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Research article

The generalized quadratic Gauss sums and its sixth power mean

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Abstract: In this article, we using elementary methods, the number of the solutions of some congruence equations and the properties of the Legendre's symbol to study the computational problem of the sixth power mean of a certain generalized quadratic Gauss sums, and to give an exact calculating formula for it.

Keywords: congruence equation; a certain generalized quadratic Gauss sums; the sixth power mean; calculating formula
Mathematics Subject Classification: 11L03, 11L07

1. Introduction

Let $q \geq 3$ is a positive integer. For any integers m, n and any Dirichlet character χ mod q (see the definition in [6]), we define a generalized quadratic Gauss sums $G(m, n, \chi; q)$ as

$$G(m, n, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^2 + na}{q}\right),$$

where as usual, $e(y) = e^{2\pi i y}$, and $i^2 = -1$.

If taking $n = 0$, then $G(m, 0, \chi; q)$ is the quadratic Gauss sums. That is,

$$G(m, 0, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^2}{q}\right).$$

About the properties of $G(m, n, \chi; q)$, some authors obtained many interesting results. For example, if $q = p$ is an odd prime, then from A. Weil's important work [1] we can get the estimate

$$|G(m, n, \chi; p)| \leq 2 \cdot \sqrt{p}$$

for all integer m and n with $(m, n, p) = 1$.

Li Xiaoxue and Xu Zhefeng [9] obtained the identity

$$\sum_{m=1}^{p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2+a}{p}\right)\right|^4$$
$$=\begin{cases} p^3-3p^2+2\left(\frac{-1}{p}\right)p^2-p-8\left(\frac{-1}{p}\right)p, & \text{if } \chi=\chi_0; \\ 2p^3-3p^2, & \text{if } \chi(-1)=-1; \\ 2p^3-3p^2-4\left(\frac{-1}{p}\right)p^2-p\left|\sum_{a=1}^{p-1}\chi\left(a+\overline{a}\right)\right|^2, & \text{if } \chi(-1)=1 \text{ and } \chi\neq\chi_0, \end{cases}$$

where χ_0 denotes the principal character mod p, $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p, and $\overline{a}$ denotes the inverse of a. That is, $a\cdot\overline{a}\equiv 1 \bmod p$.

Zhang Wenpeng and Lin Xin [7] proved the following conclusion:

Let p be an odd prime, α be any positive integer with $\alpha\geq 2$. Then for any integer n with $(n,p)=1$, if λ is any primitive character mod p^α(see the definition in [6]), then we have

$$\sideset{}{'}\sum_{m=1}^{p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^2+na}{p^\alpha}\right)\right|^4 = p^{2\alpha}\phi\left(p^\alpha\right)\left(\alpha+1-\frac{5}{p-1}\right);$$

If λ is any non-primitive character mod p^α, then we have the identity

$$\sideset{}{'}\sum_{m=1}^{p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^2+na}{p^\alpha}\right)\right|^4 = p^{2\alpha}\phi\left(p^\alpha\right),$$

where $\phi(n)$ denotes the Euler function.

For any integer m with $(m,p)=1$, Zhang Wenpeng [8] given identities

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^4$$
$$=\begin{cases} (p-1)\left[3p^2-6p-1+4\left(\frac{m}{p}\right)\sqrt{p}\right], & \text{if } p\equiv 1 \bmod 4; \\ (p-1)\left(3p^2-6p-1\right), & \text{if } p\equiv 3 \bmod 4. \end{cases}$$

For $(m,p)=1$ and $p\equiv 3 \bmod 4$, one has the identity

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^6 = (p-1)\left(10p^3-25p^2-4p-1\right).$$

If $m=0$, then $G(0,n,\chi;q)$ becomes classical Gauss sums $G(n,\chi;q)$. That is,

$$G(0,n,\chi;q)=G(n,\chi;q)=\sum_{a=1}^{q}\chi(a)e\left(\frac{na}{q}\right).$$

If χ is a primitive character mod q, then $G(n,\chi;q) = \overline{\chi}(n) \cdot \tau(\chi)$, where

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right),$$

$\chi \cdot \overline{\chi} = \chi_0$ and $|\tau(\chi)| = \sqrt{q}$, see Theorem 8.15 in [7].

Some papers related to Gauss sums, Kloosterman sums and two-term exponential sums can also be found in [3–5], we will not go through them here.

In this paper, we are going to be interested in the computational problem of the $2k$-th power mean of $G(m,n,\chi;p)$. That is,

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2+a}{p}\right) \right|^{2k}, \tag{1}$$

where p is an odd prime and k is a positive integer.

When $k \geq 3$, no one seems to research the calculation problem of (1) so far, at least we have not seen any paper about it. Of course, the study of this problem is meaningful, mainly in the following three aspects:

First of all, Gauss sums plays an important role in the study of number theory, many analytic number theory problems are closely related to it, such as the quadratic reciprocal formula of the Legendre's symbol, Dirichlet L-functions and so on. So it is necessary to study its various properties.

Secondly, due to the irregular properties of the value distribution of the generalized quadratic Gauss sums, it is difficult to obtain satisfactory results in the study of some number theory problems, and then we can use the exact calculation of the mean value instead. In this way, satisfactory results can be obtained.

Third, it is always worthwhile to be able to give an exact calculating formula for some discrete sum problems.

It is because of these reasons, we think it is necessary to further study the various higher power mean of the generalized quadratic Gauss sums. In this paper, we will use elementary methods, the number of the solutions of some congruence equations mod$p$ and the properties of the Legendre's symbol to study the calculating problem of (1) with $k = 3$, and give an exact calculating formula for it. That is, we will prove the following main result:

Theorem. For any prime $p \neq 2$, we have the identity

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2+a}{p}\right) \right|^{6} = p(p-1)\left[5p^3 - 27p^2 + 44p + 2 + 8\left(\frac{-1}{p}\right)\right].$$

Some notes: In the theorem, we only discussed the special case. That is, the modulo is an odd prime p. So we naturally ask:

(A). What happens if the modulo is a composite number?

(B). For integer $k \geq 4$, whether there exists a calculating formula for (1)?

These are two open problems. We need to further study.

2. Some lemmas

In this section, we first give two simple lemmas, which are necessary in the proof of our theorem. At the same time, in the proofs of these lemmas, we need some knowledge of elementary and analytic number theory, which can be found in [1,2], so we do not need to repeat them here. First we have the following:

Lemma 1. Let p is an odd prime, then we have the identity

$$\sum_{\substack{a=1 \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = p^3+6p^2-25p-2-8\left(\frac{-1}{p}\right).$$

Proof. From the properties of the reduced residue system modulo p we know that if d, e passed through a reduced residue system mod p, then da, eb also pass through a reduced residue system modulo p, so notice the symmetry of a and b, and d and e, we have the identity

$$\begin{aligned}
\sum_{\substack{a=1 \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 &= \sum_{\substack{a=1 \\ a^2+b^2+c^2\equiv a^2d^2+b^2e^2+1 \bmod p \\ c\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= \sum_{\substack{a=1 \\ a^2+b^2+d^2e^2\equiv a^2d^2+b^2e^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} \left(1+\left(\frac{a}{p}\right)\right)\cdot\left(1+\left(\frac{b}{p}\right)\right)\cdot\left(1+\left(\frac{d}{p}\right)\right)\cdot\left(1+\left(\frac{e}{p}\right)\right) \\
&= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} \left(1+2\left(\frac{d}{p}\right)+\left(\frac{de}{p}\right)+2\left(\frac{a}{p}\right)+\left(\frac{ab}{p}\right)\right) \\
&+ \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} \left(4\left(\frac{ad}{p}\right)+2\left(\frac{ade}{p}\right)+2\left(\frac{dab}{p}\right)+\left(\frac{abde}{p}\right)\right) \\
&\equiv W_1+2W_2+W_3+2W_4+W_5+4W_6+2W_7+2W_8+W_9, \qquad (2)
\end{aligned}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol modulo p.

Now we calculating the exact values of W_i $(1\le i\le 9)$ in (2) respectively. From the properties of the reduced residue system modulo p we have

$$\begin{aligned}
W_1 &= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = (p-1)(3p-5)+\sum_{\substack{a=1 \\ a+b\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1} 1 \\
&= (p-1)(3p-5)+(p-1)(p-2)^2-(p-2)^2+(p-2)
\end{aligned}$$

$$= p^3 - 3p^2 + 5p - 5. \tag{3}$$

$$\begin{aligned}
W_2 &= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \left(\frac{d}{p}\right) = (p-1)(2p-3) + (p-1)\sum_{d=2}^{p-1}\left(\frac{d}{p}\right) \\
&= 2(p-1)(p-2) = 2p^2 - 6p + 4. \qquad (4)
\end{aligned}$$

$$\begin{aligned}
W_3 &= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \left(\frac{de}{p}\right) = (p-1)^2 + 2(p-1)\sum_{d=2}^{p-1}\left(\frac{d}{p}\right) \\
&\quad + \sum_{\substack{a=1 \\ a+b\equiv de-1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=2}^{p-1} \sum_{e=2}^{p-1} \left(\frac{de}{p}\right) \\
&= (p-1)(p-3) + \sum_{\substack{a=0 \\ a+b\equiv de-1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=2}^{p-1} \sum_{e=2}^{p-1} \left(\frac{de}{p}\right) - \sum_{\substack{b=1 \\ b\equiv de-1 \bmod p}}^{p-1} \sum_{d=2}^{p-1} \sum_{e=2}^{p-1} \left(\frac{de}{p}\right) \\
&= (p-1)(p-3) + (p-1) - \sum_{d=2}^{p-1} \sum_{d=2}^{p-1} \left(\frac{de}{p}\right) + \sum_{\substack{d=2 \\ de\equiv 1 \bmod p}}^{p-1} \sum_{e=2}^{p-1} \left(\frac{de}{p}\right) \\
&= p^2 - 2p - 1. \qquad (5)
\end{aligned}$$

$$\begin{aligned}
W_4 &= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \left(\frac{a}{p}\right) = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{a}{p}\right) + \sum_{a=1}^{p-1} \sum_{e=2}^{p-1} \left(\frac{a}{p}\right) \\
&\quad + \sum_{b=1}^{p-1} \sum_{d=2}^{p-1} 1 + \sum_{\substack{a=1 \\ a(d-1)+b\equiv de-1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=2}^{p-1} \sum_{e=2}^{p-1} \left(\frac{a}{p}\right) \\
&= (p-1)(p-2) + \sum_{\substack{a=1 \\ a(d-1)+b\equiv de-1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{d=2}^{p-1} \sum_{e=2}^{p-1} \left(\frac{a}{p}\right) - \sum_{\substack{a=1 \\ a(d-1)\equiv de-1 \bmod p}}^{p-1} \sum_{d=2}^{p-1} \sum_{e=2}^{p-1} \left(\frac{a}{p}\right) \\
&= (p-1)(p-2) - \sum_{d=2}^{p-1} \sum_{e=2}^{p-1} \left(\frac{(de-1)(d-1)}{p}\right) \\
&= p(p-2) - \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \left(\frac{(de-1)(d-1)}{p}\right) = p^2 - 2p - 1. \qquad (6)
\end{aligned}$$

$$W_5 = \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \left(\frac{ab}{p}\right) = \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=2}^{p-1} \sum_{e=2}^{p-1} \left(\frac{ab}{p}\right)$$

$$
\begin{aligned}
&= \sum_{\substack{a=1 \\ a+b\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\left(\frac{ab(d-1)(e-1)}{p}\right) \\
&= \sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{b\,(de-1-b)\,(d-1)(e-1)}{p}\right) \\
&= \sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(d-1)(e-1)}{p}\right)\sum_{b=1}^{p-1}\left(\frac{b\,(de-1-b)}{p}\right). \qquad (7)
\end{aligned}
$$

Note that identity

$$
\sum_{a=0}^{p-1}\left(\frac{a^2+b}{p}\right) = \sum_{a=1}^{p-1}\left(\frac{a(a+b)}{p}\right) = \begin{cases} p-1, & \text{if } (b,p)=p; \\ -1, & \text{if } (b,p)=1. \end{cases} \qquad (8)
$$

From (7) and (8) we have

$$
\begin{aligned}
W_5 &= \sum_{\substack{d=2 \\ (de-1,p)=1}}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(d-1)(e-1)}{p}\right)\sum_{b=1}^{p-1}\left(\frac{b\,(1-b)}{p}\right) + \sum_{b=1}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(e-1)(1-\overline{e})}{p}\right) \\
&= p\sum_{e=2}^{p-1}\left(\frac{e}{p}\right) - \left(\frac{-1}{p}\right)\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(d-1)(e-1)}{p}\right) = -p - \left(\frac{-1}{p}\right). \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
W_6 &= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\left(\frac{ad}{p}\right) = \sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\left(\frac{d}{p}\right) + \sum_{\substack{a=1 \\ a(d-1)+b\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{ad}{p}\right) \\
&= -(p-1) + \sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(de-1-b)\,d(d-1)}{p}\right) \\
&= -(p-1) - \sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(de-1)\,d(d-1)}{p}\right) \\
&= -(p-1) - \sum_{d=2}^{p-1}\sum_{e=1}^{p-1}\left(\frac{(e-1)\,d(d-1)}{p}\right) + \sum_{d=2}^{p-1}\left(\frac{d}{p}\right) \\
&= -p + \left(\frac{-1}{p}\right)\sum_{d=1}^{p-1}\left(\frac{d(d-1)}{p}\right) = -p - \left(\frac{-1}{p}\right). \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
W_7 &= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\left(\frac{ade}{p}\right) = \sum_{a=1}^{p-1}\sum_{e=2}^{p-1}\left(\frac{ae}{p}\right) + \sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\left(\frac{d}{p}\right) \\
&\quad + \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{ade}{p}\right)
\end{aligned}
$$

$$\begin{aligned}
&= -(p-1) + \sum_{\substack{a=1\\ a+b\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{ad(d-1)e}{p}\right)\\
&= -(p-1) + \sum_{\substack{a=1\\ a+b\equiv de-1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{ad(d-1)e}{p}\right) - \sum_{\substack{a=1\\ a\equiv de-1 \bmod p}}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{ad(d-1)e}{p}\right)\\
&= -(p-1) - \sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(de-1)d(d-1)e}{p}\right)\\
&= -(p-1) - \sum_{d=2}^{p-1}\sum_{e=1}^{p-1}\left(\frac{(de-1)d(d-1)e}{p}\right) + \sum_{d=2}^{p-1}\left(\frac{d}{p}\right)\\
&= -p - \sum_{d=2}^{p-1}\sum_{e=1}^{p-1}\left(\frac{(d-\overline{e})d(d-1)}{p}\right) = -p - \left(\frac{-1}{p}\right). \qquad (11)
\end{aligned}$$

$$\begin{aligned}
W_8 &= \sum_{\substack{a=1\\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\left(\frac{abd}{p}\right) = \sum_{\substack{a=1\\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{abd}{p}\right)\\
&= \sum_{\substack{a=1\\ a+b\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{ab(d-1)(e-1)d}{p}\right)\\
&= \sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(de-1-b)b(d-1)(e-1)d}{p}\right)\\
&= \sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\left(\frac{-b^2(d-1)(\overline{d}-1)d}{p}\right) - \sum_{\substack{d=2\\ (de-1,p)=1}}^{p-1}\sum_{e=2}^{p-1}\left(\frac{-(d-1)(e-1)d}{p}\right)\\
&= (p-1)(p-2) - \sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\left(\frac{-(d-1)(e-1)d}{p}\right) + (p-2)\\
&= p(p-2) + \sum_{d=1}^{p-1}\left(\frac{d(d-1)}{p}\right) = p^2-2p-1. \qquad (12)
\end{aligned}$$

$$\begin{aligned}
W_9 &= \sum_{\substack{a=1\\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\left(\frac{abde}{p}\right) = \sum_{\substack{a=1\\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{abde}{p}\right)\\
&= \sum_{\substack{a=1\\ a+b\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{abd(d-1)e(e-1)}{p}\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(de-1-b)(d-1)(e-1)bde}{p}\right)\\
&= \sum_{\substack{d=2\\(de-1,p)=1}}^{p-1}\sum_{e=2}^{p-1}\sum_{b=1}^{p-1}\left(\frac{-b(b-de+1)(d-1)(e-1)de}{p}\right)\\
&\quad + \sum_{\substack{d=2\\de\equiv 1 \bmod p}}^{p-1}\sum_{e=2}^{p-1}\sum_{b=1}^{p-1}\left(\frac{-b^2(d-1)(e-1)}{p}\right)\\
&= -\left(\frac{-1}{p}\right)\sum_{\substack{d=2\\(de-1,p)=1}}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(d-1)(e-1)de}{p}\right)-(p-1)\\
&= -\left(\frac{-1}{p}\right)\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\left(\frac{(d-1)(e-1)de}{p}\right)+\sum_{d=2}^{p-1}\left(\frac{d}{p}\right)-(p-1)\\
&= -p-\left(\frac{-1}{p}\right). \qquad (13)
\end{aligned}$$

Now combining (2)–(6), (9)–(13) we may immediately deduce the identity

$$\sum_{\substack{a=1\\a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = p^3+6p^2-25p-2-8\left(\frac{-1}{p}\right).$$

This proves Lemma 1.

Lemma 2. Let p is an odd prime, then we have the identity

$$\sum_{\substack{a=1\\a+b+c\equiv d+e+1 \bmod p\\a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = 6p^2-21p+19.$$

Proof. For all integers $1 \le a,b,c,d,e \le p-1$, from the properties of the reduced residue system modulo p we know that the conditions $a+b+c \equiv d+e+1 \bmod p$, $a^2+b^2+c^2 \equiv d^2+e^2+1 \bmod p$ and $abc \equiv de \bmod p$ are equivalent to $a+b+c \equiv ad+be+1 \bmod p$, $a^2+b^2+c^2 \equiv a^2d^2+b^2e^2+1 \bmod p$ and $c \equiv de \bmod p$, they are equivalent to $a+b+de \equiv ad+be+1 \bmod p$, $ab+(a+b)de \equiv abde+ad+be \bmod p$, or $a+b+de \equiv ad+be+1 \bmod p$, $ab+(a+b)de \equiv abde+a+b+de-1 \bmod p$, or $a(d-1)+b(e-1) \equiv de-1 \bmod p$, $(a-1)(b-1)(de-1) \equiv 0 \bmod p$. So from these conditions we have

$$\sum_{\substack{a=1\\a+b+c\equiv d+e+1 \bmod p\\a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = \sum_{\substack{a=1\\a(d-1)+b(e-1)\equiv de-1 \bmod p\\(a-1)(b-1)(de-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1. \qquad (14)$$

Now we calculate (14) according to $a \equiv 1 \bmod p$, $b \equiv 1 \bmod p$ and $de \equiv 1 \bmod p$; $a \equiv 1 \bmod p$, $b \equiv 1 \bmod p$ and $p \nmid (de-1)$; $p \nmid (a-1)$, $b \equiv 1 \bmod p$ and $de \equiv 1 \bmod p$; $a \equiv 1 \bmod p$, $p \nmid (b-1)$ and $p \nmid (de-1)$; $p \nmid (a-1)$, $p \nmid (b-1)$ and $de \equiv 1 \bmod p$. From (14) we have

$$\begin{aligned}
\sum_{\substack{a=1 \\ a+b+c\equiv d+e+1 \bmod p \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 &= \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p \\ (a-1)(b-1)(de-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= \sum_{\substack{d=1 \\ d+\overline{d}-2\equiv 0 \bmod p}}^{p-1} 1 + \sum_{\substack{d=1 \\ d+e-2\equiv de-1 \bmod p \\ (de-1,p)=1}}^{p-1}\sum_{e=1}^{p-1} 1 + 2\sum_{\substack{a=2 \\ a(d-1)+\overline{d}-1\equiv 0 \bmod p}}^{p-1}\sum_{d=1}^{p-1} 1 \\
&\quad +2\sum_{\substack{a=2 \\ a(d-1)+e-1\equiv de-1 \bmod p \\ (de-1,p)=1}}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 + \sum_{\substack{a=2 \\ a(d-1)+b(\overline{d}-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=2}^{p-1}\sum_{d=1}^{p-1} 1 \\
&= 1+2(p-2)+4(p-2)+2(p-2)(2p-5)+(p-2)(2p-5) \\
&= 6p^2-21p+19.
\end{aligned}$$

This proves Lemma 2.

3. Proof of the theorem

In this section, we use the two basic lemmas of the previous section to prove our main result. In fact from Lemma 1, Lemma 2, the properties of the reduced residue system modulo p, the trigonometric identity

$$\sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) = \begin{cases} p, & \text{if } p \mid m; \\ 0, & \text{if } p \nmid m \end{cases}$$

and the orthogonality of characters modulo p

$$\sum_{\chi \bmod p} \chi(a) = \begin{cases} p-1, & \text{if } a \equiv 1 \bmod p; \\ 0, & \text{otherwise} \end{cases}$$

we have

$$\begin{aligned}
&\frac{1}{p(p-1)}\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a+ma^2}{p}\right)\right|^6 \\
&= \sum_{\substack{a=1 \\ abc\equiv def \bmod p \\ a^2+b^2+c^2\equiv d^2+e^2+f^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{a+b+c-d-e-f}{p}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1 \\ abc\equiv de \bmod p \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p}}^{p-1} e\left(\frac{f(a+b+c-d-e-1)}{p}\right) \\
&= p\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1 \\ abc\equiv de \bmod p \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p \\ a+b+c\equiv d+e+1 \bmod p}}^{p-1} 1 - \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1 \\ abc\equiv de \bmod p \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p}}^{p-1} 1 \\
&= p\left(6p^2-21p+19\right) - \left[p^3+6p^2-25p-2-8\left(\frac{-1}{p}\right)\right] \\
&= 5p^3-27p^2+44p+2+8\left(\frac{-1}{p}\right).
\end{aligned}
$$

This completes the proof of our theorem.

4. Conclusions

The main result of this paper is a theorem. It gives an exact calculating formula for the sixth power mean of the generalized quadratic Gauss sums. At the same time, we also proposed two open problems. We deeply believe that the research work in this paper and the proposal of the open problems in this paper will contribute to the further research in the related fields.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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