Research article

CKV-type $B$-matrices and error bounds for linear complementarity problems

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Abstract: In this paper, we introduce a new subclass of $P$-matrices called Cvetković-Kostić-Varga type $B$-matrices (CKV-type $B$-matrices), which contains DZ-type-$B$-matrices as a special case, and present an infinity norm bound for the inverse of CKV-type $B$-matrices. Based on this bound, we also give an error bound for linear complementarity problems of CKV-type $B$-matrices. It is proved that the new error bound is better than that provided by Li et al. [24] for DZ-type-$B$-matrices, and than that provided by M. García-Esnaola and J.M. Peña [10] for $B$-matrices in some cases. Numerical examples demonstrate the effectiveness of the obtained results.

Keywords: CKV-type $B$-matrices; linear complementarity problems; error bounds; infinity norm; $P$-matrices

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1. Introduction

Given an $n \times n$ real matrix $A$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem is to find a vector $x \in \mathbb{R}^n$ satisfying

$$x \geq 0, Ax + q \geq 0, (Ax + q)^T x = 0$$

(1.1)

or to show that no such vector $x$ exists. We denote the problem (1.1) and its solution by LCP($A, q$) and $x^*$, respectively. The LCP($A, q$), as one of the fundamental problems in optimization and mathematical programming, has various applications in the quadratic programming, the optimal stopping, the Nash equilibrium point of a bimatrix game, the network equilibrium problem, the contact problem, and the free boundary problem for journal bearing, for details, see [1, 2, 26].

It is well known that the LCP($A, q$) has a unique solution $x^*$ for any $q \in \mathbb{R}^n$ if and only if $A$ is a $P$-matrix [2]. Here, a real square matrix $A$ is called a $P$-matrix if all its principal minors are positive. For this case, an important topic in the study of the LCP($A, q$) concerns the bound of $\|x - x^*\|_\infty$, since
it can be used as termination criteria for iterative algorithms and can be used to measure the sensitivity of the solution of LCP$(A, q)$ in response to a small perturbation, e.g., [18, 19, 28, 34]. When the matrix $A$ is a $P$-matrix, Chen and Xiang [3] gave the following error bound for the LCP$(A, q)$:

$$
\|x - x^*\|_{\infty} \leq \max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_{\infty} \|r(x)\|_{\infty},
$$

(1.2)

where $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$ for each $i \in N := \{1, \ldots, n\}$, $d = (d_1, d_2, \ldots, d_n) \in [0,1]^n$, and $r(x) = \min\{x, Ax + q\}$ in which the min operator denotes the componentwise minimum of two vectors. Furthermore, to avoid the high-cost computations of the inverse matrix in (1.2), some easily computable bounds for the LCP$(A, q)$ were derived for the different subclass of $P$-matrices, such as $B$-matrices [10, 20], doubly $B$-matrices [6], $SB$-matrices [7, 8], $MB$-matrices [4], $B$-Nekrasov matrices [11, 21], weakly chained diagonally dominant $B$-matrices [22, 32, 35], $B^k_n$-matrices [12, 13, 27], and so on [9, 14–17, 23, 36].

Recently, Li et al. [24] presented a new subclass of $P$-matrices called Dashnic-Zusmanovich type $B$-matrices (DZ-type-$B$-matrices), and provided an error bound for the LCP$(A, q)$ when $A$ is a DZ-type-$B$-matrix.

**Definition 1.1.** [33] A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, with $n \geq 2$, is a DZ-type matrix if for each $i \in N$, there exists $j \in N$, $j \neq i$ such that

$$
(a_{ij} - r_i^j(A)) |a_{jj}| > |a_{ij}| r_j(A),
$$

where $r_i^j(A) = r_i(A) - |a_{ij}|$ and $r_j(A) = \sum_{j \in N \setminus \{i\}} |a_{ij}|$.

**Definition 1.2.** [24] Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be written in the form

$$
A = B^+ + C,
$$

(1.3)

where

$$
B^+ = [b_{ij}] = \begin{bmatrix}
    a_{11} - r_1^1 & \cdots & a_{1n} - r_1^n \\
    \vdots & \ddots & \vdots \\
    a_{n1} - r_n^1 & \cdots & a_{nn} - r_n^n
\end{bmatrix}, \quad
C = \begin{bmatrix}
    r_1^+ & \cdots & r_1^+ \\
    \vdots & \ddots & \vdots \\
    r_n^+ & \cdots & r_n^+
\end{bmatrix},
$$

and $r_i^+ := \max\{0, a_{ij} | j \neq i\}$. Then, $A$ is called a DZ-type-$B$-matrix if $B^+$ is a DZ-type matrix with all positive diagonal entries.

**Theorem 1.1.** [24, Theorem 6] Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a DZ-type-$B$-matrix, and $B^+ = [b_{ij}]$ be the matrix of (1.3). Then,

$$
\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_{\infty} \leq (n - 1) \cdot \max_{a \in N} \min_{j \in \gamma_i(B^+)} \zeta_{ij}(B^+),
$$

(1.4)

where

$$
\gamma_i(B^+) := \left\{ j \in N \setminus \{i\} : (|b_{ij} - r_i^j(B^+)| |b_{jj}| > |b_{ij}| r_j(B^+)) \right\},
$$

and

$$
\zeta_{ij}(B^+) := \frac{\left(b_{ii} - r_i^j(B^+)\right) b_{ij} \max\left\{\frac{1}{b_{ii} - r_i^j(B^+)}, 1\right\} + b_{jj}|b_{ij}| \max\left\{\frac{1}{b_{jj}}, 1\right\}}{(b_{ii} - r_i^j(B^+)) b_{jj} - |b_{ij}| r_j(B^+)}.
$$
Very recently, Cvetković et al. [5] proposed a new subclass of $H$-matrices called CKV-type matrices, which generalizes CKV matrices (also known as $\Sigma$-SDD matrices in the literature) and DZ-type matrices.

**Definition 1.3.** [5] A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, with $n \geq 2$, is called a CKV-type matrix if $N_A = \emptyset$ or $S^+_i(A)$ is not empty for all $i \in N_A$, where $N_A := \{i \in N : |a_{ii}| \leq r_i(A)\}$ and

$$S^+_i(A) := \left\{ S \in \Sigma(i) : |a_{ii}| > r^S_i(A), \text{ and for all } j \in \overline{S} \right\}$$

with $\Sigma(i) := \{S \subseteq N : i \in S\}$ and $r^S_i(A) := \sum_{j \in S \setminus \{i\}} |a_{ij}|$.

Motivated by the definition of DZ-type-$B$-matrices, two meaningful questions naturally arise: can we get a more general subclass of $P$-matrices using CKV-type matrices, and can we obtain a sharper error bound than the bound (1.4) for the linear complementarity problem of DZ-type-$B$-matrices? To answer these questions, in Section 2, we present a new class of matrices: CKV-type $B$-matrices, and prove that it is a subclass of $P$-matrices containing DZ-type-$B$-matrices and $S$-$B$-matrices. Meanwhile, we give an upper bound for the infinity norm for the inverse of CKV-type $B$-matrices. In Section 3, we give an error bound for the LCP($A$, $q$) when $A$ is a CKV-type $B$-matrix, consequently, for the LCP($A$, $q$) when $A$ is a DZ-type-$B$-matrix, and some comparisons with other results are also discussed. Finally, in Section 4, numerical examples are given to illustrate the corresponding theoretical results.

### 2. CKV-type $B$-matrices

Using CKV-type matrices, we first give the definition of CKV-type $B$-matrices.

**Definition 2.1.** A matrix $A \in \mathbb{R}^{n \times n}$ is called a CKV-type $B$-matrix if $B^+$ given by (1.3) is a CKV-type matrix with positive diagonal entries.

To show that a CKV-type $B$-matrix is a $P$-matrix, we recall the following results.

**Lemma 2.1.** [5, Theorem 6] Every CKV-type matrix is a nonsingular $H$-matrix.

**Lemma 2.2.** [29, Corollary 2.4] If $A$ is a real nonsingular $M$-matrix and $P$ is a nonnegative matrix with rank$(P) = 1$, then $A + P$ is a $P$-matrix.

**Proposition 2.1.** If $A$ is a CKV-type $B$-matrix, then $A$ is a $P$-matrix.

**Proof.** Let $A$ be written in the form $A = B^+ + C$ as shown in (1.3). It follows from (1.3) and Definition 2.1 that $B^+$ is a $Z$-matrix (all non-diagonal entries are non-positive [1]) with positive diagonal entries and $C$ is a nonnegative matrix of rank 1. By Lemma 2.1, we know that $B^+$ is a nonsingular $H$-matrix, and thus the conclusion follows from Lemma 2.2. \qed

As shown in [5, 33], the relations of strictly diagonally dominant (SDD) matrices, doubly strictly diagonally dominant (DSDD) matrices, $S$-strictly diagonally dominant ($S$-SDD) matrices, DZ-type matrices, and CKV-type matrices are:
According to [24] and the above relations, we give a figure to illustrate the relations among $B$-matrices, $DZ$-type-$B$-matrices, $DB$-matrices, $SB$-matrices, CKV-type $B$-matrices. Here, the notions of $B$-matrices, $DB$-matrices, and of $SB$-matrices are listed as follows. Let $A = B^+ + C \in \mathbb{R}^{n \times n}$, where $B^+$ is defined by (1.3). Then, $A$ is called

- a $B$-matrix if $B^+$ is SDD with all positive diagonal entries [30];
- a $DB$-matrix if $B^+$ is DSDD with all positive diagonal entries [29];
- a $SB$-matrix if $B^+$ is $S$-SDD with all positive diagonal entries for a given non-empty proper subset $S$ of $N$ [25].

![Figure 1. Relations of CKV-type $B$-matrices and some existing subclasses of $P$-matrices.](image)

Next, we give a sufficient and necessary condition for a CKV-type $B$-matrix. Before that, a lemma is needed.

**Lemma 2.3.** [5, Remark 9] A matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, with $n \geq 2$, is called a CKV-type matrix if $S^*_i(A)$ given by Definition 1.3 is not empty for all $i \in N$. Especially, if $A$ is an SDD matrix, then for all $i \in N$, all proper subsets $S$ containing $i$ belong to $S^*_i(A)$.

**Proposition 2.2.** Given any diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ with $d_i \in [0, 1]$ for all $i \in N$, and let $I$ be the identity matrix, then $A$ is a CKV-type $B$-matrix if and only if $I - D + DA$ is a CKV-type $B$-matrix.

**Proof.** Sufficiency is clearly established. We next show the necessity. Suppose $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a CKV-type $B$-matrix. Then, $A = B^+ + C$, where $B^+$ is the matrix of (1.3). Let $\bar{A} := I - D + DA = [\bar{a}_{ij}]$ and $\bar{A} := I - D + DA = \bar{B} + \bar{C}$, where

$$
\bar{B}^+ = \begin{bmatrix}
\bar{a}_{11} - \bar{r}_1^+ & \cdots & \bar{a}_{1n} - \bar{r}_1^+ \\
\vdots & \ddots & \vdots \\
\bar{a}_{n1} - \bar{r}_n^+ & \cdots & \bar{a}_{nn} - \bar{r}_n^+
\end{bmatrix}, \quad \bar{C} = \begin{bmatrix}
\bar{r}_1^+ & \cdots & \bar{r}_1^+ \\
\vdots & \ddots & \vdots \\
\bar{r}_n^+ & \cdots & \bar{r}_n^+
\end{bmatrix}.
$$
and \( \bar{r}_i^+ := \max\{0, \bar{a}_{ij} \mid j \neq i \} \).

Note that
\[
\bar{a}_{ij} = \begin{cases} 
1 - d_i + d_i a_{ii}, & i = j, \\
1 - d_i + a_{ij}, & i \neq j.
\end{cases}
\]

It follows that
\[
\bar{r}_i^+ := \max\{0, \bar{a}_{ij} \mid j \neq i \} = \max\{0, d_i a_{ij} \mid j \neq i \} = d_i \max\{0, a_{ij} \mid j \neq i \} = d_i r_i^+,
\]
and
\[
\bar{a}_{ij} - \bar{r}_i^+ = \begin{cases} 
1 - d_i + d_i (a_{ii} - r_i^+) & i = j, \\
1 - d_i + (a_{ij} - r_i^+) & i \neq j.
\end{cases}
\]

Since \( A = B^+ + C \), it holds that
\[
\bar{A} = I - D + DA = I - D + D(B^+ + C) = (I - D + DB^+) + DC.
\]

Note that \( B^+ = [b_{ij}] \) and \( b_{ij} = a_{ij} - r_i^+ \). Hence, from (2.1) and (2.2), we easily obtain that
\[
\bar{B}^+ = I - D + DB^+ \quad \text{and} \quad \bar{C} = DC.
\]

Let \( \bar{B}^+ = [\bar{b}_{ij}] \). Then,
\[
\bar{b}_{ij} = \begin{cases} 
1 - d_i + d_i b_{ii}, & i = j, \\
1 - d_i + b_{ij}, & i \neq j.
\end{cases}
\]

Since \( A \) is a CKV-type \( B \)-matrix, then \( B^+ = [b_{ij}] \) is a CKV-type matrix with positive diagonal entries. Thus, by Lemma 2.3, it follows that for each \( i \in N \), there exists \( S \in S^*_i(B^+) \), which implies that
\[
\begin{cases} 
|b_{ii}| > r_i^+(B^+), \\
(|b_{ii} - r_i^+(B^+)| - r_i^+(B^+)) > r_i^+(B^+) r_j^+(B^+) \quad \text{for all} \quad j \in \bar{S}.
\end{cases}
\]

Hence, for each \( i \in N \), it follows from (2.3) that
\[
\bar{b}_{ii} - r_i^+(B^+) = 1 - d_i + d_i (b_{ii} - r_i^+(B^+)) > 0,
\]
and that for all \( j \in \bar{S} \), if \( d_i \neq 0 \) and \( d_j \neq 0 \), then
\[
\left( [\bar{b}_{ii}] - r_i^+(B^+) \right) \left( [\bar{b}_{jj}] - r_j^+(B^+) \right) = \left[ 1 - d_i + d_i (b_{ii} - r_i^+(B^+)) \right] \left[ 1 - d_j + d_j (b_{jj} - r_j^+(B^+)) \right] \geq d_i (b_{ii} - r_i^+(B^+)) d_j (b_{jj} - r_j^+(B^+))
\]
\[
> d_i r_i^+(B^+) d_j r_j^+(B^+)
\]
\[
= r_i^+(B^+) r_j^+(B^+),
\]
and if \( d_i = 0 \) or \( d_j = 0 \), then
\[
\left( [\bar{b}_{ii}] - r_i^+(B^+) \right) \left( [\bar{b}_{jj}] - r_j^+(B^+) \right) = \left[ 1 - d_i + d_i (b_{ii} - r_i^+(B^+)) \right] \left[ 1 - d_j + d_j (b_{jj} - r_j^+(B^+)) \right]
\]
These mean that \( S \in S_i^*(\overline{B}^+) \) for each \( i \in N \). Therefore, from Definition 1.3, \( \overline{B}^+ \) is a CKV-type matrix with positive diagonal entries, and consequently, \( \overline{A} = I - D + DA \) is a CKV-type \( B \)-matrix from Definition 2.1.

In the following, we give an infinity norm bound for the inverse of CKV-type \( B \)-matrices. First, two lemmas are listed.

**Lemma 2.4.** [5] Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2, \) be a CKV-type \( B \)-matrix. Then

\[
\|A^{-1}\|_\infty \leq \max \min_{i \in N} \max_{j \in S} \beta_{ij}^S(A),
\]

where \( S_i^*(A) \) is given by Definition 1.3, and

\[
\beta_{ij}^S(A) = \frac{|a_{ij}| - r_i^+(A) + r_j^+(A)}{|a_{ii}| - r_i^+(A) - r_j^+(A) - r_i^+(A)r_j^+(A)}. \]

**Lemma 2.5.** [11] Suppose \( P = (p_1, \ldots, p_n)^T e, \) where \( e = (1, \ldots, 1) \) and \( p_i \geq 0 \) for all \( i \in N, \) then

\[
\|(I + P)^{-1}\|_\infty \leq n - 1.
\]

**Theorem 2.1.** Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n}, n \geq 2, \) be a CKV-type \( B \)-matrix, and \( B^+ = [b_{ij}] \) be the matrix of (1.3). Then

\[
\|A^{-1}\|_\infty \leq (n - 1) \cdot \max \min_{i \in N} \max_{j \in S} \beta_{ij}^S(B^+),
\]

where \( S_i^*(B^+) \) is defined as in Definition 1.3, and

\[
\beta_{ij}^S(B^+) = \frac{|b_{ij}| - r_j^+(B^+) + r_i^+(B^+)}{|b_{ii}| - r_i^+(B^+) - r_j^+(B^+) - r_i^+(B^+)r_j^+(B^+)}. \]

**Proof.** Since \( A \) is a CKV-type \( B \)-matrix, so \( B^+ \) is a CKV-type matrix with positive diagonal entries and also a \( Z \)-matrix. By Corollary 4 of [31], we know that \( B^+ \) is an \( M \)-matrix and thus \( (B^+)^{-1} \) is nonnegative. Hence, from \( A = B^+ + C \) in which \( B^+ \) and \( C \) are given by (1.3), we have

\[
A^{-1} = \left( B^+ \left( I + (B^+)^{-1}C \right) \right)^{-1} = \left( I + (B^+)^{-1}C \right)^{-1} (B^+)^{-1},
\]

which implies that

\[
\|A^{-1}\|_\infty \leq \|(I + (B^+)^{-1}C)^{-1}\|_\infty \cdot \|(B^+)^{-1}\|_\infty. \tag{2.4}
\]

Note that \( C = (r_1^+, \ldots, r_n^+)^T e \) is nonnegative. Therefore, \( (B^+)^{-1}C \) can be written as \( (p_1, \ldots, p_n)^T e, \) where \( p_i \geq 0 \) for all \( i \in N. \) By Lemma 2.5, we get

\[
\|(I + (B^+)^{-1}C)^{-1}\|_\infty \leq n - 1. \tag{2.5}
\]

Since \( B^+ \) is a CKV-type matrix, it follows from Lemma 2.4 that

\[
\|(B^+)^{-1}\|_\infty \leq \max_{i \in N} \min_{S \in S_i^*(B^+)} \max_{j \in S} \beta_{ij}^S(B^+). \tag{2.6}
\]

Hence, from (2.4), (2.5), and (2.6), the conclusion follows. \( \square \)
3. Error bounds for the linear complementarity problem

Based on Theorem 2.1, we give in this section an upper bound of \( \|(I - D + DA)^{-1}\|_\infty \) when \( A \) is a CKV-type \( B \)-matrix, and give some comparisons with other results. Before that, a useful lemma is needed.

Lemma 3.1. [20, Lemma 3] Let \( \gamma > 0 \) and \( \eta \geq 0 \). Then for any \( x \in [0, 1] \),

\[
\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min\{\gamma, 1\}}
\]

and

\[
\frac{\eta x}{1 - x + \gamma x} \leq \frac{\eta}{\gamma}.
\]

Theorem 3.1. Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \), \( n \geq 2 \), be a CKV-type \( B \)-matrix, and \( B^* = [b_{ij}] \) be the matrix of (1.3). Then

\[
\max_{d \in [0, 1]^p} \|(I - D + DA)^{-1}\|_\infty \leq (n - 1) \cdot \max_{i \in N} \min_{S \in S_i^*(B^* \}) \max_{j \in \overline{S}} \alpha^S_{ij}(B^*),
\]

where \( S_i^*(B^*) \) is defined as in Definition 1.3, and

\[
\alpha^S_{ij}(B^*) = \frac{\left( b_{ii} - r^S_i(B^*) \right) \left( b_{jj} - r^S_j(B^*) \right) \max \left\{ \frac{1}{b_{ii} - r^S_i(B^*)}, 1 \right\} + \left( b_{jj} - r^S_j(B^*) \right) r^S_i(B^*) \max \left\{ \frac{1}{b_{jj} - r^S_j(B^*)}, 1 \right\}}{\left( b_{ii} - r^S_i(B^*) \right) \left( b_{jj} - r^S_j(B^*) \right) - r^S_i(B^*) r^S_j(B^*)}.
\]

Proof. Since \( A \) is a CKV-type \( B \)-matrix, by Proposition 2.2, it follows that \( I - D + DA \) is also a CKV-type \( B \)-matrix. Taking into account that \( A = B^* + C \) in which \( B^* \) and \( C \) are defined as (1.3), then

\[
I - D + DA = I - D + D(B^* + C) = I - D + DB^* + DC.
\]

Denote \( \overline{B}^* := I - D + DB^* = [\overline{b}_{ij}] \) and \( \overline{C} = DC \). Then, from Theorem 2.1, we have

\[
\|(I - D + DA)^{-1}\|_\infty \leq (n - 1) \cdot \max_{i \in N} \min_{S \in S_i^*(B^* \}) \max_{j \in \overline{S}} \beta^S_{ij}(\overline{B}^*).
\]

By Lemma 3.1, it follows that for all \( i \in N \), \( j \in \overline{S} \),

\[
\beta^S_{ij}(\overline{B}^*) = \frac{\left| \overline{b}_{ij} - r^S_i(\overline{B}^*) \right| + r^S_j(\overline{B}^*)}{\left( b_{ii} - r^S_i(B^*) \right) \left( b_{jj} - r^S_j(B^*) \right) - r^S_i(B^*) r^S_j(B^*)} = \frac{1 - d_i + d_j \left( b_{ii} - r^S_i(B^*) \right) + d_i r^S_j(\overline{B}^*)}{\left( b_{ii} - r^S_i(B^*) \right) \left( b_{jj} - r^S_j(B^*) \right) - r^S_i(B^*) r^S_j(B^*)} = \frac{1}{1 - d_i + d_j \left( b_{ii} - r^S_i(B^*) \right)} + \frac{d_i r^S_j(\overline{B}^*)}{\left( b_{ii} - r^S_i(B^*) \right) \left( b_{jj} - r^S_j(B^*) \right) - r^S_i(B^*) r^S_j(B^*)}.
\]
\[
\frac{1}{b_{ii} - r_i^b(B^+)} + \frac{1}{b_{jj} - r_j^\bar{B}(B^+)} \geq 1 - \frac{r_i^b(B^+)}{b_{ii} - r_i^b(B^+)} \frac{r_j^\bar{B}(B^+)}{b_{jj} - r_j^\bar{B}(B^+)}
\]

\[
\max \left\{ \frac{1}{b_{ii} - r_i^b(B^+)}, 1 \right\} + \max \left\{ \frac{1}{b_{jj} - r_j^\bar{B}(B^+)}, 1 \right\} \geq \max \left\{ \frac{1}{b_{ii} - r_i^b(B^+)}, 1 \right\} \frac{r_i^b(B^+)}{b_{ii} - r_i^b(B^+)} \frac{r_j^\bar{B}(B^+)}{b_{jj} - r_j^\bar{B}(B^+)}
\]

\[
\left( b_{ii} - r_i^b(B^+) \right) \left( b_{jj} - r_j^\bar{B}(B^+) \right) \max \left\{ \frac{1}{b_{ii} - r_i^b(B^+)}, 1 \right\} + \left( b_{jj} - r_j^\bar{B}(B^+) \right) \max \left\{ \frac{1}{b_{jj} - r_j^\bar{B}(B^+)}, 1 \right\}
\]

\[
= \left( b_{ii} - r_i^b(B^+) \right) \left( b_{jj} - r_j^\bar{B}(B^+) \right) - r_i^b(B^+) r_j^\bar{B}(B^+)
\]

\[
= \alpha_{ij}^S(B^+).
\]

Furthermore, by the proof of Proposition 2.2, \( S_i^s(B^+) \subseteq S_i^s(\bar{B}^+) \) for each \( i \in N \). Thus,

\[
\max_{i \in N} \min_{S \in S_i^s(B^+)} \max_{j \in \bar{S}} \beta_{ij}^s(\bar{B}^+) \leq \max_{i \in N} \min_{S \in S_i^s(B^+)} \max_{j \in S} \alpha_{ij}^S(B^+). \tag{3.3}
\]

Hence, the conclusion follows from (3.2) and (3.3). \( \square \)

**Remark 3.1.** Note that, if \( b_{ii} - r_i^b(B^+) \leq 1 \) and \( b_{jj} - r_j^\bar{B}(B^+) \leq 1 \), then

\[
\alpha_{ij}^S(B^+) = \frac{b_{jj} - r_j^\bar{B}(B^+) + r_i^b(B^+)}{b_{ii} - r_i^b(B^+) - r_j^\bar{B}(B^+)}. \]

If \( b_{ii} - r_i^b(B^+) > 1 \) and \( b_{jj} - r_j^\bar{B}(B^+) \leq 1 \), then

\[
\alpha_{ij}^S(B^+) = \frac{\left( b_{ii} - r_i^b(B^+) \right) \left( b_{jj} - r_j^\bar{B}(B^+) \right) + r_i^b(B^+)}{b_{jj} - r_j^\bar{B}(B^+) - r_i^b(B^+)}. \]

If \( b_{ii} - r_i^b(B^+) \leq 1 \) and \( b_{jj} - r_j^\bar{B}(B^+) > 1 \), then

\[
\alpha_{ij}^S(B^+) = \frac{b_{jj} - r_j^\bar{B}(B^+) + \left( b_{jj} - r_j^\bar{B}(B^+) \right) r_i^b(B^+)}{\left( b_{ii} - r_i^b(B^+) \right) - r_i^b(B^+)}. \]

If \( b_{ii} - r_i^b(B^+) > 1 \) and \( b_{jj} - r_j^\bar{B}(B^+) > 1 \), then

\[
\alpha_{ij}^S(B^+) = \frac{\left( b_{ii} - r_i^b(B^+) \right) \left( b_{jj} - r_j^\bar{B}(B^+) \right) + \left( b_{jj} - r_j^\bar{B}(B^+) \right) r_i^b(B^+)}{\left( b_{ii} - r_i^b(B^+) \right) - r_i^b(B^+)}. \]

Since a DZ-type-B matrix is a CKV-type B-matrix, the bound (3.1) can also be used to estimate \( \max_{d \in [0,1]^n} \| (I - D + DA)^{-1} \|_\infty \) when \( A \) is a DZ-type-B matrix. The following theorem provides that the bound (3.1) is better than the bound (1.4) in Theorem 1.1 (Theorem 6 of [24]).

**Theorem 3.2.** Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) be a DZ-type-B matrix, and \( B^+ = [b_{ij}] \) be the matrix of (1.3). Then (3.1) holds. Furthermore,

\[
\max_{i \in N} \min_{S \in S_i^s(B^+)} \max_{j \in S} \zeta_{ij}(B^+) \leq \max_{i \in N} \min_{S \in S_i^s(B^+)} \max_{j \in \gamma_i(B^+)} \zeta_{ij}(B^+),
\]

where \( \gamma_i(B^+) \) and \( \zeta_{ij}(B^+) \) are given by Theorem 1.1, \( S_i^s(B^+) \) and \( \alpha_{ij}^S(B^+) \) are defined in Theorem 3.1.
Proof. For each \( i \in N \), note that
\[
\gamma_i(B^+) := \left\{ j \in N \mid \{ [b_{ii} - r_i^j(B^+)] |b_{jj}| > |b_{ij}|r_j(B^+) \} \right\},
\]
and
\[
S_i^*(B^+) := \left\{ S \in \Sigma(i) : |b_{ii}| > r_i^S(B^+) \right\}, \quad \text{for all } j \in \overline{S},
\]
\[
(b_{ii} - r_i^j(B^+))(b_{jj} - r_j^S(B^+)) \max \left\{ \frac{1}{b_{ii} - r_i^j(B^+)} , 1 \right\} + (b_{jj} - r_j^S(B^+))^2 \max \left\{ \frac{1}{b_{jj} - r_j^S(B^+)} , 1 \right\}
\]
\[
\left( b_{ii} - r_i^j(B^+) \right) \left( b_{jj} - r_j^S(B^+) \right) - r_i^S(B^+)r_j^S(B^+)
\]
\[
\frac{\left( b_{ii} - r_i^j(B^+) \right) b_{jj} \max \left\{ \frac{1}{b_{ii} - r_i^j(B^+)} , 1 \right\} + b_{jj}|b_{ij}| \max \left\{ \frac{1}{b_{ij}} , 1 \right\}}{b_{ii} - r_i^j(B^+)} b_{jj} - |b_{ij}|r_j(B^+)
\]
\[
= \zeta_{ij}(B^+).
\]
It is easy to see that \( j \in \gamma_i(B^+) \) is equivalent to \( S = N \setminus \{ j \} \in S_i^*(B^+) \). Therefore, for each \( i \in N \),
\[
\min_{S \in S_i^*(B^+)} \max_{j \in S} \alpha_{ij}^S(B^+) = \min \left\{ \min_{S = N \setminus \{ j \} \in S_i^*(B^+)} \max_{j \in S} \alpha_{ij}^S(B^+), \min_{S \in S_i^*(B^+)} \max_{j \in S} \alpha_{ij}^S(B^+) \right\}
\]
\[
= \min \left\{ \min_{j \notin \gamma_i(B^+)} \zeta_{ij}(B^+), \min_{S \in S_i^*(B^+) \setminus \{ N \setminus \{ j \} \} \in S_i^*(B^+)} \max_{j \in S} \alpha_{ij}^S(B^+) \right\}
\]
\[
\leq \min_{j \notin \gamma_i(B^+)} \zeta_{ij}(B^+).
\]
This completes the proof. \( \square \)

Particularly, for \( B \)-matrices, as an important subclass of CKV-type \( B \)-matrices, we next show that the bound (3.1) is better than that given by García-Esnaola and Peña in [10] in some cases.

**Theorem 3.3.** [10, Theorem 2.3] Let \( A \in \mathbb{R}^{n \times n} \) be a \( B \)-matrix, and \( B^+ = [b_{ij}] \) be the matrix of (1.3). Let \( \beta_i := b_{ii} - r_i(B^+) \) and \( \beta := \min_{i \in N} \beta_i \). Then
\[
\max_{d \in [0,1]^n} \| (I - D + DA)^{-1} \|_\infty \leq (n - 1) \cdot \frac{1}{\min \{\beta, 1\}}.
\]  \hspace{1cm} (3.4)
\textbf{Theorem 3.4.} Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a $B$-matrix, and $B^+ = [b_{ij}]$ be the matrix of (1.3). Let $\beta_i := b_{ii} - r_i(B^+)$, $\beta := \min \{\beta_i\}$, and $S$ be a nonempty proper subset of $N$. Then (3.1) holds. Furthermore, if $b_{ii} - r_i^\delta(B^+) \leq 1$ for all $i \in N$ and $b_{jj} - r_j^\delta(B^+) \leq 1$ for all $j \in S$, then,

$$\max_{i \in N} \min_{S \in S^*(B^+)} \max_{j \in S} \alpha_{ij}^\delta(B^+) \leq \frac{1}{\min \{\beta_i\}},$$

(3.5)

and if $\beta_i > 1$ for each $i \in N$, then

$$\max_{i \in N} \min_{S \in S^*(B^+)} \max_{j \in S} \alpha_{ij}^\delta(B^+) \geq \frac{1}{\min \{\beta_i\}},$$

(3.6)

where $\alpha_{ij}^\delta(B^+)$ is defined as in Theorem 3.1.

\textbf{Proof.} By the fact that a $B$-matrix is a CKV-type $B$-matrix, we know that (3.1) holds directly. We now prove that (3.5) and (3.6) hold. For each $i \in N$, $S \in S^*(B^+)$, and $j \in S$, if $b_{ii} - r_i^\delta(B^+) \leq 1$ and $b_{jj} - r_j^\delta(B^+) \leq 1$, then from Remark 3.1 that

$$\alpha_{ij}^\delta(B^+) = \frac{b_{jj} - r_j^\delta(B^+) + r_i^\delta(B^+)}{\left(b_{ii} - r_i^\delta(B^+)\right)\left(b_{jj} - r_j^\delta(B^+)\right) - r_i^\delta(B^+)r_j^\delta(B^+)}.$$ 

If $b_{jj} - r_j^\delta(B^+) < b_{ii} - r_i^\delta(B^+)$, then

$$\left(b_{jj} - r_j^\delta(B^+)\right) - r_j^\delta(B^+) < \left(b_{ii} - r_i^\delta(B^+)\right) - r_i^\delta(B^+),$$

which implies that

$$\left(b_{jj} - r_j^\delta(B^+)\right) - r_j^\delta(B^+) + r_i^\delta(B^+) \left(b_{jj} - r_j^\delta(B^+)\right) - r_i^\delta(B^+)r_j^\delta(B^+) < \left(b_{ii} - r_i^\delta(B^+)\right)\left(b_{jj} - r_j^\delta(B^+)\right) - r_i^\delta(B^+)r_j^\delta(B^+),$$

i.e.

$$\left[b_{jj} - r_j^\delta(B^+)\right] \left[b_{jj} - r_j^\delta(B^+)\right] + r_i^\delta(B^+) \left[b_{jj} - r_j^\delta(B^+)\right] < \left(b_{ii} - r_i^\delta(B^+)\right)\left(b_{jj} - r_j^\delta(B^+)\right) - r_i^\delta(B^+)r_j^\delta(B^+).$$

It follows that

$$\frac{\left(b_{jj} - r_j^\delta(B^+)\right) + r_i^\delta(B^+)}{\left(b_{ii} - r_i^\delta(B^+)\right)\left(b_{jj} - r_j^\delta(B^+)\right) - r_i^\delta(B^+)r_j^\delta(B^+)} < \frac{1}{\left(b_{jj} - r_j^\delta(B^+)\right) - r_i^\delta(B^+)}$$

$$= \frac{1}{b_{jj} - r_j^\delta(B^+)}$$

$$= \max \left\{ \frac{1}{b_{ii} - r_i^\delta(B^+)} \cdot \frac{1}{b_{jj} - r_j^\delta(B^+)} \right\}$$

$$\leq \frac{1}{\min \{\beta_i\}}.$$ (3.7)
If \( b_{jj} - r_j(B^+) \geq b_{ii} - r_i(B^+) \), then

\[
(b_{jj} - r_j^\delta(B^+)) - r_j^\delta(B^+) \geq (b_{ii} - r_i^\delta(B^+)) - r_i^\delta(B^+),
\]

implying that

\[
(b_{ii} - r_i^\delta(B^+))(b_{jj} - r_j^\delta(B^+)) - (b_{jj} - r_j^\delta(B^+)) r_j^\delta(B^+) + (b_{ii} - r_i^\delta(B^+)) r_j^\delta(B^+) - (r_j^\delta(B^+))^2 \leq (b_{ii} - r_i^\delta(B^+))(b_{jj} - r_j^\delta(B^+)) - r_j^\delta(B^+) r_j^\delta(B^+),
\]

i.e.

\[
\left[ (b_{ii} - r_i^\delta(B^+)) - r_j^\delta(B^+) \right] \cdot \left[ (b_{jj} - r_j^\delta(B^+)) + r_j^\delta(B^+) \right] \leq (b_{ii} - r_i^\delta(B^+))(b_{jj} - r_j^\delta(B^+)) - r_j^\delta(B^+) r_j^\delta(B^+).
\]

It holds that

\[
\frac{(b_{jj} - r_j^\delta(B^+)) + r_j^\delta(B^+)}{(b_{ii} - r_i^\delta(B^+))(b_{jj} - r_j^\delta(B^+)) - r_j^\delta(B^+) r_j^\delta(B^+)} \leq \frac{1}{(b_{ii} - r_i^\delta(B^+)) - r_j^\delta(B^+)} = \frac{1}{b_{ii} - r_i(B^+)} = \max \left\{ \frac{1}{b_{ii} - r_i(B^+), b_{jj} - r_j(B^+)} \right\} \leq \frac{1}{\min[\beta, 1]},
\]

(3.8)

Hence, (3.5) follows from (3.7) and (3.8).

If \( \beta_i := b_{ii} - r_i(B^+) > 1 \) for each \( i \in N \), then

\[ b_{ii} - r_i^\delta(B^+) \geq b_{ii} - r_i(B^+) \geq 1 \text{ and } b_{jj} - r_j^\delta(B^+) \geq b_{jj} - r_j(B^+) > 1. \]

By Remark 3.1, we can see that

\[
\alpha_{ij}^\delta(B^+) = \frac{(b_{ii} - r_i^\delta(B^+))(b_{jj} - r_j^\delta(B^+)) + (b_{jj} - r_j^\delta(B^+)) r_j^\delta(B^+)}{(b_{ii} - r_i^\delta(B^+))(b_{jj} - r_j^\delta(B^+)) - r_j^\delta(B^+) r_j^\delta(B^+)} \geq \frac{(b_{ii} - r_i^\delta(B^+))(b_{jj} - r_j^\delta(B^+)) + r_j^\delta(B^+) r_j^\delta(B^+)}{(b_{ii} - r_i^\delta(B^+))(b_{jj} - r_j^\delta(B^+)) - r_j^\delta(B^+) r_j^\delta(B^+)} \geq 1.
\]

Therefore,

\[
\max_{i \in N} \min_{S \in S_{\mathbf{K}(B^+)}^j} \max_{j \in S} \alpha_{ij}^\delta(B^+) \geq 1 = \frac{1}{\min[\beta, 1]}.
\]

The proof is complete. \( \Box \)

Remark from Theorem 3.4 that we can take the minimum of bounds (3.1) and (3.4) to estimate the error bound for the LCP(\( A, q \)) with \( A \) being a B-matrix, that is,

\[
\max_{\delta \in [0, 1]} ||(I - D + DA)^{-1}||_\infty \leq (n - 1) \cdot \min \left\{ \frac{1}{\min[\beta, 1]}, \max_{\delta \in [0, 1]} \min_{i \in N} \min_{S \in S_{\mathbf{K}(B^+)}^j} \max_{j \in S} \alpha_{ij}^\delta(B^+) \right\}.
\]
4. Numerical examples

In this section, three examples are given to show the advantage of the bound (3.1) in Theorem 3.1.

Example 4.1. Consider the following matrix

\[ A = \begin{bmatrix} 4 & 0 & -2 & -2 \\ 0 & 3 & -2 & -2 \\ -1 & -1 & 6 & -2 \\ -1 & -1 & -2 & 6 \end{bmatrix}. \]

Obviously, \( B^+ = A \) and \( C = 0 \). It is easy to verify that \( B^+ \) is not a DZ-type matrix and an \( S \)-SDD matrix, consequently, not a SDD matrix and a DSDD matrix. Hence, \( A \) is not a DZ-type-\( B \)-matrix and a \( S \)-\( B \)-matrix, and thus not a \( B \)-matrix and a \( D \)-\( B \)-matrix. So we cannot use the error bounds in [6–8, 10, 20, 24] to estimate \( \max_{d \in [0, 1]} ||(I - D + DA)^{-1}||_{\infty} \). However, by calculations, one has that \( B^+ \) is a CKV-type matrix with positive diagonal entries, and thus \( A \) is a CKV-type \( B \)-matrix. So by the bound (3.1) in Theorem 3.1, we get

\[ \max_{d \in [0, 1]} ||(I - D + DA)^{-1}||_{\infty} \leq 21. \]

Example 4.2. Consider the following matrix

\[ A = \begin{bmatrix} 3 & 0 & -2 & -2 \\ 0 & 3 & -2 & -2 \\ -1 & -1 & 6 & 0 \\ -1 & -1 & 0 & 6 \end{bmatrix}. \]

Note that \( r_i^+ := \max\{0, a_{ij} | j \neq i\} = 0 \) for \( i = 1, 2, 3, 4 \). Hence, \( B^+ = A \) and \( C = 0 \). By calculations, we have that \( A \) is a DZ-type-\( B \)-matrix, and thus it is a CKV-type \( B \)-matrix. By the bound (3.1) in Theorem 3.1, we have

\[ \max_{d \in [0, 1]} ||(I - D + DA)^{-1}||_{\infty} \leq 12.6, \]

while by the bound (1.4) in Theorem 1.1, it holds that

\[ \max_{d \in [0, 1]} ||(I - D + DA)^{-1}||_{\infty} \leq 27. \]

Obviously, the bound (3.1) is sharper than bound (1.4) in Theorem 1.1 (Theorem 6 of [24]).

Example 4.3. Consider the \( B \)-matrix

\[ A = \begin{bmatrix} 3 & -1 & -1 & -\frac{1}{2} \\ -1 & 3 & -1 & -\frac{1}{2} \\ -1 & -1 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

Note that \( B^+ = A \), \( C = 0 \). Then, by the bound (3.4) in Theorem 3.3, we have

\[ \max_{d \in [0, 1]} ||(I - D + DA)^{-1}||_{\infty} \leq 6. \]
In addition, $A$ is also a CKV-type $B$-matrix. By calculations, for $i = 1, 2, 3$, take $S = \{1, 2, 3\}$ and $\overline{S} = \{4\}$, it follows that $b_{ii} - r_i^B(B^+) \leq 1$ for all $i \in \{1, 2, 3, 4\}$ and $b_{44} - r_4^B(B^+) \leq 1$; and for $i = 4$, take $S = \{4\}$ and $\overline{S} = \{1, 2, 3\}$, it follows that $b_{44} - r_4^B(B^+) \leq 1$ for all $i \in \{1, 2, 3, 4\}$ and $b_{jj} - r_j^B(B^+) \leq 1$ for all $j \in \overline{S}$, which satisfy the hypothesis of Theorem 3.4. Therefore, by Theorem 3.4, we get

$$\max_{d \in [0,1]^d} \| (I - D + DA)^{-1} \|_\infty \leq 4.5,$$

which is smaller than the bound (3.4) in Theorem 3.3 (Theorem 2.3 of [10]).

5. Conclusions

In this paper, on the basis of the class of CKV-type matrices, a new subclass of $P$-matrices: CKV-type $B$-matrices, containing $B$-matrices, $DB$-matrices, $SB$-matrices as well as DZ-type-$B$-matrices, is introduced, and an upper bound for the infinity norm for the inverse of CKV-type $B$-matrices is provided. Then, by this bound, an error bound for the corresponding LCP($A, q$) is given. We also proved that the new error bound is sharper than those of [10] and [24] in some cases, and give numerical examples to show the advantage of our results.

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Conflict of interest

The authors declare no conflict of interest.

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