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Research article

Intuitionistic fuzzy normed prime and maximal ideals

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Abstract: Motivated by the new notion of intuitionistic fuzzy normed ideal, we present and investigate some associated properties of intuitionistic fuzzy normed ideals. We describe the intrinsic product of any two intuitionistic fuzzy normed subsets and show that the intrinsic product of intuitionistic fuzzy normed ideals is a subset of the intersection of these ideals. We specify the notions of intuitionistic fuzzy normed prime ideal and intuitionistic fuzzy normed maximal ideal, we present the conditions under which a given intuitionistic fuzzy normed ideal is considered to be an intuitionistic fuzzy normed prime (maximal) ideal. In addition, the relation between the intuitionistic characteristic function and prime and maximal ideals is generalized. Finally, we characterize relevant properties of intuitionistic fuzzy normed prime ideals and intuitionistic fuzzy normed maximal ideals.

Keywords: intuitionistic fuzzy normed ring; intuitionistic fuzzy normed ideal; intuitionistic fuzzy normed prime ideal; intuitionistic fuzzy normed maximal ideal
Mathematics Subject Classification: 03E72, 03F55

1. Introduction

The notion of intuitionistic fuzzy normed subring and intuitionistic fuzzy normed ideal was characterized by Abed Alhaleem and Ahmad in [10], after that the necessity has arisen to introduce the concepts of intuitionistic fuzzy normed prime ideals and intuitionistic fuzzy normed maximal ideals. Following the work of Emniyent and Şahin in [17] which outlined the concepts of fuzzy normed prime ideal and maximal ideal we implement the conception of intuitionistic fuzzy to prime and maximal normed ideals. After the establishment of fuzzy set by Zadeh [28] which showed that the membership of an element in a fuzzy set is at intervals [0,1], many researchers investigated on the properties of fuzzy set because it handles uncertainty and vagueness, and due to its applications in many fields of studies. A lot of work has been done on various aspects and for the last 50 years, the relation betwee maximal and prime ideals has become the core of many researchers work. Swamy and Swamy in 1988 [27] presented the conceptions of fuzzy ideal and fuzzy prime ideal with truth values in a complete lattice fulfilling the infinite distributive law. Later, many researchers studied the generalization of fuzzy ideals and fuzzy prime (maximal) ideals of rings: Dixit et al [16], Malik and Mordeson in [22] and Mukherjee and Sen in [24]. The notion of intuitionistic fuzzy set was initiated by Atanassov [6], as a characterization of fuzzy set which assigned the degree of membership and the degree of non-membership for set elements, he also delineated some operations and connections over basic intuitionistic fuzzy sets. In [5], Atanassov introduced essential definitions and properties of the interval-valued intuitionistic fuzzy sets and the explanation of mostly extended modal operator through interval-valued intuitionistic fuzzy sets were presented in [4], and some of its main properties were studied. Banerjee and Basnet [13] investigated intuitionistic fuzzy rings and intuitionistic fuzzy ideals using intuitionistic fuzzy sets. In 2005 [20], an identification of intuitionistic fuzzy ideals, intuitionistic fuzzy prime ideals and intuitionistic fuzzy completely prime ideals was given. In [14], Bakhadach et al. implemented the terms of intuitionistic fuzzy ideals and intuitionistic fuzzy prime (maximal) ideals, investigated these notions to show new results using the intuitionistic fuzzy points and membership and nonmembership functions. The paper comprises the following: we begin with the preliminary section, we submit necessary notations and elementary outcomes. In Section 3, we characterize some properties of intuitionistic fuzzy normed ideals and identify the image and the inverse image of intuitionistic fuzzy normed ideals. In Section 4, we describe the notions of intuitionistic fuzzy normed prime ideals and intuitionistic fuzzy normed maximal ideals and we characterize the relation between the intuitionistic characteristic function and prime (maximal) ideals. In Section 5, the conclusions are outlined.

2. Preliminaries

We first include some definitions needed for the subsequent sections:

Definition 2.1. [25] *A linear space L is called a normed space if for any element r there is a real number $||r||$ satisfying:*

- $||r|| \geq 0$ *for every $r \in L$, when $r = 0$ then $||r|| = 0$;*
- $||\alpha.r|| = |\alpha|.||r||$;
- $||r + v|| \leq ||r|| + ||v||$ *for all $r, v \in L$.*

Definition 2.2. [18] *A ring R is said to be a normed ring (NR) if it possesses a norm $|| \ ||$, that is, a non-negative real-valued function $|| \ || : NR \to \mathbb{R}$ such that for any $r, v \in R$,*

1) $||r|| = 0 \Leftrightarrow r = 0$,
2) $||r + v|| \leq ||r|| + ||v||$,
3) $||r|| = ||-r||$, *(and hence $||1_A|| = 1 = ||-1||$ if identity exists), and*
4) $||rv|| \leq ||r|| \, ||v||$.

Definition 2.3. [1] *Let $* : [0, 1] \times [0, 1] \to [0, 1]$ be a binary operation. Then $*$ is a t-norm if $*$ conciliates the conditions of commutativity, associativity, monotonicity and neutral element 1.*
We shortly use t-norm and write $r * v$ instead of $*(r, v)$.

Two examples of continuous t-norm are: $r * v = rv$ and $r * v = min\{r, v\}$ [26].

Proposition 2.4. [21] *A t-norm T has the property, for every $r, v \in [0,1]$*

$$T(r,v) \leq min(r,v)$$

Definition 2.5. [19] *Let $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ be a binary operation. Then $\diamond$ is a s-norm if $\diamond$ conciliates the conditions of commutativity, associativity, monotonicity and neutral element 0.*
We shortly use s-norm and write $r \diamond v$ instead of $\diamond(r,v)$.

Two examples of continuous s-norm are: $r \diamond v = min(r+v, 1)$ and $r \diamond v = max\{r, v\}$ [26].

Proposition 2.6. [21] *A s-norm S has the property, for every $r, v \in [0,1]$*

$$max(r,v) \leq S(r,v)$$

Definition 2.7. [28] *A membership function $\mu_A(r) : X \rightarrow [0,1]$ specifies the fuzzy set A over X, where $\mu_A(r)$ defines the membership of an element $r \in X$ in a fuzzy set A.*

Definition 2.8. [6] *An intuitionistic fuzzy set A in set X is in the form $IFS\ A = \{(r, \mu_A(r), \gamma_A(r) : r \in X\}$, such that the degree of membership is $\mu_A(r) : X \rightarrow [0,1]$ and the degree of non-membership is $\gamma_A(r) : X \rightarrow [0,1]$, where $0 \leq \mu_A(r) + \gamma_A(r)) \leq 1$ for all $r \in X$. We shortly use $A = (\mu_A, \gamma_A)$.*

Definition 2.9. [7] *Let A be an intuitionistic fuzzy set in a ring R, we indicate the (α,β)-cut set by $A_{\alpha,\beta} = \{r \in R : \mu_A \geq \alpha$ and $\gamma_A \leq \beta\}$ such that $\alpha + \beta \leq 1$ and $\alpha, \beta \in [0,1]$.*

Definition 2.10. [23] *The support of an intuitionistic fuzzy set A, is denoted by $A_\circ$ and defined as $A_\circ = \{r : \mu_A(r) > 0$ and $\gamma_A(r) < 1\}$.*

Definition 2.11. [2] *The complement, union and intersection of two IFS $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$, in a ring R, are defined as follows:*

1) $A^c = \{\langle r, \gamma_A(r), \mu_A(r)\rangle : r \in R\}$,
2) $A \cup B = \{\langle r, max(\mu_A(r), \mu_B(r)), min(\gamma_A(r), \gamma_B(r))\rangle : r \in R\}$,
3) $A \cap B = \{\langle r, min(\mu_A(r), \mu_B(r)), max(\gamma_A(r), \gamma_B(r))\rangle : r \in R\}$.

Definition 2.12. [12] *Let NR be a normed ring. Then an IFS $A = \{(r, \mu_A(r), \gamma_A(r)) : r \in NR\}$ of NR is an intuitionistic fuzzy normed subring ($IFNSR$) of NR if:*

i. $\mu_A(r - v) \geq \mu_A(r) * \mu_A(v)$,
ii. $\mu_A(rv) \geq \mu_A(r) * \mu_A(v)$,
iii. $\gamma_A(r - v) \leq \gamma_A(r) \diamond \gamma_A(v)$,
iv. $\gamma_A(rv) \leq \gamma_A(r) \diamond \gamma_A(v)$.

Definition 2.13. [9] *Let NR be a normed ring. Then an IFS $A = \{(r, \mu_A(r), \gamma_A(r)) : r \in NR\}$ of NR is an intuitionistic fuzzy normed ideal ($IFNI$) of NR if:*

i. $\mu_A(r - v) \geq \mu_A(r) * \mu_A(v)$,
ii. $\mu_A(rv) \geq \mu_A(r) \diamond \mu_A(v)$,
iii. $\gamma_A(r - v) \leq \gamma_A(r) \diamond \gamma_A(v))$,
iv. $\gamma_A(rv)) \leq \gamma_A(r) * \gamma_A(v)\}$.

Definition 2.14. [3] *If A and B are two fuzzy subsets of the normed ring NR. Then the product $A \circ B(r)$ is defined by:*

$$A \circ B(r) = \begin{cases} \overset{\diamond}{_{r=vz}}(\mu_A(v) * \mu_B(z)) & , \text{if } r=vz \\ 0 & , \text{otherwise} \end{cases}$$

Definition 2.15. [22] *A fuzzy ideal A (non-constant) of a ring R is considered to be a fuzzy prime ideal if $B \circ C \subseteq A$ for a fuzzy ideals B, C of R indicates that either $B \subseteq A$ or $C \subseteq A$.*

3. Properties of intuitionistic fuzzy normed ideal

In this section, we characterize several properties of intuitionistic fuzzy normed ideals and elementary results are obtained.

Definition 3.1. [8] *Let A and B be two intuitionistic fuzzy subsets of the normed ring NR. The operations are defined as:*

$$\mu_{A \circledast B}(r) = \begin{cases} \overset{\diamond}{_{r=vz}}(\mu_A(v) * \mu_B(z)) & , \text{if } r=vz \\ 0 & , \text{otherwise} \end{cases}$$

and

$$\gamma_{A \otimes B}(r) = \begin{cases} \overset{*}{_{r=vz}}(\gamma_A(v) \diamond \gamma_B(z)) & , \text{if } r=vz \\ 1 & , \text{otherwise} \end{cases}$$

Therefore, the intrinsic product of A and B is considered to be the intuitionistic fuzzy normed set $A \circ B = (\mu_{A \circledast B}, \gamma_{A \otimes B}) = (\mu_A \circledast \mu_B, \gamma_A \otimes \gamma_B)$.

Theorem 3.2. [10] *Let A and B be two intuitionistic fuzzy ideals of a normed ring NR. Then $A \cap B$ is an intuitionistic fuzzy normed ideal of NR.*

Example 3.1. *Let $NR = \mathbb{Z}$ the ring of integers under ordinary addition and multiplication of integers. Define the intuitionistic fuzzy normed subsets as $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$, by*

$$\mu_A(r) = \begin{cases} 0.7 & , \text{if } r \in 5\mathbb{Z} \\ 0.2 & , \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_A(r) = \begin{cases} 0.1 & , \text{if } r \in 5\mathbb{Z} \\ 0.4 & , \text{otherwise} \end{cases}$$

$$\mu_B(r) = \begin{cases} 0.8 & , \text{if } r \in 5\mathbb{Z} \\ 0.3 & , \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_B(r) = \begin{cases} 0.2 & , \text{if } r \in 5\mathbb{Z} \\ 0.7 & , \text{otherwise} \end{cases}$$

As $\mu_{A\cap B}(r) = min\{\mu_A(r), \mu_B(r)\}$ and $\gamma_{A\cap B}(r) = max\{\gamma_A(r), \gamma_B(r)\}$. Then,

$$\mu_{A\cap B}(r) = \begin{cases} 0.7 & , \text{if } r \in 5\mathbb{Z} \\ 0.2 & , \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_{A\cap B}(r) = \begin{cases} 0.2 & , \text{if } r \in 5\mathbb{Z} \\ 0.7 & , \text{otherwise} \end{cases}$$

It can be verified that A, B and $A \cap B$ are intuitionistic fuzzy normed ideals of NR.

Lemma 3.3. *Let A and B be an intuitionistic fuzzy normed right ideal and an intuitionistic fuzzy normed left ideal of a normed ring NR, respectively, then $A \circ B \subseteq A \cap B$ i.e, $A \circledast B(r) \leq A \cap B(r) \leq A \otimes B(r)$, where*

$$\begin{aligned} A \cap B(r) &= \{(r, \mu_{A\cap B}(r), \gamma_{A\cap B}(r)) : r \in NR\} \\ &= \{(r, min\{\mu_A(r), \mu_B(r)\}, max\{\gamma_A(r), \gamma_B(r)\}) : r \in NR\}. \end{aligned}$$

Proof. Let $A \cap B$ be an intuitionistic fuzzy normed ideal of NR. Assume that A is an intuitionistic fuzzy normed right ideal and B is an intuitionistic fuzzy normed left ideal. Let $\mu_{A \circledast B}(r) = \underset{r=vz}{\diamond}(\mu_A(v) * \mu_B(z))$ and let $\gamma_{A \otimes B}(r) = \underset{r=vz}{*}(\gamma_A(v) \diamond \gamma_B(z))$.

Since, A is an intuitionistic fuzzy normed right ideal and B is an intuitionistic fuzzy normed left ideal, we have

$$\mu_A(v) \le \mu_A(vz) = \mu_A(r) \quad and \quad \mu_B(z) \le \mu_B(vz) = \mu_B(r)$$

and

$$\gamma_A(r) = \gamma_A(vz) \ge \gamma_A(v) \quad and \quad \gamma_B(r) = \gamma_B(vz) \ge \gamma_B(z).$$

Thus,

$$\begin{aligned} \mu_{A \circledast B}(r) &= \underset{r=vz}{\diamond}(\mu_A(v) * \mu_B(z)) \\ &= min(\mu_A(v), \mu_B(z)) \\ &\le min(\mu_A(r), \mu_B(r)) \\ &\le \mu_{A \cap B}(r) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \gamma_{A \otimes B}(r) &= \underset{r=vz}{*}(\gamma_A(v) \diamond \gamma_B(z)) \\ &= max(\gamma_A(v), \gamma_B(z)) \\ &\ge max(\gamma_A(r), \gamma_B(r)) \\ &\ge \gamma_{A \cap B}(r). \end{aligned} \tag{3.2}$$

By (3.1) and (3.2) the proof is concluded. □

Remark 3.4. *The union of two intuitionistic fuzzy normed ideals of a ring NR needs not be always intuitionistic fuzzy normed ideal.*

Example 3.2. *Let $NR = \mathbb{Z}$ the ring of integers under ordinary addition and multiplication of integers. Let the intuitionistic fuzzy normed subsets $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$, define by*

$$\mu_A(r) = \begin{cases} 0.85 & , if\ r \in 3\mathbb{Z} \\ 0.3 & , otherwise \end{cases} \quad and \quad \gamma_A(r) = \begin{cases} 0.2 & , if\ r \in 3\mathbb{Z} \\ 0.4 & , otherwise \end{cases}$$

$$\mu_B(r) = \begin{cases} 0.75 & , if\ r \in 2\mathbb{Z} \\ 0.35 & , otherwise \end{cases} \quad and \quad \gamma_B(r) = \begin{cases} 0.3 & , if\ r \in 2\mathbb{Z} \\ 0.5 & , otherwise \end{cases}$$

It can be checked that A and B are intuitionistic fuzzy normed ideals of NR.

As $\mu_{A \cup B}(r) = max\{\mu_A(r), \mu_B(r)\}$ and $\gamma_{A \cup B}(r) = min\{\gamma_A(r), \gamma_B(r)\}$. Then,

$$\mu_{A \cup B}(r) = \begin{cases} 0.85 & , if\ r \in 3\mathbb{Z} \\ 0.75 & , if\ r \in 2\mathbb{Z} - 3\mathbb{Z} \\ 0.35 & , if\ r \notin 2\mathbb{Z}\ or\ r \notin 3\mathbb{Z} \end{cases} \quad and \quad \gamma_{A \cup B}(r) = \begin{cases} 0.2 & , if\ r \in 3\mathbb{Z} \\ 0.3 & , if\ r \in 2\mathbb{Z} - 3\mathbb{Z} \\ 0.4 & , if\ r \notin 2\mathbb{Z}\ or\ r \notin 3\mathbb{Z} \end{cases}$$

Let $r = 15$ and $v = 4$, then $\mu_{A \cup B}(15) = 0.85$, $\mu_{A \cup B}(4) = 0.75$ and $\gamma_{A \cup B}(15) = 0.2$, $\gamma_{A \cup B}(4) = 0.3$.

Hence, $\mu_{A \cup B}(15 - 4) = \mu_{A \cup B}(11) = 0.35 \ngeq \mu_{A \cup B}(15) * \mu_{A \cup B}(4) = min\{0.85, 0.75\}$ and $\gamma_{A \cup B}(15 - 4) = \gamma_{A \cup B}(11) = 0.4 \nleq \gamma_{A \cup B}(15) \diamond \gamma_{A \cup B}(4) = max\{0.2, 0.3\}$. Thus, the union of two intuitionistic fuzzy normed ideals of NR need not be an intuitionistic fuzzy normed ideal.

Proposition 3.5. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy normed ideal of a ring NR, then we have for all $r \in NR$:*

i. $\mu_A(0) \geq \mu_A(r)$ *and* $\gamma_A(0) \leq \gamma_A(r)$,
ii. $\mu_A(-r) = \mu_A(r)$ *and* $\gamma_A(-r) = \gamma_A(r)$,
iii. *If* $\mu_A(r - v) = \mu_A(0)$ *then* $\mu_A(r) = \mu_A(v)$,
iv. *If* $\gamma_A(r - v) = \gamma_A(0)$ *then* $\gamma_A(r) = \gamma_A(v)$.

Proof. i. As A is an intuitionistic fuzzy normed ideal, then

$$\mu_A(0) = \mu_A(r - r) \geq \mu_A(r) * \mu_A(r) = \mu_A(r)$$

and

$$\gamma_A(0) = \gamma_A(r - r) \leq \gamma_A(r) \diamond \gamma_A(r) = \gamma_A(r)$$

ii. $\mu_A(-r) = \mu_A(0 - r) \geq \mu_A(0) * \mu_A(r) = \mu_A(r)$ and $\mu_A(r) = \mu_A(0 - (-r)) \geq \mu_A(0) * \mu_A(-r) = \mu_A(-r)$. Therefore, $\mu_A(-r) = \mu_A(r)$
also,
$\gamma_A(-r) = \gamma_A(0 - r) \leq \gamma_A(0) \diamond \gamma_A(r) = \gamma_A(r)$ and $\gamma_A(r) = \gamma_A(0 - (-r)) \leq \gamma_A(0) \diamond \gamma_A(-r) = \gamma_A(-r)$. Therefore, $\gamma_A(-r) = \gamma_A(r)$.

iii. Since $\mu_A(r - v) = \mu_A(0)$, then

$$\mu_A(v) = \mu_A(r - (r - v)) \geq \mu_A(r) * \mu_A(r - v) = \mu_A(r) * \mu_A(0) \geq \mu_A(r)$$

similarly

$$\mu_A(r) = \mu_A((r - v) - (-v)) \geq \mu_A(r - v) * \mu_A(-v) = \mu_A(0) * \mu_A(v) \geq \mu_A(v)$$

Consequently, $\mu_A(r) = \mu_A(v)$.

iv. same as in iii. □

Proposition 3.6. *Let A be an intuitionistic fuzzy normed ideal of a normed ring NR, then* $\triangle A = (\mu_A, \mu_A^c)$ *is an intuitionistic fuzzy normed ideal of NR.*

Proof. Let $r, v \in NR$

$$\begin{aligned} \mu_A^c(r - v) &= 1 - \mu_A(r - v) \\ &\leq 1 - min\{\mu_A(r), \mu_A(v)\} \\ &= max\{1 - \mu_A(r), 1 - \mu_A(v)\} \\ &= max\{\mu_A^c(r), \mu_A^c(v)\} \end{aligned}$$

Then $\mu_A^c(r - v) \leq \mu_A^c(r) \diamond \mu_A^c(v)$.

$$\begin{aligned} \mu_A^c(rv) &= 1 - \mu_A(rv) \\ &\leq 1 - max\{\mu_A(r), \mu_A(v)\} \\ &= min\{1 - \mu_A(r), 1 - \mu_A(v)\} \\ &= min\{\mu_A^c(r), \mu_A^c(v)\} \end{aligned}$$

Then $\mu_A^c(rv) \leq \mu_A^c(r) * \mu_A^c(v)$.
Accordingly, $\triangle A = (\mu_A, \mu_A^c)$ is an intuitionistic fuzzy normed ideal of NR. □

Proposition 3.7. *If A is an intuitionistic fuzzy normed ideal of a normed ring NR, then* $\Diamond A = (\gamma_A^c, \gamma_A)$ *is an intuitionistic fuzzy normed ideal of NR.*

Proof. Let $r, v \in NR$

$$\begin{aligned} \gamma_A^c(r-v) &= 1-\gamma_A(r-v) \\ &\geq 1-max\{\gamma_A(r), \gamma_A(v)\} \\ &= min\{1-\gamma_A(r), 1-\gamma_A(v)\} \\ &= min\{\gamma_A^c(r), \gamma_A^c(v)\} \end{aligned}$$

Then $\gamma_A^c(r-v) \geq \gamma_A^c(r) * \gamma_A^c(v)$.

$$\begin{aligned} \gamma_{A^c}(rv) &= 1-\gamma_A(rv) \\ &\geq 1-min\{\gamma_A(r), \gamma_A(v)\} \\ &= max\{1-\mu_A(r), 1-\gamma_A(v)\} \\ &= max\{\gamma_A^c(r), \gamma_A^c(v)\} \end{aligned}$$

Then $\gamma_A^c(rv) \geq \gamma_A^c(r) \diamond \gamma_A^c(v)$.
Therefore, $\Diamond A = (\gamma_A^c, \gamma_A)$ is an intuitionistic fuzzy normed ideal of NR. □

Proposition 3.8. *An* $IFS\ A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy normed ideal of NR if the fuzzy subsets* μ_A *and* γ_A^c *are intuitionistic fuzzy normed ideals of NR.*

Proof. Let $r, v \in NR$

$$\begin{aligned} 1-\gamma_A(r-v) &= \gamma_A^c(r-v) \\ &\geq min\{\gamma_A^c(r), \gamma_A^c(v)\} \\ &= min\{(1-\gamma_A(r)), (1-\gamma_A(v))\} \\ &= 1-max\{\gamma_A(r), \gamma_A(v)\} \end{aligned}$$

Then, $\gamma_A(r-v) \leq \gamma_A(r) \diamond \gamma_A(v)$.

$$\begin{aligned} 1-\gamma_A(rv) &= \gamma_A^c(rv) \\ &\geq max\{\gamma_A^c(r), \gamma_A^c(v)\} \\ &= max\{(1-\gamma_A(r)), (1-\gamma_A(v))\} \\ &= 1-min\{\gamma_A(r), \gamma_A(v)\} \end{aligned}$$

Then, $\gamma_A(rv) \leq \gamma_A(r) * \gamma_A(v)$.

Consequently, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normed ideal of NR. □

Definition 3.9. *Let A be a set (non-empty) of the normed ring NR, the intuitionistic characteristic function of A is defined as* $\lambda_A = (\mu_{\lambda_A}, \gamma_{\lambda_A})$*, where*

$$\mu_{\lambda_A}(r) = \begin{cases} 1 & , if \quad r \in A \\ 0 & , if \quad r \notin A \end{cases} \quad and \quad \gamma_{\lambda_A}(r) = \begin{cases} 0 & , if \quad r \in A \\ 1 & , if \quad r \notin A \end{cases}$$

Lemma 3.10. *Let A and B be intuitionistic fuzzy sets of a normed ring NR, then:*
(i) $\lambda_A \cap \lambda_B = \lambda_{A\cap B}$ *(ii)* $\lambda_A \circ \lambda_B = \lambda_{A\circ B}$ *(iii) If* $A \subseteq B$*, then* $\lambda_A \subseteq \lambda_B$

Theorem 3.11. *For a non-empty subset A of NR, A is a subring of NR if and only if* $\lambda_A = (\mu_{\lambda_A}, \gamma_{\lambda_A})$ *is an intuitionistic fuzzy normed subring of NR.*

Proof. Suppose A to be a subring of NR and let $r, v \in NR$. If $r, v \in A$, then by the intuitionistic characteristic function properties $\mu_{\lambda_A}(r) = 1 = \mu_{\lambda_A}(v)$ and $\gamma_{\lambda_A}(r) = 0 = \gamma_{\lambda_A}(v)$. As A is a subring, then $r - v$ and $rv \in A$. Thus, $\mu_{\lambda_A}(r - v) = 1 = 1 * 1 = \mu_{\lambda_A}(r) * \mu_{\lambda_A}(v)$ and $\mu_{\lambda_A}(rv) = 1 = 1 * 1 = \mu_{\lambda_A}(r) * \mu_{\lambda_A}(v)$, also $\gamma_{\lambda_A}(r - v) = 0 = 0 \diamond 0 = \gamma_{\lambda_A}(r) \diamond \gamma_{\lambda_A}(v)$ and $\gamma_{\lambda_A}(rv) = 0 = 0 \diamond 0 = \gamma_{\lambda_A}(r) \diamond \gamma_{\lambda_A}(v)$. This implies,

$$\begin{aligned} \mu_{\lambda_A}(r - v) &\geq \mu_{\lambda_A}(r) * \mu_{\lambda_A}(v) \quad && \textit{and} \quad \mu_{\lambda_A}(rv) \geq \mu_{\lambda_A}(r) * \mu_{\lambda_A}(v), \\ \gamma_{\lambda_A}(r - v) &\leq \gamma_{\lambda_A}(r) \diamond \gamma_{\lambda_A}(v) \quad && \textit{and} \quad \gamma_{\lambda_A}(rv) \leq \gamma_{\lambda_A}(r) \diamond \gamma_{\lambda_A}(v). \end{aligned}$$

Similarly we can prove the above expressions if $r, v \notin A$.

Hence, $\lambda_A = (\mu_{\lambda_A}, \gamma_{\lambda_A})$ is an intuitionistic fuzzy normed subring of NR.

Conversely, we hypothesise that the intuitionistic characteristic function $\lambda_A = (\mu_{\lambda_A}, \gamma_{\lambda_A})$ is an intuitionistic fuzzy normed subring of NR. Let $r, v \in A$, then $\mu_{\lambda_A}(r) = 1 = \mu_{\lambda_A}(v)$ and $\gamma_{\lambda_A}(r) = 0 = \gamma_{\lambda_A}(v)$. So,

$$\begin{aligned} &\mu_{\lambda_A}(r - v) \geq \mu_{\lambda_A}(r) * \mu_{\lambda_A}(v) \geq 1 * 1 \geq 1, && \textit{also} \quad \mu_{\lambda_A}(r - v) \leq 1, \\ &\mu_{\lambda_A}(rv) \geq \mu_{\lambda_A}(r) * \mu_{\lambda_A}(v) \geq 1 * 1 \geq 1, && \textit{also} \quad \mu_{\lambda_A}(rv) \leq 1, \\ &\gamma_{\lambda_A}(r - v) \leq \gamma_{\lambda_A}(r) \diamond \gamma_{\lambda_A}(v) \leq 0 \diamond 0 \leq 0, && \textit{also} \quad \gamma_{\lambda_A}(r - v) \geq 0, \\ &\gamma_{\lambda_A}(rv) \leq \gamma_{\lambda_A}(r) \diamond \gamma_{\lambda_A}(v) \leq 0 \diamond 0 \leq 0, && \textit{also} \quad \gamma_{\lambda_A}(rv) \geq 0, \end{aligned}$$

then $\mu_{\lambda_A}(r - v) = 1$, $\mu_{\lambda_A}(rv) = 1$ and $\gamma_{\lambda_A}(r - v) = 0$, $\gamma_{\lambda_A}(rv) = 0$, which implies that $r - v$ and $rv \in A$. Therefore, A is a subring of NR. □

Theorem 3.12. *Let I be a non-empty subset of a normed ring NR, then I is an ideal of NR if and only if $\lambda_I = (\mu_{\lambda_I}, \gamma_{\lambda_I})$ is an intuitionistic fuzzy normed ideal of NR.*

Proof. Let I be an ideal of NR and let $r, v \in NR$.
Case I. If $r, v \in I$ then $rv \in I$ and $\mu_{\lambda_I}(r) = 1$, $\mu_{\lambda_I}(v) = 1$ and $\gamma_{\lambda_I}(r) = 0$, $\gamma_{\lambda_I}(v) = 0$. Thus, $\mu_{\lambda_I}(rv) = 1$ and $\gamma_{\lambda_I}(rv) = 0$. Accordingly, $\mu_{\lambda_I}(rv) = 1 = \mu_{\lambda_I}(r) \diamond \mu_{\lambda_I}(v)$ and $\gamma_{\lambda_I}(rv) = 0 = \gamma_{\lambda_I}(r) * \gamma_{\lambda_I}(v)$.
Case II. If $r \notin I$ or $v \notin I$ so $rv \notin I$, then $\mu_{\lambda_I}(r) = 0$ or $\mu_{\lambda_I}(v) = 0$ and $\gamma_{\lambda_I}(r) = 1$ or $\gamma_{\lambda_I}(v) = 1$. So, $\mu_{\lambda_I}(rv) = 1 \geq \mu_{\lambda_I}(r) \diamond \mu_{\lambda_I}(v)$ and $\gamma_{\lambda_I}(rv) = 0 \leq \gamma_{\lambda_I}(r) * \gamma_{\lambda_I}(v)$. Hence, $\lambda_I = (\mu_{\lambda_I}, \gamma_{\lambda_I})$ is an intuitionistic fuzzy normed ideal of NR.

On the hand, we suppose $\lambda_I = (\mu_{\lambda_I}, \gamma_{\lambda_I})$ is an intuitionistic fuzzy normed ideal of NR. The proof is similar to the second part of the proof of Theorem 3.11. □

Proposition 3.13. *If A is an intuitionistic fuzzy normed ideal of NR, then A_* is an ideal of NR where A_* is defined as,*

$$A_* = \{r \in NR : \mu_A(r) = \mu_A(0) \quad \textit{and} \quad \gamma_A(r) = \gamma_A(0)\}$$

Proof. See [10] (p. 6) □

Lemma 3.14. *Let A and B be two intuitionistic fuzzy normed left (right) ideal of NR. Therefore, $A_* \cap B_* \subseteq (A \cap B)_*$.*

Proof. Let $r \in A_* \cap B_*$, then $\mu_A(r) = \mu_A(0)$, $\mu_B(r) = \mu_B(0)$ and $\gamma_A(r) = \gamma_A(0)$, $\gamma_B(r) = \gamma_B(0)$.

$$\begin{aligned} \mu_{A \cap B}(r) &= min\{\mu_A(r), \mu_B(r)\} \\ &= min\{\mu_A(0), \mu_B(0)\} \\ &= \mu_{A \cap B}(0) \end{aligned}$$

and

$$\begin{aligned}\gamma_{A\cap B}(r) &= max\{\gamma_A(r), \gamma_B(r)\}\\ &= max\{\gamma_A(0), \gamma_B(0)\}\\ &= \gamma_{A\cap B}(0)\end{aligned}$$

So, $r \in (A \cap B)_*$. Thus, $A_* \cap B_* \subseteq (A \cap B)_*$. □

Theorem 3.15. *Let $f : NR \to NR'$ be an epimorphism mapping of normed rings. If A is an intuitionistic fuzzy normed ideal of the normed ring NR, then $f(A)$ is also an intuitionistic fuzzy normed ideal of NR'.*

Proof. Suppose $A = \{(r, \mu_A(r), \gamma_A(r)) : r \in NR\}$,
$f(A) = \{(v, \overset{\diamond}{_{f(r)=v}}\ \mu_A(r), \overset{*}{_{f(r)=v}}\ \gamma_A(r) : r \in NR, v \in NR'\}$.
Let $v_1, v_2 \in NR'$, then there exists $r_1, r_2 \in NR$ such that $f(r_1) = v_1$ and $f(r_2) = v_2$.

i.

$$\begin{aligned}\mu_{f(A)}(v_1 - v_2) &= \overset{\diamond}{_{f(r_1-r_2)=v_1-v_2}}\ \mu_A(r_1 - r_2)\\ &\geq \overset{\diamond}{_{f(r_1)=v_1, f(r_2)=v_2}}\ (\mu_A(r_1) * \mu_A(r_2))\\ &\geq (\overset{\diamond}{_{f(r_1)=v_1}}\ \mu_A(r_1)) * (\overset{\diamond}{_{f(r_2)=v_2}}\ \mu_A(r_2))\\ &\geq \mu_{f(A)}(v_1) * \mu_{f(A)}(v_2)\end{aligned}$$

ii.

$$\begin{aligned}\mu_{f(A)}(v_1 v_2) &= \overset{\diamond}{_{f(r_1 r_2)=v_1 v_2}}\ \mu_A(r_1 r_2)\\ &\geq \overset{\diamond}{_{f(r_2)=v_2}}\ \mu_A(r_2)\\ &\geq \mu_{f(A)}(v_2)\end{aligned}$$

iii.

$$\begin{aligned}\gamma_{f(A)}(v_1 - v_2) &= \overset{*}{_{f(r_1-r_2)=v_1-v_2}}\ \gamma_A(r_1 - r_2)\\ &\leq \overset{*}{_{f(r_1)=v_1, f(r_2)=v_2}}\ (\gamma_A(r_1) \diamond \gamma_A(r_2))\\ &\leq (\overset{*}{_{f(r_1)=v_1}}\ \gamma_A(r_1)) \diamond (\overset{*}{_{f(r_2)=v_2}}\ \gamma_A(r_2))\\ &\leq \gamma_{f(A)}(v_1) \diamond \gamma_{f(A)}(v_2)\end{aligned}$$

iv.

$$\begin{aligned}\gamma_{f(A)}(v_1 v_2) &= \overset{*}{_{f(r_1 r_2)=v_1 v_2}}\ \gamma_A(r_1 r_2)\\ &\leq \overset{*}{_{f(r_2)=v_2}}\ \gamma_A(r_2)\\ &\leq \gamma_{f(A)}(v_2)\end{aligned}$$

Hence, $f(A)$ is an intuitionistic fuzzy normed left ideal. Similarly, it can be justified that $f(A)$ is an intuitionistic fuzzy normed right ideal. Then, $f(A)$ is a intuitionistic fuzzy normed ideal of NR'. □

Proposition 3.16. *Define $f : NR \to NR'$ to be an epimorphism mapping. If B is an intuitionistic fuzzy normed ideal of the normed ring NR', then $f^{-1}(B)$ is also an intuitionistic fuzzy normed ideal of NR.*

Proof. Suppose $B = \{(v, \mu_B(v), \gamma_B(v)) : v \in NR'\}$, $f^{-1}(B) = \{(r, \mu_{f^{-1}(B)}(r), \gamma_{f^{-1}(B)}(r) : r \in NR\}$, where $\mu_{f^{-1}(B)}(r) = \mu_B(f(r))$ and $\gamma_{f^{-1}(B)}(r) = \gamma_B(f(r))$ for every $r \in NR$. Let $r_1, r_2 \in NR$, then

i.
$$\begin{aligned}\mu_{f^{-1}(B)}(r_1 - r_2) &= \mu_B(f(r_1 - r_2))\\ &= \mu_B(f(r_1) - f(r_2))\\ &\geq \mu_B(f(r_1)) * \mu_B(f(r_2))\\ &\geq \mu_{f^{-1}(B)}(r_1) * \mu_{f^{-1}(B)}(r_2)\end{aligned}$$

ii.
$$\begin{aligned}\mu_{f^{-1}(B)}(r_1 r_2) &= \mu_B(f(r_1 r_2))\\ &= \mu_B(f(r_1)f(r_2))\\ &\geq \mu_B(f(r_2))\\ &\geq \mu_{f^{-1}(B)}(r_2)\end{aligned}$$

iii.
$$\begin{aligned}\gamma_{f^{-1}(B)}(r_1 - r_2) &= \gamma_B(f(r_1 - r_2))\\ &= \gamma_B(f(r_1) - f(r_2))\\ &\leq \gamma_B(f(r_1)) \diamond \gamma_B(f(r_2))\\ &\leq \gamma_{f^{-1}(B)}(r_1) \diamond \gamma_{f^{-1}(B)}(r_2)\end{aligned}$$

iv.
$$\begin{aligned}\gamma_{f^{-1}(B)}(r_1 r_2) &= \gamma_B(f(r_1 r_2))\\ &= \gamma_B(f(r_1)f(r_2))\\ &\leq \gamma_B(f(r_2))\\ &\leq \gamma_{f^{-1}(B)}(r_2)\end{aligned}$$

Therefore, $f^{-1}(B)$ is an intuitionistic fuzzy normed left ideal of NR. Similarly, it can be justified that $f^{-1}(B)$ is an intuitionistic fuzzy normed right ideal. So, $f^{-1}(B)$ is a intuitionistic fuzzy normed ideal of NR. □

4. Intuitionistic fuzzy normed prime ideal and intuitionistic fuzzy normed maximal ideal

In what follows, we produce the terms of intuitionistic fuzzy normed prime ideals and intuitionistic fuzzy normed maximal ideals and we investigate some associated properties.

Definition 4.1. *An intuitionistic fuzzy normed ideal $A = (\mu_A, \gamma_A)$ of a normed ring NR is said to be an intuitionistic fuzzy normed prime ideal of NR if for an intuitionistic fuzzy normed ideals $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ of NR where $B \circ C \subseteq A$ indicates that either $B \subseteq A$ or $C \subseteq A$, which imply that $\mu_B \subseteq \mu_A$ and $\gamma_A \subseteq \gamma_B$ or $\mu_C \subseteq \mu_A$ and $\gamma_A \subseteq \gamma_C$.*

Proposition 4.2. *An intuitionistic fuzzy normed ideal $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normed prime ideal if for any two intuitionistic fuzzy normed ideals $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ of NR satisfies:*

i. $\mu_A \supseteq \mu_{B \circledast C}$ *i.e.* $\mu_A(r) \geq \underset{r=vz}{\overset{\diamond}{}}(\mu_A(v) * \mu_B(z))$;
ii. $\gamma_A \subseteq \gamma_{B \otimes C}$ *i.e.*$\gamma_A(r) \leq \underset{r=vz}{\overset{*}{}}(\gamma_A(v) \diamond \gamma_B(z))$.

Theorem 4.3. *Let A be an intuitionistic fuzzy normed prime ideal of NR. Then* $|\mathrm{Im}\ \mu_A| = |\mathrm{Im}\ \gamma_A| = 2$; in other words A is two-valued.

Proof. As A is not constant, $|\mathrm{Im}\,\mu_A| \geq 2$. assume that $|\mathrm{Im}\,\mu_A| \geq 3$. $A_{\alpha,\beta} = \{r \in R : \mu_A \geq \alpha \text{ and } \gamma_A \leq \beta\}$ where $\alpha + \beta \leq 1$. Let $r \in NR$ and let B and C be two intuitionistic fuzzy subsets in NR, such that: $\mu_A(0) = s$ and $k = glb\{\mu_A(r) : r \in NR\}$, so there exists $t, \alpha \in \mathrm{Im}(\mu_A)$ such that $k \leq t < \alpha < s$ with $\mu_B(r) = \frac{1}{2}(t + \alpha)$, $\mu_C(r) = \begin{cases} s & , if\ r \in A_{\alpha,\beta} \\ k & , if\ r \notin A_{\alpha,\beta} \end{cases}$ and $\gamma_A(0) = c$ and $h = lub\{\gamma_A(r) : r \in NR\}$, then there exists $d, \beta \in \mathrm{Im}(\gamma_A)$ such that $c < \beta < d \leq h$ with $\gamma_B(r) = \frac{1}{2}(d + \beta)$ and $\gamma_C(r) = \begin{cases} c & , if\ r \in A_{\alpha,\beta} \\ h & , if\ r \notin A_{\alpha,\beta} \end{cases}$ for all $r \in NR$. Clearly B is an intuitionistic fuzzy normed ideal of NR. Now we claim that C is an intuitionistic fuzzy normed ideal of NR.

Let $r, v \in NR$, if $r, v \in A_{\alpha,\beta}$ then $r - v \in A_{\alpha,\beta}$ and $\mu_C(r - v) = s = \mu_C(r) * \mu_C(v)$, $\gamma_C(r - v) = c = \gamma_C(r) \diamond \gamma_C(v)$. If $r \in A_{\alpha,\beta}$ and $v \notin A_{\alpha,\beta}$ then $r - v \notin A_{\alpha,\beta}$ so, $\mu_C(r - v) = k = \mu_C(r) * \mu_C(v)$, $\gamma_C(r - v) = h = \gamma_C(r) \diamond \gamma_C(v)$. If $r, v \notin A_{\alpha,\beta}$ then $r - v \notin A_{\alpha,\beta}$ so, $\mu_C(r - v) \geq k = \mu_C(r) * \mu_C(v)$, $\gamma_C(r - v) \leq h = \gamma_C(r) \diamond \gamma_C(v)$. Hence, $\mu_C(r - v) \geq \mu_C(r) * \mu_C(v)$ and $\gamma_C(r - v) \leq \gamma_C(r) \diamond \gamma_C(v)$ for all $r, v \in NR$.

Now if $r \in A_{\alpha,\beta}$ then $rv \in A_{\alpha,\beta}$, thus $\mu_C(rv) = s = \mu_C(r) \diamond \mu_C(v)$ and $\gamma_C(rv) = c = \gamma_C(r) * \gamma_C(v)$. If $r \notin A_{\alpha,\beta}$, then $\mu_C(rv) \geq k = \mu_C(r) \diamond \mu_C(v)$ and $\gamma_C(rv) \leq h = \gamma_C(r) * \gamma_C(v)$. Therefore C is an intuitionistic fuzzy normed ideal of NR.

To prove that $B \circ C \subseteq A$. Let $r \in NR$, we discuss the following cases:

(i) If $r = 0$, consequently

$$\mu_{B \circledast C}(0) = \overset{\diamond}{\underset{r = uv}{}} (\mu_B(u) * \mu_C(v)) \leq \frac{1}{2}(t + \alpha) < s = \mu_A(0);$$

$$\gamma_{B \otimes C}(r) = \overset{*}{\underset{r = uv}{}} (\gamma_B(u) \diamond \gamma_C(v)) \geq \frac{1}{2}(d + \beta) > c = \gamma_A(0).$$

(ii) If $r \neq 0$, $r \in A_{\alpha,\beta}$. Then $\mu_A(r) \geq \alpha$ and $\gamma_A(r) \leq \beta$. Thus,

$$\mu_{B \circledast C}(r) = \overset{\diamond}{\underset{r = uv}{}} (\mu_B(u) * \mu_C(v)) \leq \frac{1}{2}(t + \alpha) < \alpha \leq \mu_A(r);$$

$$\gamma_{B \otimes C}(r) = \overset{*}{\underset{r = uv}{}} (\gamma_B(u) \diamond \gamma_C(v)) \geq \frac{1}{2}(d + \beta) > \beta \geq \gamma_A(r).$$

Since $\mu_B(u) * \mu_C(v) \leq \mu_B(u)$ and $\gamma_B(u) \diamond \gamma_C(v) \geq \gamma_B(u)$.

(iii) If $r \neq 0$, $r \notin A_{\alpha,\beta}$. Then in that case $u, v \in NR$ such that $r = uv$, $u \notin A_{\alpha,\beta}$ and $v \notin A_{\alpha,\beta}$. Then,

$$\mu_{B \circledast C}(r) = \overset{\diamond}{\underset{r = uv}{}} (\mu_B(u) * \mu_C(v)) = k \leq \mu_A(r);$$

$$\gamma_{B \otimes C}(r) = \overset{*}{\underset{r = uv}{}} (\gamma_B(u) \diamond \gamma_C(v)) = h \geq \gamma_A(r).$$

Therefore, in any case $\mu_{B \circledast C}(r) \leq \mu_A(r)$ and $\gamma_{B \otimes C}(r) \geq \gamma_A(r)$ for all $r \in NR$. Hence, $B \circ C \subseteq A$.

Let $a, b \in NR$ such that $\mu_A(a) = t$, $\mu_A(b) = \alpha$ and $\gamma_A(a) = d$, $\gamma_A(b) = \beta$. Thus, $\mu_B(a) = \frac{1}{2}(t + \alpha) > t = \mu_A(r)$ and $\gamma_B(a) = \frac{1}{2}(d + \beta) < d = \gamma_A(r)$ which implies that $B \nsubseteq A$. Also, $\mu_A(b) = \alpha$ and $\gamma_A(b) = \beta$ imply that $b \in A_{\alpha,\beta}$ so, $\mu_C(b) = s > \alpha$ and $\gamma_C(b) = c < \beta$, so $C \nsubseteq A$. Therefore, neither $B \nsubseteq A$ nor

$C \nsubseteq A$. This indicates that A could not be an intuitionistic fuzzy normed prime ideal of NR, so its a contradiction. Thus, $|\text{Im } \mu_A| = |\text{Im}\gamma_A| = 2$. □

Proposition 4.4. *If A is an intuitionistic fuzzy normed prime ideal of NR, so the following are satisfied:*

i. $\mu_A(0_{NR}) = 1$ *and* $\gamma_A(0_{NR}) = 0$*;*
ii. $\text{Im}(\mu_A) = \{1, \alpha\}$ and $\text{Im}(\gamma_A) = \{0, \beta\}$, where $\alpha, \beta \in [0, 1]$;
iii. A_* is a prime ideal of NR.

Theorem 4.5. *Let A be a fuzzy subset of NR where A is two-valued,* $\mu_A(0) = 1$ *and* $\gamma_A(0) = 0$*, and the set* $A_* = \{r \in NR : \mu_A(r) = \mu_A(0) \text{ and } \gamma_A(r) = \gamma_A(0)\}$ *is a prime ideal of NR. Hence, A is an intuitionistic fuzzy normed prime ideal of NR.*

Proof. We have $\text{Im}(\mu_A) = \{1, \alpha\}$ and $\text{Im}(\gamma_A) = \{0, \beta\}$. Let $r, v \in NR$. If $r, v \in A_*$, then $r - v \in A_*$ so, $\mu_A(r - v) = 1 = \mu_A(r) * \mu_A(v)$ and $\gamma_A(r - v) = 0 = \gamma_A(r) \diamond \gamma_A(v)$. If $r, v \notin A_*$, then $\mu_A(r - v) = \alpha \geq \mu_A(r) * \mu_A(v)$ and $\gamma_A(r - v) = \beta \leq \gamma_A(r) \diamond \gamma_A(v)$.

Therefore, for all $r, v \in NR$,

$$\mu_A(r - v) \geq \mu_A(r) * \mu_A(v)$$
$$\gamma_A(r - v) \leq \gamma_A(r) \diamond \gamma_A(v)$$

Similarly,

$$\mu_A(rv) \geq \mu_A(r) \diamond \mu_A(v)$$
$$\gamma_A(rv) \leq \gamma_A(r) * \gamma_A(v)$$

Thus A is an intuitionistic fuzzy ideal of NR.

Assume B and C be fuzzy ideals of NR where $B \circ C \subseteq A$. Assume that $B \nsubseteq A$ and $C \nsubseteq A$. Then, we have $r, v \in NR$ in such a way that $\mu_B(r) > \mu_A(r)$ and $\gamma_B(r) < \gamma_A(r)$, $\mu_C(v) > \mu_A(v)$ and $\gamma_C(r) < \gamma_A(r)$, so for all $a \in A_*$, $\mu_A(a) = 1 = \mu_A(0)$ and $\gamma_A(a) = 0 = \gamma_A(0)$, $r \notin A_*$ and $v \notin A_*$. Since, A_* is a prime ideal of NR, we have $n \in NR$ in such a way that $rnv \notin A_*$. Let $a = rnv$ then $\mu_A(a) = \mu_A(r) = \mu_A(v) = \alpha$ and $\gamma_A(a) = \gamma_A(r) = \gamma_A(v) = \beta$, now

$$\begin{aligned}
\mu_{B \circledast C}(a) &= \underset{a=st}{\diamond}(\mu_B(s) * \mu_C(t)) \\
&\geq \mu_B(r) * \mu_C(nv) \\
&\geq \mu_B(r) * \mu_C(v) \\
&> \alpha = \mu_A(a) \quad [Since, \mu_B(r) \geq \mu_A(r) = \alpha \text{ and } \mu_C(nv) \geq \mu_C(v) \geq \mu_A(v) = \alpha].
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{B \otimes C}(a) &= \underset{a=st}{*}(\gamma_B(s) \diamond \gamma_C(t)) \\
&\leq \gamma_B(r) \diamond \gamma_C(nv) \\
&\leq \gamma_B(r) \diamond \gamma_C(v) \\
&< \beta = \gamma_A(a) \quad [Since, \gamma_B(r) \leq \gamma_A(r) = \beta \text{ and } \gamma_C(nv) \leq \gamma_C(v) \leq \gamma_A(v) = \beta].
\end{aligned}$$

Which means that $B \circ C \nsubseteq A$. Which contradicts with the hypothesis that $B \circ C \subseteq A$. Therefore, either $B \subseteq A$ or $C \subseteq A$. Then A is an intuitionistic fuzzy normed prime ideal. □

Theorem 4.6. *Let P be a subset (non-empty) of NR. P is a prime ideal if and only if the intuitionistic characteristic function* $\lambda_P = (\mu_{\lambda_P}, \gamma_{\lambda_P})$ *is an intuitionistic fuzzy normed prime ideal.*

Proof. presume that P is a prime ideal of NR. So by Theorem 3.12, λ_P is an intuitionistic fuzzy normed ideal of NR. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any intuitionistic fuzzy normed ideals of NR with $A \circ B \subseteq \lambda_P$ while $A \nsubseteq \lambda_P$ and $B \nsubseteq \lambda_P$. Then there exist $r, v \in NR$ such that

$$\mu_A(r) \neq 0, \gamma_A(r) \neq 1 \quad and \quad \mu_B(v) \neq 0, \gamma_B(v) \neq 1$$

but

$$\mu_{\lambda_P}(r) = 0, \gamma_{\lambda_P}(r) = 1 \quad and \quad \mu_{\lambda_P}(v) = 0, \gamma_{\lambda_P}(v) = 1$$

Therefore, $r \notin P$ and $v \notin P$. Since P is a prime ideal, there exist $n \in NR$ such that $rnv \notin P$.

Let $a = rnv$, then $\mu_{\lambda_P}(a) = 0$ and $\gamma_{\lambda_P}(a) = 1$. Thus, $\mu_{A\circledast b}(a) = 0$ and $\gamma_{A\otimes B}(a) = 1$. but

$$\begin{aligned} \mu_{A\circledast B}(a) &= \underset{a=st}{\diamond}(\mu_A(s) * \mu_B(t)) \\ &\geq \mu_A(r) * \mu_B(nv) \\ &\geq \mu_A(r) * \mu_B(v) \\ &\geq min\{\mu_A(r), \mu_B(v)\} \\ &\neq 0 \quad [Since, \mu_A(r) \neq 0 \; and \; \mu_B(v) \neq 0]. \end{aligned}$$

and

$$\begin{aligned} \gamma_{A\otimes B}(a) &= \underset{a=st}{*}(\gamma_A(s) \diamond \gamma_B(t)) \\ &\leq \gamma_A(r) \diamond \gamma_B(nv) \\ &\leq \gamma_A(r) \diamond \gamma_B(v) \\ &\leq max\{\gamma_A(r), \gamma_B(v)\} \\ &\neq 1 \quad [Since, \gamma_A(r) \neq 1 \; and \; \gamma_B(v) \neq 1]. \end{aligned}$$

This is a contradiction with $\mu_{\lambda_P}(a) = 0$ and $\gamma_{\lambda_P}(a) = 1$. Thus for any intuitionistic fuzzy normed ideals A and B of NR we have $A \circ B \subseteq \lambda_P$ imply that $A \subseteq \lambda_P$ or $B \subseteq \lambda_P$. So, $\lambda_P = (\mu_{\lambda_P}, \gamma_{\lambda_P})$ is an intuitionistic fuzzy normed prime ideal of NR.

Conversely, suppose λ_P is an intuitionistic fuzzy normed prime ideal. Let A and B be two intuitionistic fuzzy normed prime ideal of NR such that $A \circ B \subseteq P$. Let $r \in NR$, suppose $\mu_{\lambda_A \circledast \lambda_B}(r) \neq 0$ and $\gamma_{\lambda_A \otimes \lambda_B}(r) \neq 1$, then $\mu_{\lambda_A \circledast \lambda_B}(r) = \underset{r=cd}{\diamond}(\mu_{\lambda_A}(c) * \mu_{\lambda_B}(d)) \neq 0$ and $\gamma_{\lambda_A \otimes \lambda_B}(r) = \underset{r=cd}{*}(\gamma_{\lambda_A}(c) \diamond \gamma_{\lambda_B}(d)) \neq 1$. Then we have $c, d \in NR$ such that $r = cd$, $\mu_{\lambda_A}(c) \neq 0$, $\mu_{\lambda_B}(d) \neq 0$ and $\gamma_{\lambda_A}(c) \neq 1$, $\gamma_{\lambda_B}(d) \neq 1$. Then, $\mu_{\lambda_A}(c) = 1$, $\mu_{\lambda_B}(d) = 1$ and $\gamma_{\lambda_A}(c) = 0$, $\gamma_{\lambda_B}(d) = 0$. Which implies $c \in A$ and $d \in B$, therefore $r = cd \in A \circ B \subseteq P$. Then, $\mu_{\lambda_P}(r) = 1$ and $\gamma_{\lambda_P}(r) = 0$. Thus, for all $r \in NR$, $\mu_{\lambda_A \circledast \lambda_B}(r) \leq \mu_{\lambda_P}(r)$ and $\gamma_{\lambda_A \otimes \lambda_B}(r) \geq \gamma_{\lambda_P}(r)$. So, $\lambda_A \circ \lambda_B \subseteq \lambda_P$. Since λ_P is an intuitionistic fuzzy normed prime ideal. Then either $\lambda_A \subseteq \lambda_P$ or $\lambda_B \subseteq \lambda_P$. Therefore, either $A \subseteq P$ or $B \subseteq P$. Hence P is a prime ideal in NR. □

Definition 4.7. [15] *Given a ring R and a proper ideal M of R, M is a maximal ideal of R if any of the following equivalent conditions hold:*

i. *There exists no other proper ideal J of R so that $M \subsetneq J$.*
ii. *For any ideal J with $M \subseteq J$, either $J = M$ or $J = R$.*

Definition 4.8. *An intuitionistic fuzzy normed ideal A of a normed ring NR is said to be an intuitionistic fuzzy normed maximal ideal if for any intuitionistic fuzzy normed ideal B of NR, $A \subseteq B$, implies that either $B_* = A_*$ or $B = \lambda_{NR}$. Intuitionistic fuzzy normed maximal left (right) ideal are correspondingly specified.*

Proposition 4.9. *Let A be an intuitionistic fuzzy normed maximal left (right) ideal of NR. Then,* $|\mathrm{Im}\ \mu_A| = |\mathrm{Im}\ \gamma_A| = 2$

Theorem 4.10. *Let A be an intuitionistic fuzzy normed maximal left (right) ideal of a normed ring NR. Then $A_* = \{r \in NR : \mu_A(r) = \mu_A(0)$ and $\gamma_A(r) = \gamma_A(0)\}$ is a maximal left (right) ideal of NR.*

Proof. As A is not constant, $A_* \neq NR$. Then using Proposition 4.9, A is two-valued. Let $\mathrm{Im}(\mu_A) = \{1, \alpha\}$ and $\mathrm{Im}(\gamma_A) = \{0, \beta\}$, where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. Assume M to be a left ideal of NR in away that $A_* \subseteq M$. Take B be an intuitionistic fuzzy subset of NR where if $r \in M$ then $\mu_B(r) = 1$ and $\gamma_B(r) = 0$ and if $r \notin M$ then $\mu_B(r) = c$ and $\gamma_B(r) = d$, where $\alpha < c < 1$ and $0 < d < \beta$. Then B is an intuitionistic fuzzy normed left ideal. Obviously $A \subseteq B$. As A is an intuitionistic fuzzy normed maximal left ideal of NR then $A_* = B_*$ or $B = \lambda_{NR}$. If $A_* = B_*$ then $A_* = M$ given that $B_* = M$. If $B = \lambda_{NR}$ subsequently $M = NR$. Therefore, A_* is a maximal left ideal of NR. □

Theorem 4.11. *If A is an intuitionistic fuzzy normed maximal left (right) ideal of NR, then $\mu_A(0) = 1$ and $\gamma_A(0) = 0$.*

Proof. Suppose $\mu_A(0) \neq 1$ and $\gamma_A(0) \neq 0$ and B to be an intuitionistic fuzzy subset of NR defined as $B = \{r \in NR : \mu_B(r) = h$ and $\gamma_B(r) = k\}$, where $\mu_A(0) < h < 1$ and $0 < k < \gamma_A(0)$. Then, B is an intuitionistic fuzzy normed ideal of NR. We can simply check that $A \subset B$, $B \neq \lambda_{NR}$ and $B_* = \{r \in NR : \mu_B(r) = \mu_B(0)$ and $\gamma_B(r) = \gamma_B(0)\} = NR$. Hence, $A \subset B$ but $A_* \neq B_*$ and $B \neq \lambda_{NR}$ which contradicts with the assumption that A is an intuitionistic fuzzy normed maximal ideal of NR. Therefore, $\mu_A(0) = 1$ and $\gamma_A(0) = 0$. □

Theorem 4.12. *Let A be a intuitionistic fuzzy normed left (right) ideal of NR. If A_* is a maximal left (right) ideal of NR with $\mu_A(0) = 1$ and $\gamma_A(0) = 0$, then A is an intuitionistic fuzzy normed maximal left (right) ideal of NR.*

Proof. By Proposition 4.9 A is two-valued. Let $\mathrm{Im}(\mu_A) = \{1, \alpha\}$ and $\mathrm{Im}(\gamma_A) = \{0, \beta\}$, where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. Define B to be an intuitionistic fuzzy normed left ideal of NR where $A \subseteq B$. Hence, $\mu_B(0) = 1$ and $\gamma_B(0) = 0$. Let $r \in A_*$. Then $1 = \mu_A(0) = \mu_A(r) \leq \mu_B(r)$ and $0 = \gamma_A(0) = \gamma_A(r) \geq \gamma_B(r)$. Thus $\mu_B(r) = 1 = \mu_B(0)$ and $\gamma_B(r) = 0 = \gamma_B(0)$, hence $r \in B_*$ then $A_* \subseteq B_*$. Given that A_* a maximal left ideal of NR, then $A_* = B_*$ or $B_* = NR$. If $B_* = NR$ subsequently $B = \lambda_{NR}$. Therefore, A is an intuitionistic fuzzy normed maximal left ideal of NR. □

Remark 4.13. *Let $A \subseteq NR$ and let $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. Let $\lambda_{A_{\alpha,\beta}}$ be an intuitionistic fuzzy subset of NR where $\mu_{\lambda_{A_\alpha}}(r) = 1$ if $r \in A$, $\mu_{\lambda_{A_\alpha}}(r) = \alpha$ if $r \notin A$ and $\gamma_{\lambda_{A_\beta}}(r) = 0$ if $r \in A$, $\gamma_{\lambda_{A_\beta}}(r) = \beta$ if $r \notin A$. If $\alpha = 0$ and $\beta = 1$, the $\lambda_{A_{\alpha,\beta}}$ is the intuitionistic characteristic function of A, which identified by $\lambda_A = (\mu_{\lambda_A}, \gamma_{\lambda_A})$. If NR is a ring and A is an intuitionistic fuzzy normed left (right) ideal of NR, then:*

- $\mu_{\lambda_{A_\alpha}}(0) = 1$, $\gamma_{\lambda_{A_\beta}}(0) = 0$;
- $(\lambda_{A_{\alpha,\beta}})_* = A$, $[(\lambda_{A_{\alpha,\beta}})_* = \{r \in NR : \mu_{\lambda_{A_\alpha}}(r) = \mu_{\lambda_{A_\alpha}}(0), \gamma_{\lambda_{A_\beta}}(r) = \gamma_{\lambda_{A_\beta}}(0)\} = A]$;
- $\mathrm{Im}(\mu_A) = \{1, \alpha\}$ and $\mathrm{Im}(\gamma_A) = \{0, \beta\}$;
- $\lambda_{A_{\alpha,\beta}}$ is an intuitionistic fuzzy normed left (right) ideal of NR.

5. Conclusions

In this article, we defined the intrinsic product of two intuitionistic fuzzy normed ideals and proved that this product is a subset of their intersection. Also, we characterized some properties of intuitionistic fuzzy normed ideals. We initiated the concepts of intuitionistic fuzzy normed prime ideal and intuitionistic fuzzy normed maximal ideal and we established several results related to these ideals. Further, we specified the conditions under which a given intuitionistic fuzzy normed ideal is considered to be an intuitionistic fuzzy normed prime (maximal) ideal. We generalised the relation between the intuitionistic characteristic function and prime (maximal) ideals.

Conflict of interest

The author declares no conflict of interest in this paper

References

1. A. Al-Masarwah, A. G. Ahmad, Structures on doubt neutrosophic ideals of BCK/BCI-algebras under (S, T)-norms, *Neutrosophic Sets Syst.*, **33** (2020), 275–289.
2. A. U. M. Alkouri, A. R. Salleh, Complex Atanassov's intuitionistic fuzzy relation, *Abstr. Appl. Anal.*, **2013** (2013), 287382.
3. A. Uddin Ahmed, S. Rahman, B. Davvaz, On intuitionistic fuzzy idempotent, prime, strongly irreducible and t-pure ideals of semirings, *J. Intell. Fuzzy Syst.*, **33** (2017), 433–443.
4. K. Atanassov, On the most extended modal operator of first type over interval-valued intuitionistic fuzzy sets, *Mathematics*, **6** (2018), 123.
5. K. T. Atanassov, Interval valued intuitionistic fuzzy sets, In: *Intuitionistic Fuzzy Sets*, Physica, Heidelberg, 1999.
6. K. T. Atanassov, Intuitionistic fuzzy sets, In: *Intuitionistic Fuzzy Sets*, Physica, Heidelberg, 1999.
7. K. T. Atanassov, Intuitionistic fuzzy sets, *International Journal Bioautomation*, **20** (2016), 1–6.
8. L. Atanassov, On intuitionistic fuzzy versions of L. Zadehs extension principle, *Notes on intuitionistic Fuzzy Sets*, **13** (2006), 33–36.
9. M. O. Alsarahead, A. G. Ahmad, Complex intuitionistic fuzzy ideals, *AIP conf. proc.*, **1940** (2018), 020118.
10. N. Abed Alhaleem, A. G. Ahmad, Intuitionistic fuzzy normed subrings and intuitionistic fuzzy normed ideals, *Mathematics*, **8** (2020), 1594.
11. N. Abed Alhaleem, A. G. Ahmad, Intuitionistic fuzzy normal subrings over normed rings, *Int. J. Anal. Appl.*, **19** (2021), 341–359.
12. N. Abed Alhaleem, A. G. Ahmad, Intuitionistic anti fuzzy normed ideals, *Notes on intuitionistic Fuzzy Sets*, **27** (2021), 60–71.
13. B. Banerjee, D. K. Basnet, Intuitionistic fuzzy subrings and ideals, *J. Fuzzy Math.,* **11** (2003), 139–155.

14. I. Bakhadach, S. Melliani, M. Oukessou, L. Chadli Intuitionistic fuzzy ideal and intuitionistic fuzzy prime ideal in a ring, *Notes Intuitionistic Fuzzy Sets*, **22** (2016), 59–63.
15. D. S. Dummit, R. M. Foote, *Abstract algebra*, Wiley, 2004.
16. V. Dixit, R. Kumar, N. Ajmal, Fuzzy ideals and fuzzy prime ideals of a ring, *Fuzzy Set. Syst.*, **44** (1991), 127–138.
17. A. Emniyet, M. Şahin, Fuzzy normed rings, *Symmetry*, **10** (2018), 515.
18. I. Gelfand, Normierte ringe, *Rec. Math. [Mat. Sbornik] N. S.*, **9** (1941), 3–24.
19. M. M. Gupta, J. Qi, Theory of T-norms and fuzzy inference methods, *Fuzzy Set. Syst.*, **40** (1991), 431–450.
20. K. Hur, S. Y. Jang, H. W. Kang, Intuitionistic fuzzy ideals of a ring, *Pure Appl. Math.*, **12** (2005), 193–209.
21. K. H. Kim, J. G. Lee, S. M. Lee, Intuitionistic (T, S) normed fuzzy ideals of Γ- rings, *Int. Math. Forum*, **3** (2008), 115–123.
22. A. Malik, J. N. Mordeson, Fuzzy prime ideals of a ring, *Fuzzy Set. Syst.*, **37** (1990), 93–98.
23. M. F. Marashdeh, A. R. Salleh, Intuitionistic fuzzy rings, *Int. J. Algebra*, **5** (2011), 37–47.
24. T. Mukherjee, M. Sen, On fuzzy ideals of a ring I, *Fuzzy Set. Syst.*, **21** (1987), 99–104.
25. M. A. Najmark, *Normed rings*, Groningen: Noordhoff, 1964.
26. R. S. Rahmat, M. S. M. Noorani, Fixed point theorem on intuitionistic fuzzy metric spaces, *Iran. J. Fuzzy Syst.*, **3** (2006), 23–29.
27. U. Swamy, K. Swamy, Fuzzy prime ideals of rings, *J. Math. Anal. Appl.*, **134** (1988), 94–103.
28. L. A. Zadeh, Fuzzy sets, *Information and Control*, **8** (1965), 338–353.