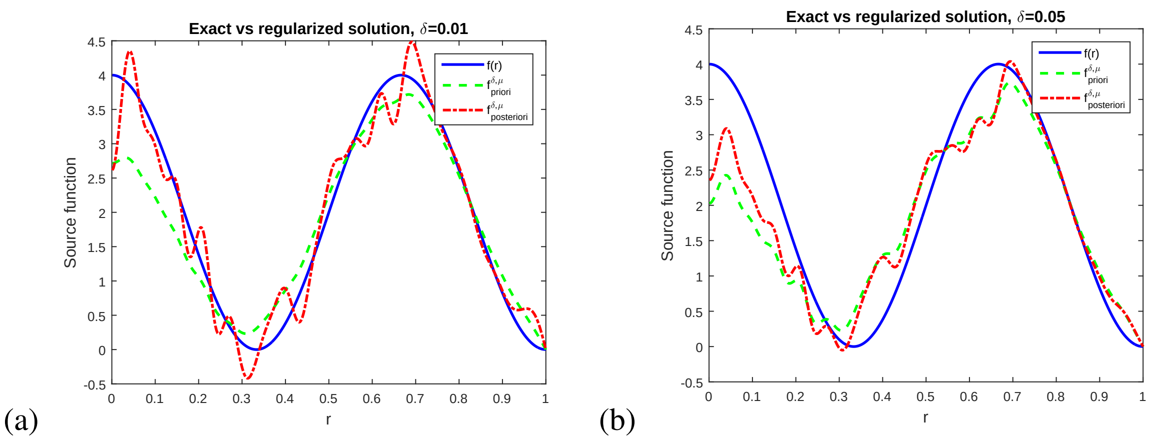
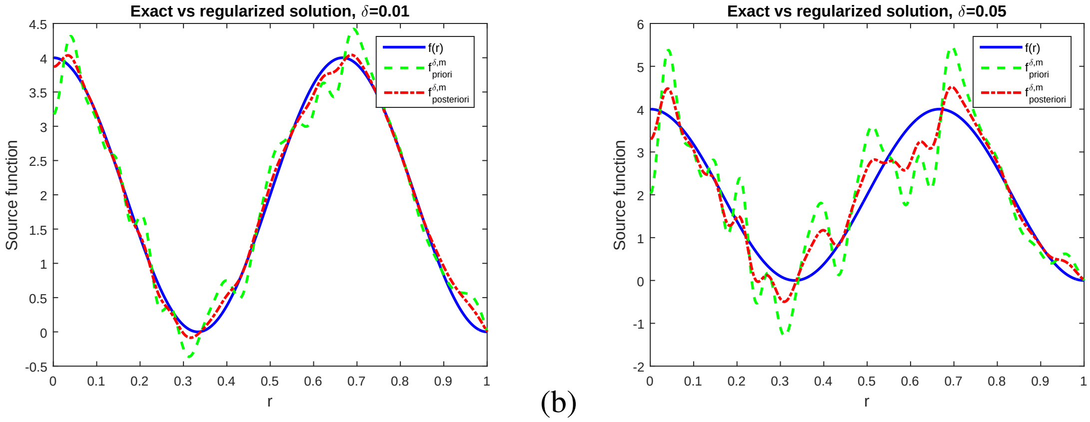


Figure 5. Example 6.2: Source function f(r) reconstructed by WTRM.
(a) $\delta = 0.01,\ \alpha = 0.65,\ \mu_{priori} = 1.1 * 10^{-4},\ \mu_{posterior} = 1.1 * 10^{-6}$;
(b) $\delta = 0.05,\ \alpha = 0.65,\ \mu_{priori} = 1.7 * 10^{-4},\ \mu_{posterior} = 1.0 * 10^{-5}$.



(a) (b)

Figure 6. Example 6.2: Source function f(r) reconstructed by FLRM.
(a) $\delta = 0.01,\ \alpha = 0.8,\ \beta = 20,\ m_{priori} = 31075,\ m_{posterior} = 5000$;
(b) $\delta = 0.05,\ \alpha = 0.8,\ \beta = 20,\ m_{priori} = 16324,\ m_{posterior} = 5000$.

Table 2. Numerical results of Example 6.2.

(a)

δ	0.01	0.05
RE_{priori}	0.1940	0.2912
$RE_{posterior}$	0.1356	0.2306

Relative error of WTRM ($\alpha = 0.65$);

(b)

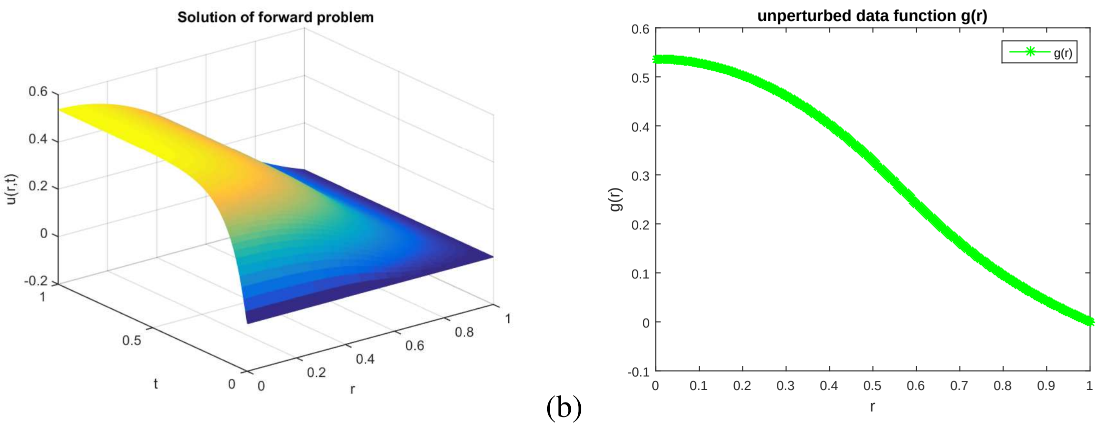
δ	0.01	0.05
RE_{priori}	0.1077	0.3394
$RE_{posterior}$	0.0411	0.1670

Relative error of FLRM ($\alpha = 0.8$).

The numerical experiments of Example 6.2 is similar to those of Example 6.1. The result of Example 6.2 is shown in Figures 4–6 and Table 2.

Example 6.3. Consider the nonsmooth source function:

$$f_3(r) = \begin{cases} 5, & 0 < r \leq 0.5, \\ 80(1-r)^4, & 0.5 < r < 1. \end{cases} \tag{6.5}$$



(a) (b)

Figure 7. Example 6.3: (a) Solution of forward problem; (b) The unperturbed data function g(r).

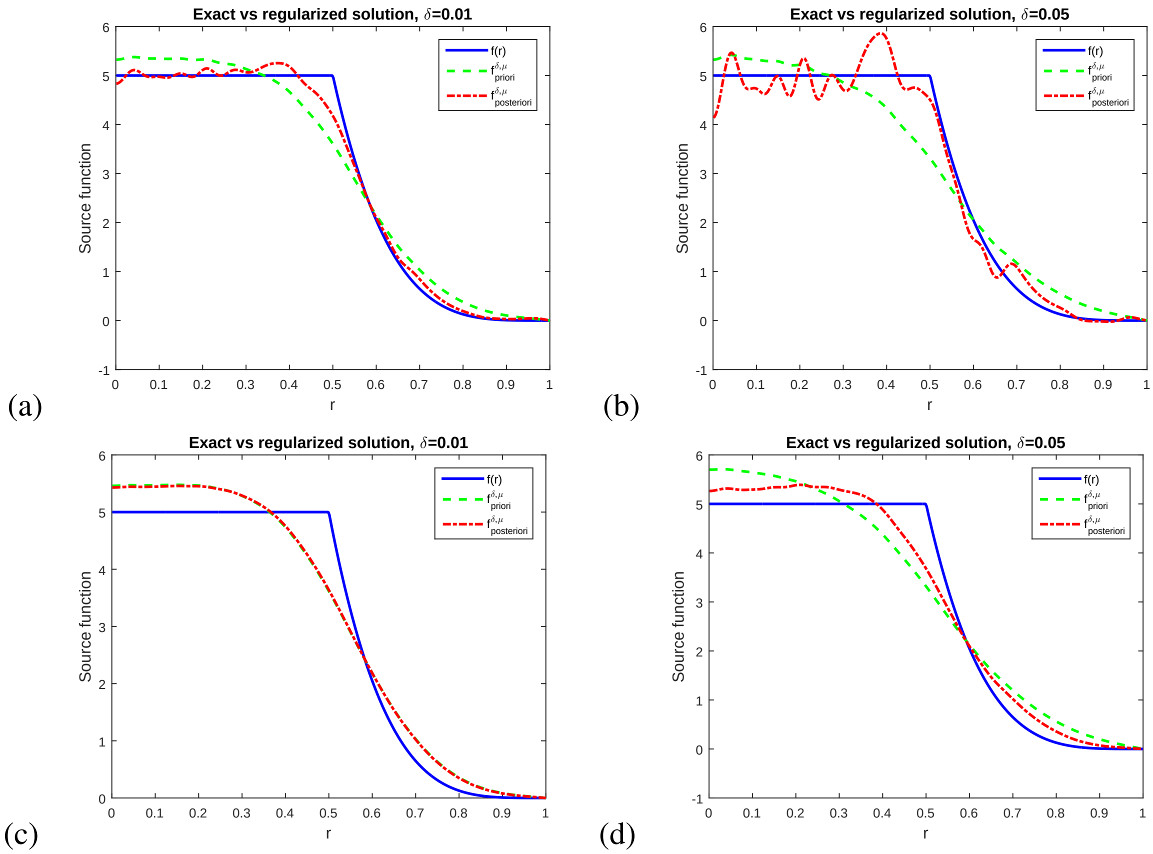


Figure 8. Example 6.3: Source function f(r) reconstructed by WTRM.
(a) $\delta = 0.01,\ \alpha = 0.65,\ \mu_{priori} = 5.3 * 10^{-4},\ \mu_{posterior} = 3.0 * 10^{-5}$;
(b) $\delta = 0.05,\ \alpha = 0.65,\ \mu_{priori} = 1.3 * 10^{-3},\ \mu_{posterior} = 3.0 * 10^{-5}$;
(c) $\delta = 0.01,\ \alpha = 1,\ \mu_{priori} = 1.1 * 10^{-4},\ \mu_{posterior} = 3.0 * 10^{-5}$;
(d) $\delta = 0.05,\ \alpha = 1,\ \mu_{priori} = 3.1 * 10^{-4},\ \mu_{posterior} = 1.0 * 10^{-5}$.

Table 3. Numerical results of Example 6.3.

(a)

δ	0.01	0.05
RE_{priori}	0.1124	0.1475
$RE_{posterior}$	0.0499	0.0849

(b)

δ	0.01	0.05
RE_{priori}	0.1194	0.1607
$RE_{posterior}$	0.1162	0.1047

(a) Relative error of weighted fractional Tikhonov method ($\alpha = 0.65$);
(b) Relative error of classic Tikhonov method ($\alpha = 1$).

In this example, we compare the numerical results of two fractional regularization methods and classical regularization methods under the same parameter choice. Figure 7 shows the solution of forward problem and the unperturbed data function $g(r)$ of Example 6.3. Figure 8 and Table 3 show the comparison of the numerical results of the weighted fractional Tikhonov regularization method and the classical Tikhonov regularization method. The result shows that the weighted fractional Tikhonov method outperforms the classical Tikhonov method under the same parameters, and the classic Tikhonov regularized solution oversmooths.

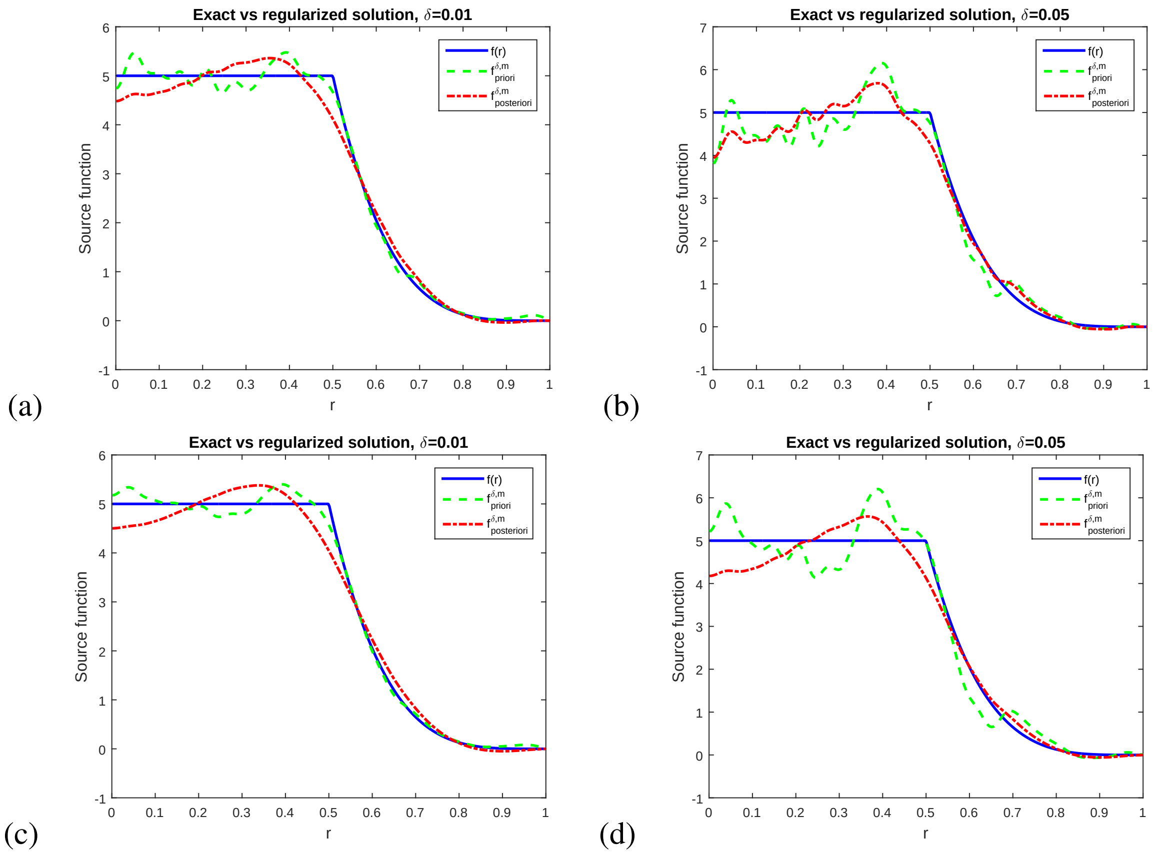


Figure 9. Example 6.3: Source function f(r) reconstructed by FLRM.
(a) $\delta = 0.01,\ \alpha = 0.8,\ \beta = 20,\ m_{priori} = 9423, m_{posterior} = 1000$;
(b) $\delta = 0.05,\ \alpha = 0.8,\ \beta = 20,\ m_{priori} = 3222,\ m_{posterior} = 1000$;
(c) $\delta = 0.01,\ \alpha = 1,\ \beta = 20,\ m_{priori} = 9423,\ m_{posterior} = 1000$;
(d) $\delta = 0.05,\ \alpha = 1,\ \beta = 20,\ m_{priori} = 3222,\ m_{posterior} = 1000$.

Table 4. Numerical results of Example 6.3.

	δ	0.01	0.05		δ	0.01	0.05
(a)	RE_{priori}	0.0453	0.1123	(b)	RE_{priori}	0.0422	0.1220
	$RE_{posterior}$	0.0699	0.0953		$RE_{posterior}$	0.0769	0.0981

(a) Relative error of fractional Landweber method ($\alpha = 0.8$);
(b) Relative error of classic Landweber method ($\alpha = 1$).

Figure 9 and Table 4 show the comparison of the numerical results of the fractional Landweber regularization method and the classical Landweber regularization method. We can see that the fractional Landweber method provides better numerical result than the classical Landweber method under the same iterative steps.

7. Conclusions

As we have seen, the fractional methods include the classical method by introducing a new fractional parameter. We have proved the error estimates for the fractional regularization methods under the the a-priori parameter choice rule and the Morozov's parameter choice rule. The error estimates are order-optimal. This shows that in theory the fractional regularization methods are not inferior to the classical regularization method. Numerical results are also displays that the fractional methods can overcome the disadvantage of over-smoothing of the classical methods. The future research will be generalize the fractional regularization methods to some fractional differential equations with important application background.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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