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Research article

Variable step size predictor design for a class of linear discrete-time censored system

Zhifang Li[1], Huihong Zhao[1,*], Hailong Meng[1] and Yong Chen[2]

[1] School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China
[2] North Automatic Control Technology Institute, Taiyuan, China

* **Correspondence:** Email: huihong1980@163.com; Tel: +8618253129269.

Abstract: We propose a novel variable step size predictor design method for a class of linear discrete-time censored system. We divide the censored system into two parts. The system measurement equation in one part doesn't contain the censored data, and the system measurement equation in the other part is the censored signal. For the normal one, we use the Kalman filtering technology to design one-step predictor. For the one that the measurement equation is censored, we determine the predictor step size according to the censored data length and give the gain compensation parameter matrix $\beta(\mathfrak{s})$ for the case predictor with obvious errors applying the minimum error variance trace, projection formula, and empirical analysis, respectively. Finally, a simulation example shows that the variable step size predictor based on empirical analysis has better estimation performance.

Keywords: censored system; Kalman filtering; variable step size predictor; minimum error variance trace; projection formula
Mathematics Subject Classification: 93E11, 93C55

1. Introduction

The censored system is also known as Tobit model, survival model or truncated model, which was first proposed as a regression model of expenditure by the Nobel Prize winner James Tobin [1]. The censored system is different from the discrete selection model and the general continuous variable selection model, which consists of two parts: one is the ideal measurement model which represents the constraint condition; the other is the actual measurement model which satisfies the constraint condition [2]. The estimation problem for censored system has been a very active area of research and has widely applications. For example, in the aspect of economics, a new estimator was proposed to estimate non-zero and unknown censoring threshold, and was applied to charitable donations [3]. In the aspect of traffic, based on the censored system, the signal strength received from the censored data in the system was used as a distance to estimate and control the vehicle position, and the estimation observer was designed [4]. In the aspect of fault detection, the discrete system with output dead-zone was studied as a special kind of censoring system, and the intermittent fault detector was designed [5]. The censored systems were divided into five categories according to different likelihood functions of models [2], the first system was the standard Tobit model, and the other four systems were also called generalized Tobit model. We focus on the state estimation problem of the first type of model, and its output equation is described as follows:

$$\tilde{y}_i = \mathcal{N} x_i + u_i \qquad u_i \sim N(0, \sigma^2),$$

$$y_i = \begin{cases} \tilde{y}_i & if\ \tilde{y}_i > \tau \\ \tau & if\ \tilde{y}_i \leq \tau \end{cases} \qquad i = 1,2,\ldots,n.$$

The previous research mainly focused on parameter estimation of the censored system, such as Bayesian estimation method [6]. In recent years, in order to solve the state estimation problem in the first type of model, numerous methods have been proposed [7–20]. In [7], Allik et al. first proposed a new Tobit Kalman filter theory, and the saturated data converged to the standard Kalman filter when there wasn’t censored. In [8], Allik et al. designed an estimation method of the censored system by using Bayesian theory, projection formula, and probability distribution. Using state augmentation technique and the orthogonality projection principle, Geng et al. designed an estimator for a class of discrete-time system with both censored and fading measurements [9]. In [10], by establishing a new weighted variance formula and introducing Bernoulli random variables, Han et al. dealt with the random matrices and the censoring phenomenon, respectively. In [11,12], the Tobit Kalman filter was extended to a linear discrete time-varying system with time-correlated multiplicative measurement noises. Based on polynomial filtering technique, Zhao proposed a novel non-Gaussian noise estimation method for a class of special censored system [13]. Based on linear Tobit Kalman filter theory, Arthur et al. designed an improved extended Tobit Kalman filter for nonlinear censored system [14]. Li et al. described the censored measurements under the Round-Robin protocol by a new periodical model, and introduced threshold-dependent linear matrix inequalities with dimension periodic variation to obtain the gain matrix of the filter and ensure the filtering performance [15]. Huang and He designed a fault detector by the extended Tobit Kalman filter theory for nonlinear system with censored data [16]. In [17], Huang et al. designed a robust recursive filter and obtained the recursive value of the upper bound of the filtering error variance for censored system, and analyzed the bounded condition of the filtering error variance. Geng et al. designed an improved recursive Tobit Kalman filter and solved the filtering problem under the effect of non-Gaussian and time-correlated additive measurement noise by using the measurement difference method and Lévy-Ito theorem [18]. In [19], Du and Li proposed a strong tracking Tobit Kalman filter which had both censored measurements and model uncertainties. Taking the orthogonal principle as the criterion of the model mismatch, the gain matrix was adjusted adaptively by introducing fading factor into the prior error variance. Loumponias considered the case of interval censoring and improved the standard Tobit Kalman filter [20]. From the present point of view, the traditional Kalman filter, extended Kalman filter, and other methods are not accurate. Although Tobit Kalman filter has been proposed in [8], it is still not ideal from the perspective of simulation effect. Therefore, we seek for a new estimation method to make the estimation accuracy better in this paper.

In this paper, first of all, we summarize the traditional estimation methods of the censored system, one of which is that we use Tobit Kalman filter to estimate the state of censored system. In addition, the unbiased minimum variance estimation method [21] proposed by Gillijns and De Moor is also used to solve the estimation problem for linear censored system. By observing the simulation verification of these two methods, it is found that the simulation effects of these two methods aren't the best. Therefore, we propose a new estimation method--variable step size prediction method. On the one hand, we use Kalman filtering technology to solve the case system without censored data. On the other hand, we propose the variable step size prediction method by introducing compensation term to solve the case system with censored data.

The structure of the paper is arranged as follows. In Section 2, we formulate the estimation problem for linear censored system. In Section 3, we introduce two classical estimation methods. In Section 4, we consider a new estimator, variable step size predictor. Finally, in Section 5, a simulation example is carried out for these methods.

2. Problem formulation

Consider a class of linear censored system, having dynamics:

$$x(\mathfrak{s}+1) = \Xi(\mathfrak{s})x(\mathfrak{s}) + \eth(\mathfrak{s}), \tag{2.1}$$

$$\H{y}(\mathfrak{s}) = \mathcal{N}(\mathfrak{s})x(\mathfrak{s}) + v(\mathfrak{s}), \tag{2.2}$$

with actual measurement

$$y(\mathfrak{s}) = \begin{cases} \H{y}(\mathfrak{s}), & \H{y}(\mathfrak{s}) > \tau, \\ \tau, & \H{y}(\mathfrak{s}) \le \tau, \end{cases} \tag{2.3}$$

where (2.1) represents the process model, (2.2) and (2.3) represent the measurement model, $\mathfrak{s} \in N$ is the discrete-time index, $x(\mathfrak{s}) \in \mathcal{R}^n$ is the state vector, $\H{y}(\mathfrak{s}) \in \mathcal{R}^p$ is the hidden measurement vector, $y(\mathfrak{s}) \in \mathcal{R}^p$ is the measurement vector of actual measurement, and $\tau = \left[\tau_1\ \tau_2\ ...\ \tau_p\right]^{\mathrm{T}}$ is the threshold vector. $\Xi(\mathfrak{s}) \in \mathcal{R}^{n\times n}$ and $\mathcal{N}(\mathfrak{s}) \in \mathcal{R}^{p\times n}$ are known matrix parameters.

Assumption 1. *The initial state $x(0)$ is unknown and uncorrelated to $\eth(\mathfrak{s})$ and $v(\mathfrak{s})$ that satisfies*

$$Ex(0) = \mathring{U}(0),$$

$$E\left[\left(x(0) - \mathring{U}(0)\right)\left(x(0) - \mathring{U}(0)\right)^{\mathrm{T}}\right] = \mathcal{J}(0).$$

Assumption 2. *The $\eth(\mathfrak{s}) \in \mathcal{R}^n$ and $v(\mathfrak{s}) \in \mathcal{R}^p$ are uncorrelated white noise with zero mean that satisfy*

$$E\eth(\mathfrak{s}) = 0,$$

$$E[\eth(\mathfrak{s})\eth^{\mathrm{T}}(j)] = Q\delta_{\mathfrak{s}j},$$

$$Ev(\mathfrak{s}) = 0,$$

$$E[v(\mathfrak{s})v^{\mathrm{T}}(j)] = R\delta_{\mathfrak{s}j},$$

where $\delta_{\mathfrak{s}\mathfrak{s}} = 1$, $\delta_{\mathfrak{s}j} = 0(\mathfrak{s} \neq j)$.

In this paper, the problem is to find the linear minimum variance estimation of state process $x(\mathfrak{s})$ of a class of discrete-time censored system. Throughout this paper, we denote $\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$ as a linear function based on measurement sequence $\{y(0), y(1), \dots y(\mathfrak{s}-1)\}$ that minimizes the error variance trace $\{E\{[x(\mathfrak{s}) - \hat{x}(\mathfrak{s}|\mathfrak{s}-1)][x(\mathfrak{s}) - \hat{x}(\mathfrak{s}|\mathfrak{s}-1)]^{\mathrm{T}}\}\}$.

3. Classical estimation methods for censored system

In this section, we first introduce the classical Tobit Kalman filter method [8]. Then, we extend the unbiased minimum variance estimation method to solve the estimation problem for linear censored system [21].

3.1. Tobit Kalman filter

Tobit Kalman filter is a new recursive estimation method for linear censored system. The Bayesian theory and projection formula are used to obtain the state estimation and error variance. In the update stage, a Bernoulli random variable is introduced to describe the censored measurement, and the probability distribution of censored variables is used to evaluate the measurement.

Based on system (2.1)–(2.3) and Tobit Kalman filter theory, the state recursive equation is given by

$$\hat{x}(\mathfrak{s}+1|\mathfrak{s}) = \Xi(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}), \tag{3.1}$$

$$\hat{x}(\mathfrak{s}|\mathfrak{s}) = \hat{x}(\mathfrak{s}|\mathfrak{s}-1) + R_{\tilde{x}\tilde{y}}(\mathfrak{s})R_{\tilde{y}\tilde{y}}^{-1}(\mathfrak{s})\left(y(\mathfrak{s}) - E\big(y(\mathfrak{s})\big)\right), \tag{3.2}$$

where

$$R_{\tilde{x}\tilde{y}}(\mathfrak{s}) = \mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})E[\mathcal{D}(\mathfrak{s})], \tag{3.3}$$

$$R_{\tilde{y}\tilde{y}}(\mathfrak{s}) = E[\mathcal{D}(\mathfrak{s})]\mathcal{N}(\mathfrak{s})\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})E[\mathcal{D}(\mathfrak{s})] + E[\mathcal{D}(\mathfrak{s})v(\mathfrak{s})v^{\mathrm{T}}(\mathfrak{s})\mathcal{D}(\mathfrak{s})], \tag{3.4}$$

$$E\big(y(\mathfrak{s})\big) = \Psi\left(\frac{\mathcal{N}(\mathfrak{s})x(\mathfrak{s})-\tau}{\sigma}\right)\left[\mathcal{N}(\mathfrak{s})x(\mathfrak{s}) + \sigma\lambda\left(\frac{\tau-\mathcal{N}(\mathfrak{s})x(\mathfrak{s})}{\sigma}\right)\right] + \Psi\left(\frac{\tau-\mathcal{N}(\mathfrak{s})x(\mathfrak{s})}{\sigma}\right)\tau, \tag{3.5}$$

$$\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1) = \Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}-1|\mathfrak{s}-1)\Xi^{\mathrm{T}}(\mathfrak{s}) + Q, \tag{3.6}$$

$$\mathcal{J}(\mathfrak{s}|\mathfrak{s}) = \big[I - R_{\tilde{x}\tilde{y}}(\mathfrak{s})R_{\tilde{y}\tilde{y}}^{-1}(\mathfrak{s})E[\mathcal{D}(\mathfrak{s})]\mathcal{N}(\mathfrak{s})\big]\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1), \tag{3.7}$$

$$E[\mathcal{D}(\mathfrak{s})] = \operatorname{diag}\begin{pmatrix} \Psi\left(\frac{\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)^{(1)}-\tau^{(1)}}{\sigma^{(1)}}\right) \\ \Psi\left(\frac{\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)^{(2)}-\tau^{(2)}}{\sigma^{(2)}}\right) \\ \vdots \\ \Psi\left(\frac{\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)^{(p)}-\tau^{(p)}}{\sigma^{(p)}}\right) \end{pmatrix}, \tag{3.8}$$

$$E[\mathcal{D}(\mathfrak{s})v(\mathfrak{s})v^{\mathrm{T}}(\mathfrak{s})\mathcal{D}(\mathfrak{s})]^{\mathrm{T}} = \operatorname{diag}\begin{pmatrix} var\left[y(\mathfrak{s})^{(1)}|\hat{x}(\mathfrak{s}|\mathfrak{s}-1)^{(1)},\sigma^{(1)}\right] \\ var\left[y(\mathfrak{s})^{(2)}|\hat{x}(\mathfrak{s}|\mathfrak{s}-1)^{(2)},\sigma^{(2)}\right] \\ \vdots \\ var\left[y(\mathfrak{s})^{(p)}|\hat{x}(\mathfrak{s}|\mathfrak{s}-1)^{(p)},\sigma^{(p)}\right] \end{pmatrix}, \tag{3.9}$$

$$var(y(\mathfrak{s})|x(\mathfrak{s}),\sigma) = \sigma^2\left[1-\lambda\left(\frac{\tau-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right)\left[\lambda\left(\frac{\tau-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right)-\left(\frac{\tau-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right)\right]\right], \tag{3.10}$$

$$\Psi\left(\frac{y(\mathfrak{s})-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right) = \int_{-\infty}^{y(\mathfrak{s})}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(z(\mathfrak{s})-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\right)^2}{2\sigma^2}}dz(\mathfrak{s}), \tag{3.11}$$

$$\lambda\left(\frac{\tau-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right) = \frac{\varphi\left(\frac{\tau-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right)}{1-\Psi\left(\frac{\tau-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right)}, \tag{3.12}$$

$$\varphi\left(\frac{y(\mathfrak{s})-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\left(y(\mathfrak{s})-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\right)^2}{2\sigma^2}}. \tag{3.13}$$

The Tobit Kalman filter estimation can be computed by the following steps:

Step 1: Set $\mathfrak{s}=0$, $\hat{x}(0|-1)=\mathring{U}(0)$, and $\mathcal{J}(0|-1)=\mathcal{J}(0)$. Compute $\lambda\left(\frac{\tau-\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)}{\sigma}\right)$ by (3.12).

Step 2: Compute $E[\mathcal{D}(\mathfrak{s})]$, $E[\mathcal{D}(\mathfrak{s})v(\mathfrak{s})v^{\mathrm{T}}(\mathfrak{s})\mathcal{D}(\mathfrak{s})]^{\mathrm{T}}$, and $var(y(\mathfrak{s})|x(\mathfrak{s}),\sigma)$ by (3.8), (3.9), and (3.10), respectively.

Step 3: Compute $R_{\tilde{x}\tilde{y}}(\mathfrak{s})$, $R_{\tilde{y}\tilde{y}}(\mathfrak{s})$, and $E\big(y(\mathfrak{s})\big)$ by (3.3), (3.4), and (3.5).

Step 4: Compute $\hat{x}(\mathfrak{s}+1|\mathfrak{s})$, $\hat{x}(\mathfrak{s}|\mathfrak{s})$, $\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)$ and $\mathcal{J}(\mathfrak{s}|\mathfrak{s})$, by (3.1), (3.2), (3.6), and (3.7), respectively.

Step 5: Let $\mathfrak{s}=\mathfrak{s}+1$. Repeat Step 1 to Step 4 till $\mathfrak{s}=N$.

3.2. Unbiased minimum variance estimation

In this section, a recursive filter for linear discrete-time censored system is obtained by using unbiased minimum variance estimation method [21]. In [21], an unbiased minimum variance estimation of the unknown disturbance is obtained by using weighted least squares method. Under the unbiased condition, the Lagrange multiplication is used to calculate the minimum trace of the error variance and obtain the gain matrix. By employing unbiased minimum variance estimation method, the one-step predictor for linear censored system is given in Theorem 1.

Theorem 1. *For system (2.1)–(2.3) with Assumption 1 and 2, the one-step predictor in the case of* $y(\mathfrak{s}) > \tau$ *is as follows:*

$$\hat{x}(\mathfrak{s}+1|\mathfrak{s}) = \Xi(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}), \tag{3.14}$$

$$\hat{x}(\mathfrak{s}|\mathfrak{s}) = \hat{x}(\mathfrak{s}|\mathfrak{s}-1) + L_0(\mathfrak{s})\varepsilon(\mathfrak{s}), \tag{3.15}$$

where

$$L_0(\mathfrak{s}) = \mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})R_\varepsilon^{-1}(\mathfrak{s}), \tag{3.16}$$

$$R_\varepsilon(\mathfrak{s}) = E[\varepsilon(\mathfrak{s})\varepsilon^{\mathrm{T}}(\mathfrak{s})] = \mathcal{N}(\mathfrak{s})\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s}) + R, \tag{3.17}$$

$$\mathcal{J}(\mathfrak{s}+1|\mathfrak{s}) = \Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s}) + Q, \tag{3.18}$$

$$\mathcal{J}(\mathfrak{s}|\mathfrak{s}) = [I - L_0(\mathfrak{s})\mathcal{N}(\mathfrak{s})]\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1), \tag{3.19}$$

$$\varepsilon(\mathfrak{s}) = y(\mathfrak{s}) - \hat{y}(\mathfrak{s}|\mathfrak{s}-1) = \mathcal{N}(\mathfrak{s})[x(\mathfrak{s}) - \hat{x}(\mathfrak{s}|\mathfrak{s}-1)] + v(\mathfrak{s}). \tag{3.20}$$

The one-step predictor in the case of $y(\mathfrak{s}) = \tau$ *is computed by the following equation:*

$$\hat{x}(\mathfrak{s}+1|\mathfrak{s}) = \Xi(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}), \tag{3.21}$$

$$\hat{x}(\mathfrak{s}|\mathfrak{s}) = \hat{x}(\mathfrak{s}|\mathfrak{s}-1) + L(\mathfrak{s})\left[y(\mathfrak{s}) - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1) - M(\mathfrak{s})\hat{f}(\mathfrak{s})\right], \tag{3.22}$$

where

$$\hat{f}(\mathfrak{s}) = F(\mathfrak{s})[y(\mathfrak{s}) - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)], \tag{3.23}$$

$$L(\mathfrak{s}) = \mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})R^{-1}(\mathfrak{s}), \tag{3.24}$$

$$F(\mathfrak{s}) = [M^{\mathrm{T}}(\mathfrak{s})R^{-1}(\mathfrak{s})M(\mathfrak{s})]^{-1}M^{\mathrm{T}}(\mathfrak{s})R^{-1}(\mathfrak{s}), \tag{3.25}$$

$$R(\mathfrak{s}) = E[e(\mathfrak{s})e^{\mathrm{T}}(\mathfrak{s})] = \mathcal{N}(\mathfrak{s})\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s}) + R, \tag{3.26}$$

$$e(\mathfrak{s}) = \mathcal{N}(\mathfrak{s})[x(\mathfrak{s}) - \hat{x}(\mathfrak{s}|\mathfrak{s}-1)] + v(\mathfrak{s}), \tag{3.27}$$

$$\mathcal{J}(\mathfrak{s}+1|\mathfrak{s}) = \Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s}) + Q, \tag{3.28}$$

$$\mathcal{J}(\mathfrak{s}|\mathfrak{s}) = \mathcal{J}(\mathfrak{s}|\mathfrak{s}-1) - L(\mathfrak{s})[R(\mathfrak{s}) - M(\mathfrak{s})\mathcal{J}^f(\mathfrak{s})M^{\mathrm{T}}(\mathfrak{s})]L^{\mathrm{T}}(\mathfrak{s}), \tag{3.29}$$

$$\mathcal{J}^f(\mathfrak{s}) = [M^{\mathrm{T}}(\mathfrak{s})R^{-1}(\mathfrak{s})M(\mathfrak{s})]^{-1}, \tag{3.30}$$

and $F(\mathfrak{s})M(\mathfrak{s}) = I$ *is a sufficient and necessary condition for unknown input estimation unbiased.*

Proof: According to

$$y(\mathfrak{s}) = \begin{bmatrix} y_{\mathfrak{s}1} \\ \vdots \\ \vdots \\ y_{\mathfrak{s}p} \end{bmatrix}, \qquad \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \vdots \\ \tau_p \end{bmatrix},$$

we define

$$M(\mathfrak{s}) = \mathrm{diag}\left[\delta(y_{\mathfrak{s}1} - \tau_1) \quad \delta(y_{\mathfrak{s}2} - \tau_2) \ldots \ldots \delta\left(y_{\mathfrak{s}p} - \tau_p\right)\right].$$

Then, the system (2.1)–(2.3) is re-expressed as

$$x(\mathfrak{s}+1) = \Xi(\mathfrak{s})x(\mathfrak{s}) + \eth(\mathfrak{s}), \tag{3.31}$$

$$y(\mathfrak{s}) = \mathcal{N}(\mathfrak{s})x(\mathfrak{s}) + v(\mathfrak{s}) + C(\mathfrak{s}, y)f(\mathfrak{s}), \tag{3.32}$$

$$C(\mathfrak{s}, y) = \begin{cases} 0, & y(\mathfrak{s}) > \tau, \\ M(\mathfrak{s}), & y(\mathfrak{s}) = \tau, \end{cases} \tag{3.33}$$

where $f(\mathfrak{s}) \in \mathcal{R}^m$ is an unknown input vector, $\Xi(\mathfrak{s})$, $\mathcal{N}(\mathfrak{s})$ and $M(\mathfrak{s})$ are observable.

When $y(\mathfrak{s}) > \tau$ and $C(\mathfrak{s}, y) = 0$, it is a conventional linear discrete system, and we can obtain (3.14)–(3.20) by Kalman filtering technology. When $y(\mathfrak{s}) = \tau$ and $C(\mathfrak{s}, y) = M(\mathfrak{s})$, it is a linear discrete system with unknown disturbance. And, we can get (3.21)–(3.30) by using the same methods in [21].

The proof is now completed.

Then according to Theorem 1, the unbiased minimum variance estimation based one-step predictor can be computed by:

Step 1: Set $\mathfrak{s} = 0$, $\hat{x}(0|-1) = \mathring{\mathrm{U}}(0)$, and $\mathcal{J}(0|-1) = \mathcal{J}(0)$, if $y(\mathfrak{s}) > \tau$, $L_0(\mathfrak{s})$, $R_\varepsilon(\mathfrak{s})$, and $\varepsilon(\mathfrak{s})$ are computed by (3.16), (3.17), and (3.20), respectively.

Step 2: Compute $\hat{x}(\mathfrak{s}+1|\mathfrak{s})$, $\hat{x}(\mathfrak{s}|\mathfrak{s})$, $\mathcal{J}(\mathfrak{s}+1|\mathfrak{s})$, and $\mathcal{J}(\mathfrak{s}|\mathfrak{s})$ by (3.14), (3.15), (3.18), and (3.19), respectively.

Step 3: If $y(\mathfrak{s}) = \tau$, compute $L(\mathfrak{s})$, $R(\mathfrak{s})$, and $e(\mathfrak{s})$ by (3.24), (3.26), and (3.27).

Step 4: Compute $\hat{f}(\mathfrak{s})$ and $F(\mathfrak{s})$ by (3.23) and (3.25), respectively.

Step 5: Compute $\hat{x}(\mathfrak{s}+1|\mathfrak{s})$, $\hat{x}(\mathfrak{s}|\mathfrak{s})$, $\mathcal{J}(\mathfrak{s}|\mathfrak{s})$, $\mathcal{J}^f(\mathfrak{s})$, and $\mathcal{J}(\mathfrak{s}+1|\mathfrak{s})$ by (3.21), (3.22), (3.28), (3.29), and (3.30), respectively.

Step 6: Let $\mathfrak{s} = \mathfrak{s}+1$. Repeat Step 1 to Step 5 till $\mathfrak{s} = N$.

4. Variable step size predictor

It is found that the estimation error will increase when measurement is censored. Note that, the multi-step prediction can skip the censored measurement and use the effective measurement. Therefore, in this section, based on classical Kalman prediction method, we design variable step size predictor.

According to the measurement model (2.3), we estimate the state of the censored system in the following two cases.

4.1. Predictor in case of $y(\mathfrak{s}) \neq \tau$

When $y(\mathfrak{s}) \neq \tau$, the system (2.1)–(2.3) equation is as follows:

$$x(\mathfrak{s}+1) = \Xi(\mathfrak{s})x(\mathfrak{s}) + \eth(\mathfrak{s}), \tag{4.1}$$

$$y(\mathfrak{s}) = \mathcal{N}(\mathfrak{s})x(\mathfrak{s}) + v(\mathfrak{s}). \tag{4.2}$$

By using Kalman filtering technology, we have

$$\hat{x}(\mathfrak{s}+1|\mathfrak{s}) = \Xi(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}), \tag{4.3}$$

$$\hat{x}(\mathfrak{s}|\mathfrak{s}) = \hat{x}(\mathfrak{s}|\mathfrak{s}-1) + K(\mathfrak{s})\varepsilon(\mathfrak{s}), \tag{4.4}$$

where

$$K(\mathfrak{s}) = \mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})[\mathcal{N}(\mathfrak{s})\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s}) + R]^{-1}, \tag{4.5}$$

$$\varepsilon(\mathfrak{s}) = y(\mathfrak{s}) - \hat{y}(\mathfrak{s}|\mathfrak{s}-1) = \mathcal{N}(\mathfrak{s})[x(\mathfrak{s}) - \hat{x}(\mathfrak{s}|\mathfrak{s}-1)] + v(\mathfrak{s}), \tag{4.6}$$

$$\mathcal{J}(\mathfrak{s}+1|\mathfrak{s}) = \Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s}) + Q, \tag{4.7}$$

$$\mathcal{J}(\mathfrak{s}|\mathfrak{s}) = [I - K(\mathfrak{s})\mathcal{N}(\mathfrak{s})]\mathcal{J}(\mathfrak{s}|\mathfrak{s}-1). \tag{4.8}$$

4.2. Predictor in case of $y(\mathfrak{s}) = \tau$

When $y(\mathfrak{s}) = \tau$, the system (2.1)–(2.3) has the following equation:

$$x(\mathfrak{s}+1) = \Xi(\mathfrak{s})x(\mathfrak{s}) + \eth(\mathfrak{s}), \tag{4.9}$$

$$y(\mathfrak{s}) = \tau. \tag{4.10}$$

Remark 1. *In this section, it can be obtained from system (2.1)–(2.3) that the system is in the censored case when* $\H{\ddot{y}}(\mathfrak{s}) \leq \tau$, *i.e.* $y(\mathfrak{s}) = \tau$. *In this case, if* $\tau \geq \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, *we can use multi-step predictor to estimate the state of the system. If* $\tau < \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, *the predictor needs to be revised.*

The estimation formula of the censored system can be described as:

If $\tau \geq \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, we use multi-step predictor to estimate the state of the system.

The recursive predictor is

$$\hat{x}(\mathfrak{s}+i+1|\mathfrak{s}) = \Xi(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s}), \quad i = 1, \ldots, d. \tag{4.11}$$

And the error variance of the predictor can be given by

$$\mathcal{J}(\mathfrak{s}+i+1|\mathfrak{s}) = \Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}+i|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s}) + Q, \quad i = 1, \ldots, d. \tag{4.12}$$

If $\tau < \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, the predictor (4.11) needs to be revised as

$$\hat{x}(\mathfrak{s}+i+1|\mathfrak{s}) = \Xi(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s}) + \beta(\mathfrak{s})\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})\big), \quad i = 1, \ldots, d. \tag{4.13}$$

The $\beta(\mathfrak{s})$ is a gain compensation parameter matrix. We will use three methods to calculate it. In case of $y(\mathfrak{s}) = \tau$, and $\tau < \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, we give two methods to calculate the gain compensation parameter matrix $\beta(\mathfrak{s})$ by using the minimum error variance trace and projection formula in Theorem 2 and 3. In Remark 2, we also summarize the third method based on empirical analysis.

The first method is based on the minimum error variance trace, and $\beta(\mathfrak{s})$ can be given by Theorem 2.

Theorem 2. Consider system (4.9)–(4.10) with Assumption 1 and 2, based on minimum trace of error variance, we have the gain compensation parameter matrix $\beta(\mathfrak{s}) = 0$.

Proof: According to (4.9)–(4.10), and (4.13), the state estimation error is computed as follows:

$$\begin{aligned}\tilde{x}(\mathfrak{s}+i+1|\mathfrak{s}) &= x(\mathfrak{s}+i+1)-\hat{x}(\mathfrak{s}+i+1|\mathfrak{s})\\ &= \Xi(\mathfrak{s})\tilde{x}(\mathfrak{s}+i|\mathfrak{s})+ð(\mathfrak{s}+i)-\beta(\mathfrak{s})\tau+\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s}).\end{aligned}$$

Moreover, the error variance can be obtained by

$$\begin{aligned}\mathcal{J}(\mathfrak{s}+i+1|\mathfrak{s}) &= E[\tilde{x}(\mathfrak{s}+i+1|\mathfrak{s})\tilde{x}^{\mathrm{T}}(\mathfrak{s}+i+1|\mathfrak{s})]\\
&= \Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}+i|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s})+E[\Xi(\mathfrak{s})\tilde{x}(\mathfrak{s}+i|\mathfrak{s})ð^{\mathrm{T}}(\mathfrak{s}+i)]+E[\Xi(\mathfrak{s})\tilde{x}(\mathfrak{s}+i|\mathfrak{s})\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})]\\
&+E[\Xi(\mathfrak{s})\tilde{x}(\mathfrak{s}+i|\mathfrak{s})\hat{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})]+E[ð(\mathfrak{s}+i)\tilde{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s})]+Q\\
&+E[ð(\mathfrak{s}+i)\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})]+E[ð(\mathfrak{s}+i)\hat{x}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})]\\
&-E[\beta(\mathfrak{s})\tau\tilde{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s})]-E[\beta(\mathfrak{s})\tau ð^{\mathrm{T}}(\mathfrak{s}+i)]+E[\beta(\mathfrak{s})\tau\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})]\\
&-E[\beta(\mathfrak{s})\tau\hat{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})]+E[\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})\tilde{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s})]\\
&+E[\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})ð^{\mathrm{T}}(\mathfrak{s}+i)]-E[\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})]\\
&+E[\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})\hat{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})]\\
&= \Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}+i|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s})+Q+\beta(\mathfrak{s})\tau\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})-\beta(\mathfrak{s})\tau\hat{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})\\
&-\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})+\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})E[\hat{x}(\mathfrak{s}+i|\mathfrak{s})\hat{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})]\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})\\
&= \Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}+i|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s})+Q+\beta(\mathfrak{s})\tau\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})-\beta(\mathfrak{s})\tau\hat{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})\\
&-\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})\\
&+\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\{E[x(\mathfrak{s}+i)x^{\mathrm{T}}(\mathfrak{s}+i)]-\mathcal{J}(\mathfrak{s}+i|\mathfrak{s})\}\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s}).\end{aligned} \tag{4.14}$$

where

$$L(\mathfrak{s}+i)=E[x(\mathfrak{s}+i)x^{\mathrm{T}}(\mathfrak{s}+i)]=\Xi(\mathfrak{s})L(\mathfrak{s}+i-1)\Xi^{\mathrm{T}}(\mathfrak{s})+Q,$$

$$L(0)=\mathcal{J}(0).$$

The goal of this theorem is to minimize trace of the error variance as follows:

$$\mathrm{trace}\{\Xi(\mathfrak{s})\mathcal{J}(\mathfrak{s}+i|\mathfrak{s})\Xi^{\mathrm{T}}(\mathfrak{s})+Q+\beta(\mathfrak{s})\tau\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})-\beta(\mathfrak{s})\tau\hat{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})-\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})\tau^{\mathrm{T}}\beta^{\mathrm{T}}(\mathfrak{s})+\beta(\mathfrak{s})\mathcal{N}(\mathfrak{s})\{L(\mathfrak{s}+i)-\mathcal{J}(\mathfrak{s}+i|\mathfrak{s})\}\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\beta^{\mathrm{T}}(\mathfrak{s})\}. \tag{4.15}$$

Take the derivative of (4.15) with respect to $\beta(\mathfrak{s})$ and make it equal to zero, we have

$$\beta(\mathfrak{s})\{2\tau\tau^{\mathrm{T}}-2\tau\hat{x}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})-2\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}+i|\mathfrak{s})\tau^{\mathrm{T}}+\mathcal{N}(\mathfrak{s})L(\mathfrak{s}+i)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})+\mathcal{N}(\mathfrak{s})L^{\mathrm{T}}(\mathfrak{s}+i)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})-\mathcal{N}(\mathfrak{s})\mathcal{J}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})-\mathcal{N}(\mathfrak{s})\mathcal{J}^{\mathrm{T}}(\mathfrak{s}+i|\mathfrak{s})\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\}=0,$$

and obviously, $\beta(\mathfrak{s})=0$. Now, the error variance is minimal.

The proof is now completed.

The second method is based on projection formula, and $\beta(\mathfrak{s})$ can be given by Theorem 3.

Theorem 3. Consider system (4.9)–(4.10) with Assumption 1 and 2, based on projection formula, we have

$$\beta(\mathfrak{s}) = \left[\Xi^{\mathfrak{s}}(\mathfrak{s})\mathring{U}(0)\tau^{\mathrm{T}} - \Xi^{\mathfrak{s}}(\mathfrak{s})\mathring{U}(0)\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\right]$$
$$\times [\tau\tau^{\mathrm{T}} - \tau\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s}) - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\tau^{\mathrm{T}} + \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})]^{-1}.$$

Proof: By applying projection formula to (4.3), we can obtain that

$$\hat{x}(\mathfrak{s}+1|\mathfrak{s}) = \Xi(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1) + E\left[x(\mathfrak{s}+1)\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)^{\mathrm{T}}\right]E\left[\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)^{\mathrm{T}}\right]^{-1}\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big).$$

Compared with (4.13), we have

$$\begin{aligned}
\beta(\mathfrak{s}) &= E\left[x(\mathfrak{s}+1)\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)^{\mathrm{T}}\right] \\
&\times E\left[\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)^{\mathrm{T}}\right]^{-1} \\
&= E\left[\big(\Xi(\mathfrak{s})x(\mathfrak{s}) + \eth(\mathfrak{s})\big)\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)^{\mathrm{T}}\right] \\
&\times E\left[\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)\big(\tau - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\big)^{\mathrm{T}}\right]^{-1} \\
&= [\Xi(\mathfrak{s})E[x(\mathfrak{s})]\tau^{\mathrm{T}} - \Xi(\mathfrak{s})E[x(\mathfrak{s})\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)]\mathcal{N}^{\mathrm{T}}(\mathfrak{s}) + E[\eth(\mathfrak{s})]\tau^{\mathrm{T}} \\
&- E[\eth(\mathfrak{s})\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)]\mathcal{N}^{\mathrm{T}}(\mathfrak{s})][\tau\tau^{\mathrm{T}} - \tau\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s}) - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\tau^{\mathrm{T}} \\
&+ \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})]^{-1}.
\end{aligned}$$

Notice that

$$\begin{aligned}
x(1) &= \Xi(\mathfrak{s})x(0) + \eth(0), \\
x(2) &= \Xi(\mathfrak{s})x(1) + \eth(1), \\
&\vdots \\
&\vdots \\
x(\mathfrak{s}) &= \Xi(\mathfrak{s})x(\mathfrak{s}-1) + \eth(\mathfrak{s}-1),
\end{aligned}$$

then

$$x(\mathfrak{s}) = \Xi^{\mathfrak{s}-1}(\mathfrak{s})x(0) + \sum_{i=0}^{\mathfrak{s}-1}\Xi^{\mathfrak{s}-1-i}(\mathfrak{s})\eth(i).$$

According to $E[x(0)] = \mathring{U}(0)$,

$$E[x(\mathfrak{s})] = \Xi^{\mathfrak{s}-1}(\mathfrak{s})\mathring{U}(0).$$

Therefore

$$\begin{aligned}
\beta(\mathfrak{s}) &= \left[\Xi^{\mathfrak{s}}(\mathfrak{s})\mathring{U}(0)\tau^{\mathrm{T}} - \Xi^{\mathfrak{s}}(\mathfrak{s})\mathring{U}(0)\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})\right] \\
&\times [\tau\tau^{\mathrm{T}} - \tau\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s}) - \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\tau^{\mathrm{T}} \\
&+ \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)\hat{x}^{\mathrm{T}}(\mathfrak{s}|\mathfrak{s}-1)\mathcal{N}^{\mathrm{T}}(\mathfrak{s})]^{-1},
\end{aligned}$$

The proof is now completed.

The third method is based on empirical analysis, and $\beta(\mathfrak{s})$ can be given by Remark 2.

Remark 2. *It can be seen from Figures 2 and 3 in Section 5 that although we can get the theoretical result of $\beta(\mathfrak{s})$ according to Theorem 2 and 3 and have certain advantages compared with Tobit Kalman filter, there is still a certain gap compared with the real measurement. Note that when $y(\mathfrak{s})=\tau$, it is not the true measurement value, and the true measurement value should be less than or equal to τ. Therefore, we design (4.13) as the estimator and use the second term on the right of the equal-sign (4.13) as the compensation for the censored measurements. In fact, when $\tau<\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, the estimator has obvious deviation. We need to adjust the parameter $\beta(\mathfrak{s})$ in time to make the next step estimation* accurate *enough. In this case, $\beta(\mathfrak{s})$ must be a positive definite matrix. At the same time, because the condition of $y(\mathfrak{s})=\tau$ is $\check{y}(\mathfrak{s})\leq\tau$, that is, when $y(\mathfrak{s})=\tau$ appears, it is equivalent to that the true measurement value increases the dimension of $\tau-\check{y}(\mathfrak{s})$. Obviously, in this case, the value of $\beta(\mathfrak{s})$ should be kept around the size of the unit matrix, so that the state estimate of the censored system will be close to the real value.*

In this paper, we take to $\beta(\mathfrak{s})$ be equal to the identity matrix.

Then, the variable step size predictor can be computed by:

Step 1: Set $\mathfrak{s}=0$, $\hat{x}(0|-1)=\mathring{U}(0)$, and $\mathcal{J}(0|-1)=\mathcal{J}(0)$, if $y(\mathfrak{s})\neq\tau$, compute $k(\mathfrak{s})$ and $\varepsilon(\mathfrak{s})$ by (4.5) and (4.6), respectively. Then compute $\hat{x}(\mathfrak{s}+1|\mathfrak{s})$, $\hat{x}(\mathfrak{s}|\mathfrak{s})$, $\mathcal{J}(\mathfrak{s}+1|\mathfrak{s})$, and $\mathcal{J}(\mathfrak{s}|\mathfrak{s})$ by (4.3), (4.4), (4.7), and (4.8).

Step 2: If $y(\mathfrak{s})=\tau$, compare the size of τ and $\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$.

Step 3: If $\tau\geq\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, compute $\hat{x}(\mathfrak{s}+1|\mathfrak{s})$ and $\mathcal{J}(\mathfrak{s}+1|\mathfrak{s})$ by (4.11) and (4.12), respectively.

Step 4: If $\tau<\mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, compute $\hat{x}(\mathfrak{s}+1|\mathfrak{s})$ and $\mathcal{J}(\mathfrak{s}+1|\mathfrak{s})$ by (4.13) and (4.14), respectively.

Step 5: Let $\mathfrak{s}=\mathfrak{s}+1$. Repeat Step 1 to Step 4 till $\mathfrak{s}=N$.

5. Simulation

In this section, the Tobit Kalman filtering, unbiased minimum variance estimation, and variable step size predictor for censored system are compared by a numerical simulation. The following example is motivated by the problem of estimating ballistic roll rates from censored magnetometer data [22], and has dynamics of the form of (2.1)–(2.3) with the parameters

$$\Xi(\mathfrak{s})=\begin{bmatrix}\cos\Delta & -\sin\Delta\\ \alpha_1\sin\Delta & \alpha_2\cos\Delta\end{bmatrix},\qquad \Delta=0.052\pi,$$

$$\mathcal{N}(\mathfrak{s})=[1\quad 2],\qquad\qquad \tau=-1,$$

the system noise $\eth(\mathfrak{s})$ and the measurement noise $v(\mathfrak{s})$ are both uncorrelated white noises, satisfying $R=1$ and $Q=[1,0;0,1]$. Set $\mathring{U}(0)=[0\quad 0]^{\mathrm{T}}$, $\alpha_1=0.9$, $\alpha_2=0.6$.

The Tobit Kalman filter based predictor and unbiased minimum variance based predictor are compared with the three estimation methods for solving the gain compensation parameter matrix $\beta(\mathfrak{s})$ of the variable step size predictor in Figures 1, 2, and 3, respectively. It is shown that the variable step size predictor based on projection formula fluctuates greatly, while the predictor based on Tobit Kalman filter and the predictor based on empirical analysis have higher estimation accuracy. In Figure 4, minimum trace of error variance based variable step size predictor, projection formula based variable step size predictor, and empirical analysis based variable step size predictor are compared. It is seen that empirical analysis based predictor has better estimation performance.

In Table 1, we take 100 sampling periods to calculate the average prediction estimation error of different methods. The second column of the table is the mean prediction error of the state element$(x_1(\mathfrak{s}) - \hat{x}_1(\mathfrak{s}))$, and the third column is the mean prediction error of the state element $(x_2(\mathfrak{s}) - \hat{x}_2(\mathfrak{s}))$. The fourth column is the mean prediction error of the state elements sum $(x_1(\mathfrak{s}) - \hat{x}_1(\mathfrak{s}))$ and $(x_2(\mathfrak{s}) - \hat{x}_2(\mathfrak{s}))$. It can be seen from the fourth column of the table, the empirical analysis based variable step size predictor has smaller estimation error and better estimation accuracy.

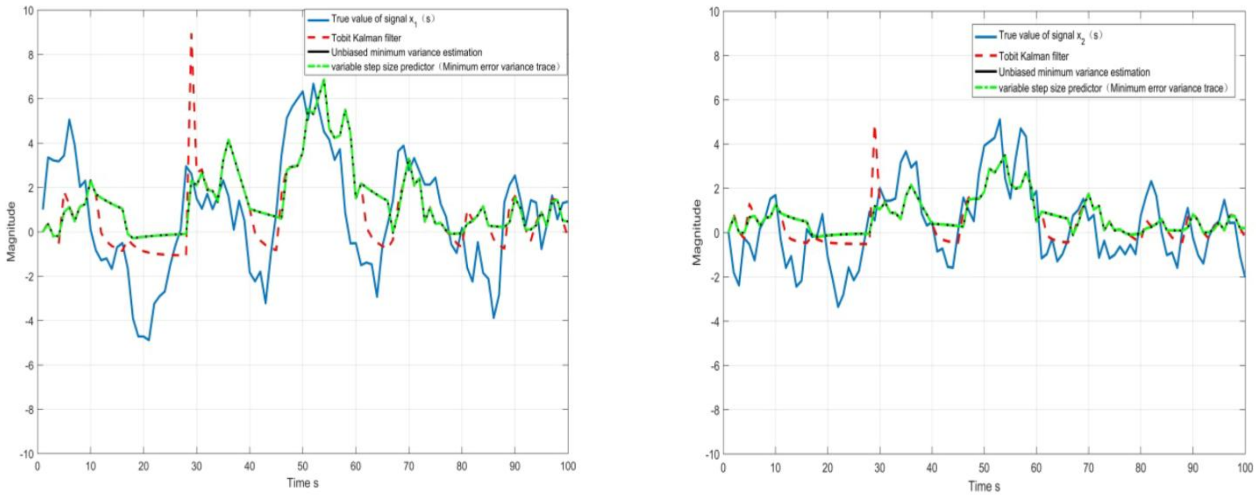

Figure 1. True value of signal $x_1(\mathfrak{s})$ and $x_2(\mathfrak{s})$ (blue solid line), Tobit Kalman filter based predictor (red dashed line), unbiased minimum variance estimation based predictor (black solid line), and minimum trace of error variance based variable step size predictor (green dash-dot line).

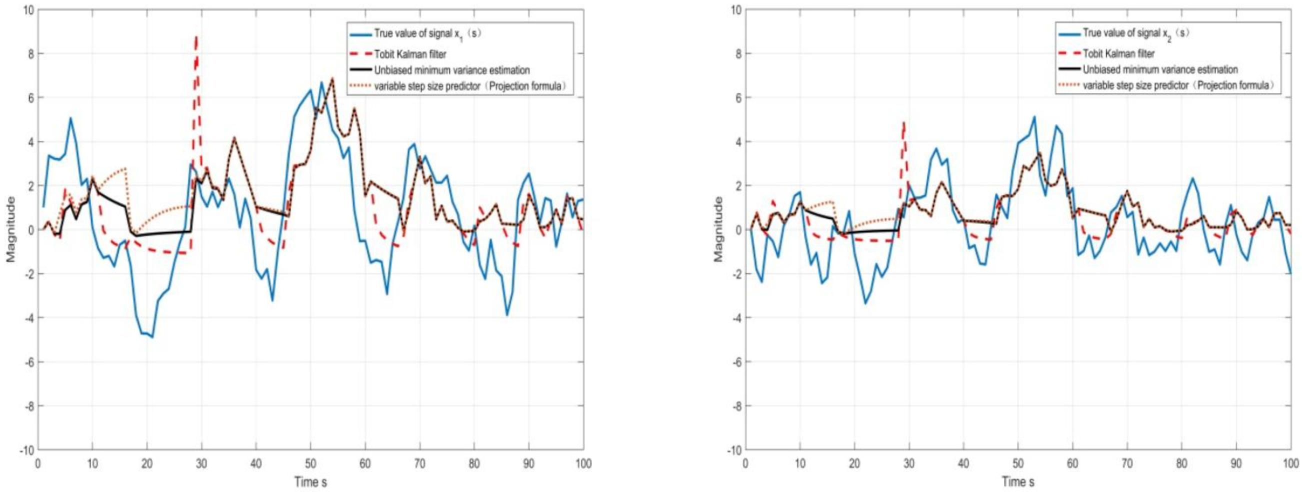

Figure 2. True value of signal $x_1(\mathfrak{s})$ and $x_2(\mathfrak{s})$ (blue solid line), Tobit Kalman filter based predictor (red dashed line), unbiased minimum variance estimation based predictor (black solid line), and projection formula based variable step size predictor (orange dotted line).

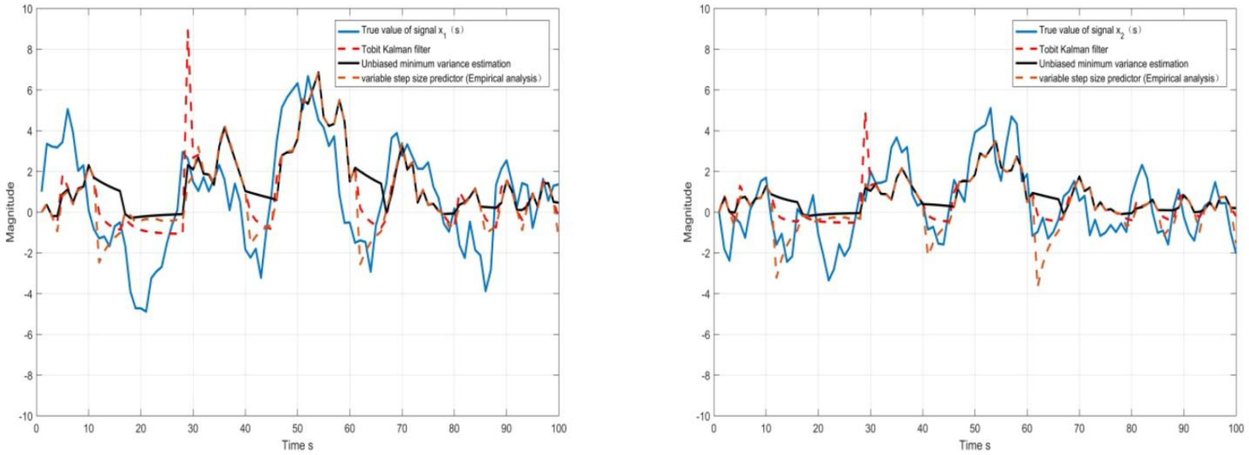

Figure 3. True value of signal $x_1(\mathfrak{s})$ and $x_2(\mathfrak{s})$ (blue solid line), Tobit Kalman filter based predictor (red dashed line), unbiased minimum variance estimation based predictor (black solid line), and empirical analysis based variable step size predictor (orange dashed line).

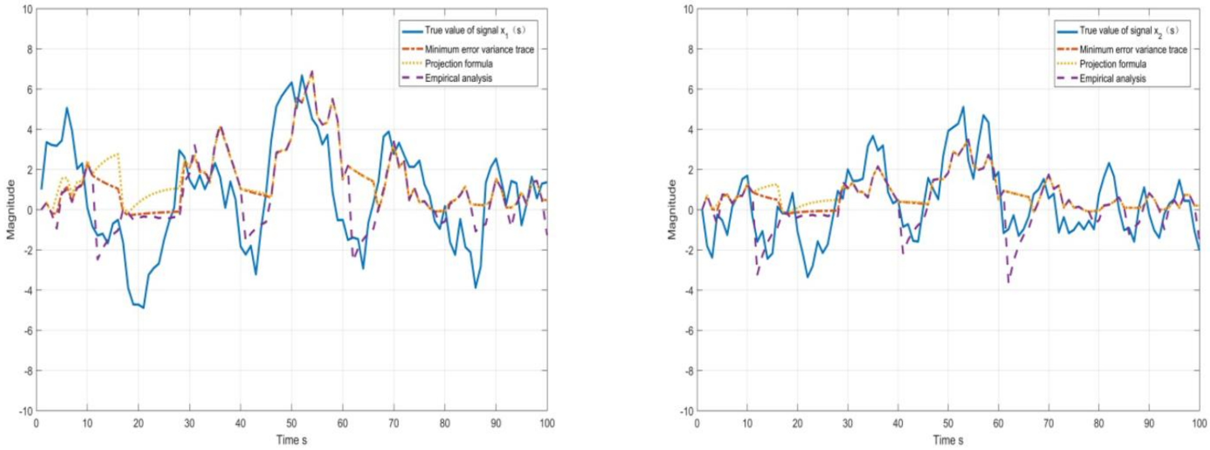

Figure 4. True value of signal $x_1(\mathfrak{s})$ and $x_2(\mathfrak{s})$ (blue solid line), minimum trace of error variance based variable step size predictor (orange dash-dot line), projection formula based variable step size predictor (yellow dotted line), and empirical analysis based variable step size predictor (purple dashed line).

Table 1. Prediction error under different estimation methods.

Estimation methods	State Mean Prediction Error $(\boldsymbol{x_1}(\mathfrak{s}) - \hat{\boldsymbol{x}}_1(\mathfrak{s}))$	State Mean Prediction Error $(\boldsymbol{x_2}(\mathfrak{s}) - \hat{\boldsymbol{x}}_2(\mathfrak{s}))$	State Mean Prediction Error $(\boldsymbol{x}(\mathfrak{s}) - \hat{\boldsymbol{x}}(\mathfrak{s}))$
Tobit Kalman filter	0.378532026	0.262740827	0.320636427
Unbiased minimum variance	0.662930028	0.399790754	0.531360391
Variable step size predictor (empirical analysis)	0.162237848	0.039515826	0.100876837

6. Conclusions

A novel estimation method of variable step size predictor is proposed for censored system. Through the recursive method and projection formula, the variable step size prediction estimation formula is obtained. When $y(\mathfrak{s}) \neq \tau$, we use Kalman filter one-step prediction method to design the estimator. When $y(\mathfrak{s}) = \tau$ and $\tau \geq \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, we design the estimator based on Kalman filter multi-step prediction method. When $y(\mathfrak{s}) = \tau$ and $\tau < \mathcal{N}(\mathfrak{s})\hat{x}(\mathfrak{s}|\mathfrak{s}-1)$, we design the estimator by using the minimum trace of error variance, projection formula, and empirical analysis, respectively. Finally, the Tobit Kalman filter based predictor, unbiased minimum variance estimation based predictor, minimum trace of error variance based method, projection formula based method, and empirical analysis based method are compared by numerical simulation. The results show that the variable step size predictor based on empirical analysis has the optimal estimation performance.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. J. Tobin, Estimation of relationships for limited dependent variables, *Econometrica*, **26** (1958), 24–36.
2. T. Amemiya, Tobit model: A survey, *J. Econometrics*, **24** (1984), 3–61.
3. R. T. Carson, Y. X. Sun, The Tobit model with a non-zero threshold, *Econom. J.*, **10** (2007), 488–502.
4. C. Miller, B. Allik, M. Piovoso, R. Zurakowski, *Estimation of Mobile Vehicle Range & Position Using The Tobit Kalman Filter*, 53rd IEEE Conference on Decision and Control, 2014, 5001–5007.
5. J. Huang, X. He, Detection of Intermittent Fault for Discrete-Time Systems with Output Dead-Zone: A Variant Tobit Kalman Filtering Approach, *J. Control Sci. Eng.*, (2017), 7849841.
6. H. Okasha, M. Nassar, S. A. Dobbah, E-Bayesian estimation of Burr Type XII model based on adaptive Type-Ⅱ progressive hybrid censored data, *AIMS Math.*, **6** (2021), 4173–4196.
7. B. Allik, C. Miller, M. Piovoso, R. Zurakowski, *Estimation of Saturated Data Using the Tobit Kalman Filter*, American Control Conference (ACC), 2014, 4151–4156.
8. B. Allik, C. Miller, M. Piovoso, R. Zurakowski, The Tobit Kalman Filter: An Estimator for Censored Measurements, *IEEE T. Contr. Syst. T.*, **24** (2016), 365–371.

9. H. Geng, Z. D. Wang, Y. Liang, Y. H. Cheng, F. E. Alsaadi, Tobit Kalman filter with fading measurements, *Signal Process.*, **140** (2017), 60–68.
10. F. Han, H. L. Dong, Z. D. Wang, G. F. Li, F. E. Alsaadi, Improved Tobit Kalman Filtering for Systems with Random Parameters via Conditional Expectation, *Signal Process.*, **147** (2018), 35–45.
11. W. L. Li, Y. M. Jia, J. P. Du, Tobit Kalman filter with time-correlated multiplicative measurement noise, *IET Control Theory Appl.*, **11** (2017), 122–128.
12. H. Geng, Z. D. Wang, Y. Liang, Y. H. Cheng, F. E. Alsaadi, Tobit Kalman Filter With Time-Correlated Multiplicative Sensor Noises Under Redundant Channel Transmission, *IEEE Sens. J.*, **17** (2017), 8367–8377.
13. H. H. Zhao, Z. F. Li, B. Li, T. X. Li, A study on input noise second-order filtering and smoothing of linear stochastic discrete systems with packet dropouts, *Adv. Differ. Equ.*, (2020), 474.
14. J. Arthur, A. Attarian, F. Hamilton, H. Tran, Nonlinear Kalman filtering for censored observations, *Appl. Math. Comput.*, **316** (2018), 155–166.
15. J. J. Li, G. L. Wei, D. R. Ding, Y. R. Li, Set-membership filtering for discrete time-varying nonlinear systems with censored measurements under Round-Robin protocol, *Neurocomputing*, **281** (2018), 20–26.
16. J. Huang, X. He, *Fault detection of nonlinear systems with missing measurements and censored data*, IECON 2017 - 43rd Annual Conference of the IEEE Industrial Electronics Society, Beijing, 2017, 5797–5802.
17. C. Huang, B. Shen, H. W. Chen, H. S. Shu, A Dynamically event-triggered approach to recursive filtering with censored measurements and parameter uncertainties, *J. Franklin Inst.*, **356** (2019), 8870–8889.
18. H. Geng, Z. D. Wang, Y. H. Cheng, F. E. Alsaadi, A. M. Dobaie, State estimation under non-Gaussian Lévy and time-correlated additive sensor noises: A modified Tobit Kalman filtering approach, *Signal Process.*, **154** (2019), 120–128.
19. Z. L. Du, X. M. Li, Strong Tracking Tobit Kalman Filter with Model Uncertainties, *Int. J. Control. Autom.*, **17** (2019), 345–355.
20. K. Loumponias, N. Vretos, G. Tsaklidis, P. Daras, An Improved Tobit Kalman Filter with Adaptive Censoring Limits, *Circuits Syst. Signal Process.*, **39** (2020), 5588–5617.
21. S. Gillijns, B. De Moor, Unbiased minimum-variance input and state estimation for linear discrete-time systems with direct feedthrough, *Automatica*, **43** (2007), 934–937.
22. B. Allik, M. Ilg, R. Zurakowski, Ballistic Roll Estimation using EKF Frequency Tracking and Adaptive Noise Cancellation, *IEEE Trans. Aerosp. Electron. Syst.*, **49** (2013), 2546–2553.