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*Research article*

# Representations of a non-pointed Hopf algebra

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**Abstract:** In this paper, we construct all the indecomposable modules of a class of non-pointed Hopf algebras, which are quotient Hopf algebras of a class of prime Hopf algebras of GK-dimension one. Then the decomposition formulas of the tensor product of any two indecomposable modules are established. Based on these results, the representation ring of the Hopf algebras is characterized by generators and some relations.

**Keywords:** Hopf algebra; representation type; indecomposable module; tensor product; representation ring
**Mathematics Subject Classification:** 16D70, 16G60, 16T05

## 1. Introduction

There are a lot of works about classification of Hopf algebras or Nichols algebras of finite GK-dimension. For example, the readers can refer to [1, 2]. In the paper [2], Liu tried to classify all prime Hopf algebras of GK-dimension one and constructed a series of new examples of non-pointed Hopf algebras $D(\underline{m}, d, \gamma)$. As a by-product, a series of finite-dimensional non-semisimple quotient Hopf algebras are obtained, which have no Chevalley property. To understand this new class of quotient Hopf algebras, we see that those quotient Hopf algebras under the conditions that $n$ is odd and $d = 2n$, denoted by $D(n)$, are just isomorphic to an extension of the generalized quaternion group algebra $\Bbbk Q_{4n}$ equipped with a nontrivial suitable coalgebraic structure. It is well known that the representations of the generalized quaternion group $Q_{4n}$ have been known for a long time. In [3, 4], all the irreducible representations of $Q_{4n}$ are given. As applications, [5] determined the complex representation rings of $Q_{4n}$ and gave the isomorphism class of the $n$-th augmentation quotient of the augmentation ideal. In [6] the group code over the generalized quaternion group $Q_{4n}$ is studied, which is based on representations of $Q_{4n}$. It is important in cryptography.

The task of this paper is to classify all the indecomposable modules of $D(n)$ explicitly. The decomposition formulas of the tensor products of them are established. Finally, we describe the representation ring of $D(n)$ by generators and generating relations. There is much effort to put into understanding and classifying all indecomposable modules of algebras of finite representation type. The readers can refer to the books [7, 8] for the representation theory of algebras and some newest results [9, 10] for example. In [11], Yang determined the representation type of a class of pointed Hopf algebras and classified all indecomposable modules of simple-pointed Hopf algebra $R(q, \alpha)$. In the paper [12], the representations of the half of the small quantum group $\mathfrak{u}_q(\mathfrak{sl}_2)$ were constructed by the technique of the deformed preprojective algebras. Furthermore, a lot of papers investigated the representation rings of various Hopf algebras, the readers can refer to [13–17]. By techniques of generators and generating relations, Su and Yang described representation rings of the weak generalized Taft Hopf algebras as well as some small quantum groups in [15, 16]. Sun et al. described the representation rings of Drinfeld doubles of Taft algebras in [17]. Motivated by the above works, we shall establish the decomposition formulas of the tensor products of the indecomposable $D(n)$-modules and determine the representation ring of $D(n)$. This can help us to understand the structure and representation theory of $D(n)$ in a better way.

The paper is organized as follows. In Section 2, we review the definition of $D(n)$ and show that $D(n)$ is of finite representation type. In Section 3, we shall construct all the indecomposable $D(n)$-modules and establish all the decomposition formulas of the tensor product of two indecomposable $D(n)$-modules. In Section 4, we characterize the representation ring of $D(n)$ by three generators and some generating relations.

For the theories of Hopf algebras and representation theory, we refer to [7, 8, 18, 19].

## 2. Preliminaries

Throughout this paper, we work over an algebraic closed field $\Bbbk$ of characteristic 0. Unless otherwise stated, all algebras, Hopf algebras, and modules are finite-dimensional over $\Bbbk$, all maps are $\Bbbk$-linear, dim and $\otimes$ stand for $\dim_{\Bbbk}$ and $\otimes_{\Bbbk}$, respectively. In this paper, we describe the representations of a quotient of the Hopf algebra $D(\underline{m}, d, \gamma)$ for the case of $m = 2$.

Firstly, we review the definition of the Hopf algebra $D(\underline{2}, d, \gamma)$ in [2]. Let $2|d$. As an algebra, it is generated by $a^{\pm 1},\ b^{\pm 1},\ c,\ u_0,\ u_1$, subject to the following relations

$$aa^{-1} = a^{-1}a = 1,\ bb^{-1} = b^{-1}b = 1,\ a^{2d} = b^2,\ c^2 = 1 - a^{2d}, \tag{2.1}$$

$$ab = ba,\ ac = ca,\ cb = -bc,\ au_k = u_k a^{-1}(k = 1, 2), \tag{2.2}$$

$$cu_0 = 2u_1 = \omega a^d u_0 c,\ cu_1 = 0 = u_1 c,\ u_0 b = a^{-2d} b u_0,\ u_1 b = -a^{-2d} b u_1, \tag{2.3}$$

$$u_0^2 = a^{-\frac{3d}{2}} b,\ u_1^2 = 0,\ u_0 u_1 = -\frac{1}{2}\omega a^{-\frac{3d}{2}} cb,\ u_1 u_0 = \frac{1}{2} a^{-\frac{3d}{2}} cb, \tag{2.4}$$

where $\omega \in \Bbbk$ is a primitive 4-th root of unity. The comultiplication $\Delta$, the counit $\epsilon$ and the antipode $S$ of $D(\underline{2}, d, \gamma)$ are given by

$$\begin{aligned}
&\Delta(a) = a \otimes a,\ \Delta(b) = b \otimes b,\ \Delta(c) = c \otimes b + 1 \otimes c,\\
&\Delta(u_0) = u_0 \otimes u_0 - u_1 \otimes a^{-d} b u_1,\ \Delta(u_1) = u_0 \otimes u_1 + u_1 \otimes a^{-d} b u_0,\\
&\epsilon(a) = \epsilon(b) = \epsilon(u_0) = 1,\ \epsilon(c) = \epsilon(u_0) = 0,\\
&S(a) = a^{-1},\ S(b) = b^{-1},\ S(c) = -cb^{-1},\ S(u_0) = a^{-\frac{3d}{2}} b u_0,\ S(u_1) = -\omega a^{-\frac{d}{2}} u_1.
\end{aligned}$$

From now on, assume that $n$ is odd and $d = 2n$. Let $D(n)$ be the quotient Hopf algebra

$$D(n) =: D(\underline{2}, d, \gamma)/(a^n - 1).$$

We claim that the Hopf algebra $D(n)$ can be viewed as the Hopf algebra generated by $x, y, z$ satisfying the following relations

$$x^{2n} = 1,\ y^2 = 0,\ x^n = z^2,\ xy = -yx,\ xz = zx^{-1},\ yz = \omega zy.$$

The comultiplication $\Delta$, the counit $\epsilon$ and the antipode $S$ are given by

$$\Delta(x) = x \otimes x,\ \Delta(y) = 1 \otimes y + y \otimes z^2,\ \Delta(z) = z \otimes z + yz \otimes yz^{-1},$$

$$\epsilon(x) = \epsilon(z) = 1,\ \epsilon(y) = 0,\ S(x) = x^{-1},\ S(y) = -yz^2,\ S(z) = z^{-1}.$$

Indeed, we define the maps $\varphi$ and $\psi$ as

$$\varphi: \quad a \mapsto xz^2,\ b \mapsto z^2,\ c \mapsto 2y,\ u_0 \mapsto z,\ u_1 \mapsto yz$$

and

$$\psi: \quad x \mapsto ab, \quad y \mapsto \frac{1}{2}c, \quad z \mapsto u_0,$$

respectively. It is straightforward to check that $\varphi$ and $\psi$ are Hopf algebra isomorphisms and $\psi \circ \varphi = id$. Hence we have the claim.

Now, we use the later generators and relations to define the Hopf algebra $D(n)$. Actually, $D(n)$ is a class of non-pointed Hopf algebras.

It is noted that the generalized quaternion group algebra $\Bbbk Q_{4n}$ of order $4n$ is defined as

$$\Bbbk Q_{4n} = \Bbbk\langle x, z |\ x^{2n} = 1,\ x^n = z^2,\ xz = zx^{-1}\rangle.$$

Obviously, it can be embedded into the algebra $D(n)$ as an algebra, but not as a Hopf algebra.

Firstly, we determine the representation type of $D(n)$ as an algebra.

**Lemma 2.1.** *The algebra $D(n)$ is of finite representation type.*

*Proof.* Let $Q_{4n}$ be the generalized quaternion group

$$\langle x, z | x^{2n} = 1,\ x^n = z^2,\ xz = zx^{-1}\rangle,$$

and $A = \Bbbk\langle y | y^2 = 0\rangle$ a $\Bbbk$-algebra. It is obvious that $A$ is of finite representation type. Define

$$y \cdot x = -y, \quad y \cdot z = \omega y.$$

Since $(y \cdot x)(y \cdot x) = 0$, $(y \cdot z)(y \cdot z) = 0$, $Q_{4n}$ can be viewed as subgroup of $\mathrm{Aut}_{\mathrm{Alg}}(A)$. Therefore, we have the skew group algebra $Q_{4n} * A$, whose multiplication is given by

$$(h * a)(k * b) = hk * (a \cdot k)b$$

for any $h, k \in Q_{4n}$, $a, b \in A$.

It is easy to see that $A$ can be viewed as the subalgebra of $Q_{4n} * A$ and

$$(x * 1)(1 * y) = x * y,\ (1 * y)(x * 1) = x * (y \cdot x) = -x * y,$$

$$(z * 1)(1 * y) = z * y,\ (1 * y)(z * 1) = z * (y \cdot z) = \omega z * y,$$

and $D(n) \cong Q_{4n} * A$ as an algebra. Consequently, $D(n)$ is of finite representation type as an algebra by [20, Theorem 1.1, Theorem 1.3(a)]. □

## 3. The indecomposable modules of $D(n)$ and their tensor products

In this section, we mainly construct all the indecomposable $D(n)$-modules and establish their tensor products. It is remarked that the simple modules of the algebra $G * B$ are completely understood, and coincide with those of the group $G$ for which $B$ acts as zero (see [21]).

Firstly we classify all the indecomposable modules of $D(n)$. Let $\xi \in \Bbbk$ be a primitive $2n$-th root of unity.

**Theorem 3.1.** *(a) There are 4 pairwise non-isomorphic 1-dimensional $D(n)$-modules $S_i$ with basis $\{v_i\}$ for $0 \le i \le 3$, the action of $D(n)$ is defined by*

$$x \cdot v_i = (-1)^i v_i,\ y \cdot v_i = 0,\ z \cdot v_i = \omega^i v_i.$$

*(b) There are $n-1$ pairwise non-isomorphic 2-dimensional simple $D(n)$-modules $M_j$ with basis $\{v_j^1, v_j^2\}$ for $1 \le j \le n-1$, the action of $D(n)$ is defined by*

$$\begin{aligned} x \cdot v_j^1 &= \xi^j v_j^1, & x \cdot v_j^2 &= \xi^{-j} v_j^2,\\ y \cdot v_j^1 &= 0, & y \cdot v_j^2 &= 0,\\ z \cdot v_j^1 &= v_j^2, & z \cdot v_j^2 &= (-1)^j v_j^1. \end{aligned}$$

*(c) There are 4 pairwise non-isomorphic 2-dimensional indecomposable projective $D(n)$-modules $P_i$ with basis $\{\mu_i^1, \mu_i^2\}$ for $0 \le i \le 3$, the action of $D(n)$ is defined by*

$$\begin{aligned} x \cdot \mu_i^1 &= (-1)^i \mu_i^1, & x \cdot \mu_i^2 &= (-1)^{i+1} \mu_i^2,\\ y \cdot \mu_i^1 &= \mu_i^2, & y \cdot \mu_i^2 &= 0,\\ z \cdot \mu_i^1 &= \omega^i \mu_i^1, & z \cdot \mu_i^2 &= -\omega^{i+1} \mu_i^2. \end{aligned}$$

*(d) There are $n-1$ pairwise non-isomorphic 4-dimensional indecomposable projective $D(n)$-modules $T_j$ with basis $\{\vartheta_j^1, \vartheta_j^2, \vartheta_j^3, \vartheta_j^4\}$ for $1 \le j \le n-1$, the action of $D(n)$ is defined by*

$$\begin{aligned} x \cdot \vartheta_j^1 &= \xi^j \vartheta_j^1, & x \cdot \vartheta_j^2 &= \xi^{-j} \vartheta_j^2, & x \cdot \vartheta_j^3 &= -\xi^j \vartheta_j^3, & x \cdot \vartheta_j^4 &= -\xi^{-j} \vartheta_j^4,\\ y \cdot \vartheta_j^1 &= \vartheta_j^3, & y \cdot \vartheta_j^2 &= \vartheta_j^4, & y \cdot \vartheta_j^3 &= 0, & y \cdot \vartheta_j^4 &= 0,\\ z \cdot \vartheta_j^1 &= \vartheta_j^2, & z \cdot \vartheta_j^2 &= (-1)^j \vartheta_j^1, & z \cdot \vartheta_j^3 &= -\omega \vartheta_j^4, & z \cdot \vartheta_j^4 &= \omega(-1)^{j+1} \vartheta_j^3. \end{aligned}$$

*Proof.* The results of $(a)$,$(b)$ are showed in [5, 6, 21].

Firstly, we construct the 2-dimensional indecomposable non-simple $D(n)$-modules. Let $x \cdot \mu^i = \lambda_i \mu^i$ for $i = 1, 2,\ \lambda_i \in \Bbbk$ and $y \cdot \mu^1 \neq 0$. Suppose that $y \cdot \mu^1 = \bar{\mu}^1$, it is obvious that $\bar{\mu}^1$ and $\mu^1$ are linearly independent. We might let $y \cdot \mu^1 = \bar{\mu}^1 =: \mu^2$ as well, then $y \cdot \mu^2 = 0$. Since $x \cdot \mu^2 = x \cdot (y \cdot \mu^1) = -y \cdot (x \cdot \mu^1) = -\lambda_1 \mu^2$, there is $\lambda_2 = -\lambda_1$. Now consider $z \cdot \mu^1$. If $z \cdot \mu^1$ and $\mu^1, \mu^2$ are linearly dependent, let $z \cdot \mu^1 = p_1 \mu^1 + p_2 \mu^2,\ p_1, p_2 \in \Bbbk$, then $z \cdot \mu^2 = z \cdot (y \cdot \mu^1) = -\omega y \cdot (z \cdot \mu^1) = -\omega p_1 \mu^2$. By $z^2 = x^n$, it's easy to see $(1 + \omega) p_1 p_2 = 0$ and $p_1^2 = \lambda_1^n$, so $p_2 = 0$ for $p_1 \neq 0$. Since $\lambda_1^{2n} = 1$, $\lambda_1^n = p_1^2 = \pm 1$. When $\lambda_1^n = 1$, $p_1 = \pm 1$, there is $\lambda_1 = \lambda_1^{-1}$ by $xz = zx^{-1}$, thus $\lambda_1 = \pm 1$. But if $\lambda_1 = -1$, then $\lambda_1^n = -1 \neq 1$, which is a contradiction. Thus $\lambda_1 = 1$. Similarly, if $\lambda_1^n = -1$, then $\lambda_1 = -1$, it also gets a contradiction. Hence we get $(c)$.

Next, let $x \cdot v_1 = \xi^i v_1$ for some $i \in \mathbb{Z}_{2n}$, $z \cdot v_1 = v_2$, $y \cdot v_1 = v_3$, here $\xi$ is a primitive $2n$-th root of unity. Then

$$x \cdot v_2 = x \cdot (z \cdot v_1) = zx^{-1} \cdot v_1 = \xi^{-i} z \cdot v_1 = \xi^{-i} v_2$$

and

$$x \cdot v_3 = -\xi^i v_3, \quad y \cdot v_2 = \omega z \cdot v_3, \quad y \cdot v_3 = 0, \quad z \cdot v_2 = (-1)^i v_1.$$

Obviously, $z \cdot v_3 \neq 0$.

If $z \cdot v_3$ and $v_1, v_2, v_3$ are linearly dependent, let $z \cdot v_3 = av_1 + bv_2 + cv_3,\ \ a, b, c \in \Bbbk$. By $xz = zx^{-1}$, it's easy to get that $a = b = 0$. Since $z^4 = 1$, we can set $z \cdot v_3 = \omega^k v_3$ for some $k \in \mathbb{Z}_4$, where $\omega$ is a primitive 4-th root of unity, then $y \cdot v_2 = \omega^{k+1} v_3$. At this time, the matrices of $x,\ y,\ z$ acting on $\{v_1,\ v_2,\ v_3\}$ are

$$x \mapsto \begin{pmatrix} \xi^i & 0 & 0 \\ 0 & \xi^{-i} & 0 \\ 0 & 0 & -\xi^i \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \omega^{k+1} & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & (-1)^i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \omega^k \end{pmatrix},$$

respectively. It is directly checked that all the generating relations are satisfied only when $i = 0$ or $i = n$. If $i = 0$, then

$$x \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \pm 1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mp\omega \end{pmatrix};$$

If $i = n$, then

$$x \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \pm\omega & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

Now we use a unified expression to describe such modules $V_k$ with a basis $\{v_k^1, v_k^2, v_k^3\}$ for $0 \leq k \leq 3$, and the matrices of $x,\ y,\ z$ acting on this basis are

$$x \mapsto \begin{pmatrix} (-1)^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-1)^{k+1} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \omega^k & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & (-1)^k & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \omega^{k-1} \end{pmatrix},$$

respectively. In fact, these modules are decomposable. For, let

$$\{\omega_k^1 := -\omega^k v_k^1 + v_k^2,\ \ \omega_k^2 := \frac{1}{2}((-1)^k v_k^1 + \omega^k v_k^2),\ \ \omega_k^3 := v_k^3 | 0 \leq k \leq 3\}$$

be another basis, then the matrices of $x,\ y,\ z$ acting on $\{\omega_k^1,\ \omega_k^2,\ \omega_k^3\}$ are

$$x \mapsto \begin{pmatrix} (-1)^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-1)^{k+1} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} -\omega^k & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^{k-1} \end{pmatrix},$$

respectively. Thus we get that

$$V_k = \Bbbk\{\omega_k^1\} \oplus \Bbbk\{\omega_k^2, \omega_k^3\} \cong S_{k+2} \oplus P_k.$$

Moreover, if $z \cdot v_3$ and $v_1, v_2, v_3$ are linearly independent, let $z \cdot v_3 = v_4$, then

$$x \cdot v_4 = -\xi^{-i} v_4, \ \ y \cdot v_2 = \omega v_4, \ \ y \cdot v_4 = 0, \ \ z \cdot v_4 = (-1)^{i+1} v_3,$$

and the matrices of $x,\ y,\ z$ acting on $\{v_1,\ v_2,\ v_3,\ v_4\}$ are

$$x \mapsto \begin{pmatrix} \xi^i & 0 & 0 & 0 \\ 0 & \xi^{-i} & 0 & 0 \\ 0 & 0 & -\xi^i & 0 \\ 0 & 0 & 0 & -\xi^{-i} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & (-1)^i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^{i+1} \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively. We set $v_4' = \omega v_4$ to get the result (d). It is noted that when $n < i < 2n$, let $\bar{v}_1 := v_2,\ \bar{v}_2 := (-1)^i v_1,\ \bar{v}_3 := v_4,\ \bar{v}_4 := (-1)^i v_3$, then the matrices of $x,\ y,\ z$ acting on this basis are

$$x \mapsto \begin{pmatrix} \xi^{2n-i} & 0 & 0 & 0 \\ 0 & \xi^{i-2n} & 0 & 0 \\ 0 & 0 & -\xi^{2n-i} & 0 \\ 0 & 0 & 0 & -\xi^{i-2n} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$z \mapsto \begin{pmatrix} 0 & (-1)^i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega(-1)^{i+1} \\ 0 & 0 & -\omega & 0 \end{pmatrix},$$

respectively. Therefore when $n < i < 2n$, the modules are isomorphic to the case of $2n-i$. Furthermore, when $i = 0$, we choose the basis

$$\{\bar{v}_1 := v_1 + v_2,\ \bar{v}_2 := v_3 + v_4,\ \bar{v}_3 := v_1 - v_2,\ \bar{v}_4 := v_3 - v_4\},$$

then the matrices of $x,\ y,\ z$ acting on this basis are

$$x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix},$$

respectively. Hence it is decomposable and isomorphic to $P_0 \oplus P_2$. Similarly, for the case $i = n$, the module is decomposable and isomorphic to $P_1 \oplus P_3$. Indeed, we choose the basis

$$\{\bar{v}_1 := v_1 + \omega v_2,\ \bar{v}_2 := v_3 + \omega v_4,\ \bar{v}_3 := v_1 - \omega v_2,\ \bar{v}_4 := v_3 - \omega v_4\},$$

then the matrices of $x,\ y,\ z$ acting on this basis are

$$x \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} -\omega & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. Therefore, we get the result ($d$).

Then we claim that $P_i(0 \leq i \leq 3)$ and $T_j(1 \leq j \leq n-1)$ are indecomposable projective modules.

In fact, we know that the primitive idempotents of $D(n)$ are listed in [6] as

$$e_0 = \frac{1}{4n}\sum_{k=0}^{2n-1} x^k(1+z),\ e_1 = \frac{1}{4n}\sum_{k=0}^{2n-1}(-x)^k(1-iz),$$

$$e_2 = \frac{1}{4n}\sum_{k=0}^{2n-1} x^k(1-z),\ e_3 = \frac{1}{4n}\sum_{k=0}^{2n-1}(-x)^k(1+iz).$$

Since

$$xe_0 = e_0,\ xe_1 = -e_1,\ xe_2 = e_2,\ xe_3 = -e_3,$$

and

$$ze_0 = e_0,\ ze_1 = \omega e_1,\ ze_2 = -e_2,\ ze_3 = -\omega e_3,$$

then $\Bbbk\{e_i,\ ye_i|0 \leq i \leq 3\}$ consist of four indecomposable modules of $D(n)$ and are isomorphic to $P_i$, respectively. Thus $D(n)e_i \cong P_i$ is an indecomposable projective module.

For $0 \leq j \leq 2n-1$, set

$$\theta_j = \frac{1}{2n}\sum_{r=0}^{2n-1} \xi^{-jr}x^r,$$

then $\{\theta_0,\ \theta_1,\ \cdots,\theta_{2n-1}\}$ is a set of orthogonal idempotents of $D(n)$. Since $x\theta_j = \xi^j\theta_j$, the matrices of $x,\ y,\ z$ act on $\{\theta_j,\ z\theta_j,\ y\theta_j,\ yz\theta_j\}$, are

$$x \mapsto \begin{pmatrix} \xi^j & 0 & 0 & 0 \\ 0 & \xi^{-j} & 0 & 0 \\ 0 & 0 & -\xi^j & 0 \\ 0 & 0 & 0 & -\xi^{-j} \end{pmatrix},\quad y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\quad z \mapsto \begin{pmatrix} 0 & (-1)^j & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega(-1)^{j+1} \\ 0 & 0 & -\omega & 0 \end{pmatrix},$$

respectively. By the result of (d), we know that $\Bbbk\{\theta_j,\ z\theta_j,\ y\theta_j,\ yz\theta_j\} \cong T_j$ for $1 \leq j \leq n-1$, so $D(n)\theta_j \cong T_j$ is an indecomposable projective module.

The straightforward verification shows that $P_i(0 \leq i \leq 3)$ and $T_j(1 \leq j \leq n-1)$ are uniserial, that is $0 \subset S_{i-1(mod\ 4)} \subset P_i$ and $0 \subset M_{n-j} \subset T_j$ are the unique composition series of $P_i$ and $T_j$, respectively. Since $D(n)$ is a Frobenius algebra and thus is self-injective, we get that all the indecomposable projective modules are indecomposable injective modules. Therefore $D(n)$ is a Nakayama algebra. By [8, Theorem V.3.5], the modules listed above are all the indecomposable modules of $D(n)$.

The proof is completed. □

**Corollary 3.2.** *(1) For all $0 \leq i \leq 3$, $P_i$ is the projective cover of $S_i$.*
*(2) For all $1 \leq j \leq n-1$, $T_j$ is the projective cover of $M_j$.*

*Proof.* The results is directly obtained by [8, Lemma 5.6]. □

Let $H$ be a Hopf algebra, $M$ and $N$ be left $H$-modules. It has been known that $M \otimes_\Bbbk N$ is a left $H$-module defined by

$$h \cdot (m \otimes n) = \sum_{(h)} h_{(1)} \cdot m \otimes h_{(2)} \cdot n$$

for all $h \in H,\ m \in M$ and $n \in N$, where $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$.

The remaining of this section is devoted to establishing all the decomposition formulas of the tensor products of two indecomposable $D(n)$-modules.

**Theorem 3.3.** *(1) (a) For* $0 \le i, j \le 3$,

$$S_i \otimes S_j \cong S_j \otimes S_i \cong S_{i+j(mod\ 4)}.$$

*(b) For* $1 \le j \le n-1$,

$$S_i \otimes M_j \cong M_j \otimes S_i \cong \begin{cases} M_j, & i = 0, 2, \\ M_{n-j}, & i = 1, 3. \end{cases}$$

*(c) For* $0 \le i, j \le 3$,

$$S_i \otimes P_j \cong P_j \otimes S_i \cong P_{i+j(mod\ 4)}.$$

*(d) For* $1 \le j \le n-1$,

$$S_i \otimes T_j \cong T_j \otimes S_i \cong \begin{cases} T_j, & i = 0, 2, \\ T_{n-j}, & i = 1, 3. \end{cases}$$

*(2) (a) For* $1 \le i, j \le n-1$,

$$M_i \otimes M_j \cong \begin{cases} M_{i+j} \oplus M_{|i-j|}, & 0 < i+j < n,\ i \ne j \\ M_{2n-(i+j)} \oplus M_{|i-j|}, & n < i+j < 2n,\ i \ne j \\ M_{|i-j|} \oplus S_1 \oplus S_3, & i+j = n,\ i \ne j \\ M_{i+j} \oplus S_0 \oplus S_2, & i = j,\quad 0 < i+j < n, \\ M_{2n-(i+j)} \oplus S_0 \oplus S_2, & i = j,\quad n < i+j < 2n. \end{cases}$$

*(b) For* $1 \le i \le n-1,\ 0 \le j \le 3$,

$$M_i \otimes P_j \cong P_j \otimes M_i \cong \begin{cases} T_i, & j = 0, 2, \\ T_{n-i}, & j = 1, 3. \end{cases}$$

*(c) For* $1 \le i, j \le n-1$,

$$M_i \otimes T_j \cong T_j \otimes M_i \cong \begin{cases} T_{i+j} \oplus T_{|i-j|}, & 0 < i+j < n,\ i \ne j \\ T_{2n-(i+j)} \oplus T_{|i-j|}, & n < i+j < 2n,\ i \ne j \\ P_1 \oplus P_3 \oplus T_{|i-j|}, & i+j = n,\ i \ne j \\ T_{i+j} \oplus P_0 \oplus P_2, & i = j,\quad 0 < i+j < n, \\ T_{2n-(i+j)} \oplus P_0 \oplus P_2, & i = j,\quad n < i+j < 2n. \end{cases}$$

*(3) (a) For* $0 \le i, j \le 3$,

$$P_i \otimes P_j \cong P_j \otimes P_i \cong P_{i+j(mod\ 4)} \oplus P_{i+j+1(mod\ 4)}.$$

*(b) For* $1 \le j \le n-1, 0 \le i \le 3$,

$$P_i \otimes T_j \cong T_j \otimes P_i \cong T_j \oplus T_{n-j}.$$

*(4) For* $1 \le i, j \le n-1$,

$$T_i \otimes T_j \cong \begin{cases} T_{i+j} \oplus T_{|i-j|} \oplus T_{n-(i+j)} \oplus T_{n-|i-j|}, & 0 < i+j < n,\ i \neq j \\ T_{2n-(i+j)} \oplus T_{|i-j|} \oplus T_{(i+j)-n} \oplus T_{n-|i-j|}, & n < i+j < 2n,\ i \neq j \\ P_0 \oplus P_1 \oplus P_2 \oplus P_3 \oplus T_{|i-j|} \oplus T_{n-|i-j|}, & i+j = n,\ i \neq j \\ P_0 \oplus P_1 \oplus P_2 \oplus P_3 \oplus T_{i+j} \oplus T_{n-(i+j)}, & i = j, \quad 0 < i+j < n, \\ P_0 \oplus P_1 \oplus P_2 \oplus P_3 \oplus T_{2n-(i+j)} \oplus T_{(i+j)-n}, & i = j, \quad n < i+j < 2n. \end{cases}$$

*Proof.* For $S_i \otimes S_j$, let $v_{ij} := v_i \otimes v_j$, then

$$x \cdot v_{ij} = (-1)^{i+j} v_{ij},\ y \cdot v_{ij} = 0,\ z \cdot v_{ij} = \omega^{i+j} v_{ij},$$

thus $S_i \otimes S_j \cong S_{i+j (mod\ 4)} \cong S_j \otimes S_i$.

For $S_i \otimes M_j$, let $v_{ij}^1 := v_i \otimes v_j^1,\ v_{ij}^2 := v_i \otimes v_j^2$, then

$$\begin{aligned} x \cdot v_{ij}^1 &= (-1)^i \xi^j v_{ij}^1, & y \cdot v_{ij}^1 &= 0, & z \cdot v_{ij}^1 &= \omega^i v_{ij}^2, \\ x \cdot v_{ij}^2 &= (-1)^i \xi^{-j} v_{ij}^2, & y \cdot v_{ij}^2 &= 0, & z \cdot v_{ij}^2 &= (-1)^j \omega^i v_{ij}^1, \end{aligned}$$

thus $S_i \otimes M_j \cong M_j$ when $i = 0, 2$ and $S_i \otimes M_j \cong M_{n-j}$ when $i = 1, 3$. Similarly, we can get the same results of $M_j \otimes S_i$.

For $S_i \otimes P_j$, let $\mu_{ij}^1 := v_i \otimes \mu_j^1,\ \mu_{ij}^2 := v_i \otimes \mu_j^2$, then

$$\begin{aligned} x \cdot \mu_{ij}^1 &= (-1)^{i+j} \mu_{ij}^1, & y \cdot \mu_{ij}^1 &= \mu_{ij}^2, & z \cdot \mu_{ij}^1 &= \omega^{i+j} \mu_{ij}^2, \\ x \cdot \mu_{ij}^2 &= (-1)^{i+j+1} \mu_{ij}^2, & y \cdot \mu_{ij}^2 &= 0, & z \cdot \mu_{ij}^2 &= -\omega^{i+j+1} \mu_{ij}^1, \end{aligned}$$

thus $S_i \otimes P_j \cong P_{i+j (mod\ 4)}$. The same results of $P_j \otimes S_i$ can be obtained in the same way.

For $S_i \otimes T_j$, let $\vartheta_{ij}^1 := v_i \otimes \vartheta_j^1,\ \vartheta_{ij}^2 := v_i \otimes \vartheta_j^2,\ \vartheta_{ij}^3 := v_i \otimes \vartheta_j^3,\ \vartheta_{ij}^4 := v_i \otimes \vartheta_j^4$, then

$$\begin{aligned} x \cdot \vartheta_{ij}^1 &= (-1)^i \xi^j \vartheta_{ij}^1, & y \cdot \vartheta_{ij}^1 &= \vartheta_{ij}^3, & z \cdot \vartheta_{ij}^1 &= \omega^i \vartheta_{ij}^2, \\ x \cdot \vartheta_{ij}^2 &= (-1)^i \xi^{-j} \vartheta_{ij}^2, & y \cdot \vartheta_{ij}^2 &= \vartheta_{ij}^4, & z \cdot \vartheta_{ij}^2 &= (-1)^j \omega^i \vartheta_{ij}^1, \\ x \cdot \vartheta_{ij}^3 &= (-1)^{i+1} \xi^j \vartheta_{ij}^3, & y \cdot \vartheta_{ij}^3 &= 0, & z \cdot \vartheta_{ij}^3 &= -\omega^{i+1} \vartheta_{ij}^4, \\ x \cdot \vartheta_{ij}^4 &= (-1)^{i+1} \xi^{-j} \vartheta_{ij}^4, & y \cdot \vartheta_{ij}^4 &= 0, & z \cdot \vartheta_{ij}^4 &= (-1)^{j+1} \omega^{i+1} \vartheta_{ij}^3, \end{aligned}$$

thus $S_i \otimes T_j \cong T_j$ when $i = 0, 2$ and $S_i \otimes T_j \cong T_{n-j}$ when $i = 1, 3$. Similarly, we get the same results of $T_j \otimes S_i$.

For $M_i \otimes M_j$, let $\omega_{ij}^1 := v_i^1 \otimes v_j^1,\ \omega_{ij}^2 := v_i^1 \otimes v_j^2,\ \omega_{ij}^3 := v_i^2 \otimes v_j^1,\ \omega_{ij}^4 := v_i^2 \otimes v_j^2$, then

$$\begin{aligned} x \cdot \omega_{ij}^1 &= \xi^{i+j} \omega_{ij}^1, & y \cdot \omega_{ij}^1 &= 0, & z \cdot \omega_{ij}^1 &= \omega_{ij}^4, \\ x \cdot \omega_{ij}^2 &= \xi^{i-j} \omega_{ij}^2, & y \cdot \omega_{ij}^2 &= 0, & z \cdot \omega_{ij}^2 &= (-1)^j \omega_{ij}^3, \\ x \cdot \omega_{ij}^3 &= \xi^{j-i} \omega_{ij}^3, & y \cdot \omega_{ij}^3 &= 0, & z \cdot \omega_{ij}^3 &= (-1)^i \omega_{ij}^2, \\ x \cdot \omega_{ij}^4 &= \xi^{-(i+j)} \omega_{ij}^4, & y \cdot \omega_{ij}^4 &= 0, & z \cdot \omega_{ij}^4 &= (-1)^{i+j} \omega_{ij}^1. \end{aligned}$$

Obviously, when $0 < i + j,\ i - j < n$, $M_i \otimes M_j \cong M_{i+j} \otimes M_{i-j}$. Besides, we need to note that when $n < i + j < 2n$, let $\bar{\omega}^1_{ij} := \omega^4_{ij},\ \bar{\omega}^2_{ij} := (-1)^{i+j}\omega^1_{ij}$, then

$$\begin{aligned} &x \cdot \bar{\omega}^1_{ij} = \xi^{2n-(i+j)}\bar{\omega}^1_{ij}, && y \cdot \bar{\omega}^1_{ij} = 0, && z \cdot \bar{\omega}^1_{ij} = \bar{\omega}^2_{ij},\\ &x \cdot \bar{\omega}^2_{ij} = \xi^{i+j-2n}\bar{\omega}^2_{ij}, && y \cdot \bar{\omega}^2_{ij} = 0, && z \cdot \bar{\omega}^2_{ij} = (-1)^{i+j}\bar{\omega}^1_{ij}. \end{aligned}$$

Therefore, $\Bbbk\{\bar{\omega}^1_{ij},\ \bar{\omega}^2_{ij}\} \cong M_{i+j}$. When $-n < i-j < 0$, $n < i-j+2n < 2n$, using the previous conclusion, we directly get that $\Bbbk\{\bar{\omega}^3_{ij} := \omega^3_{ij},\ \bar{\omega}^4_{ij} := (-1)^i\omega^2_{ij}\} \cong M_{j-i}$.

In particular, when $i+j = n$, the matrices of $x,\ y,\ z$ acting on the basis $\{\bar{\omega}^1_{ij},\ \bar{\omega}^2_{ij}\}$ are simultaneously diagonalizable. Thus the modules are isomorphic to $S_1 \oplus S_3$; when $i - j = 0$, the matrices of $x,\ y,\ z$ acting on the basis $\{\bar{\omega}^3_{ij},\ \bar{\omega}^4_{ij}\}$ are also simultaneously diagonalizable and isomorphic to $S_0 \oplus S_2$. In fact we might assume that $i \geq j$ since the result of $M_j \otimes M_i$ are the same as $M_i \otimes M_j$.

For $M_i \otimes P_j$, let $\bar{\nu}^1_{ij} := v^1_i \otimes \mu^1_j,\ \bar{\nu}^2_{ij} := \omega^j v^2_i \otimes \mu^1_j,\ \bar{\nu}^3_{ij} := v^1_i \otimes \mu^2_j,\ \bar{\nu}^4_{ij} := \omega^j v^2_i \otimes \mu^2_j$, then

$$\begin{aligned} &x \cdot \bar{\nu}^1_{ij} = (-1)^j\xi^i\bar{\nu}^1_{ij}, && y \cdot \bar{\nu}^1_{ij} = \bar{\nu}^3_{ij}, && z \cdot \bar{\nu}^1_{ij} = \bar{\nu}^2_{ij},\\ &x \cdot \bar{\nu}^2_{ij} = (-1)^j\xi^{-i}\bar{\nu}^2_{ij}, && y \cdot \bar{\nu}^2_{ij} = \bar{\nu}^4_{ij}, && z \cdot \bar{\nu}^2_{ij} = (-1)^{i+j}\bar{\nu}^1_{ij},\\ &x \cdot \bar{\nu}^3_{ij} = (-1)^{j+1}\xi^i\bar{\nu}^3_{ij}, && y \cdot \bar{\nu}^3_{ij} = 0, && z \cdot \bar{\nu}^3_{ij} = -\omega\bar{\nu}^4_{ij},\\ &x \cdot \bar{\nu}^4_{ij} = (-1)^{j+1}\xi^{-i}\bar{\nu}^4_{ij}, && y \cdot \bar{\nu}^4_{ij} = 0, && z \cdot \bar{\nu}^4_{ij} = \omega(-1)^{i+j+1}\bar{\nu}^3_{ij}. \end{aligned}$$

Thus when $j = 0, 2$, $M_i \otimes P_j \cong P_j \otimes M_i \cong T_i$; when $j = 1, 3$, $M_i \otimes P_j \cong P_j \otimes M_i \cong T_{n-i}$,

For $M_i \otimes T_j$, let $\{\bar{\vartheta}^1_{ij} := v^1_i \otimes \vartheta^1_j,\ \bar{\vartheta}^2_{ij} := v^2_i \otimes \vartheta^2_j,\ \bar{\vartheta}^3_{ij} := v^1_i \otimes \vartheta^3_j,\ \bar{\vartheta}^4_{ij} := v^2_i \otimes \vartheta^4_j,\ \bar{\vartheta}^5_{ij} := v^1_i \otimes \vartheta^2_j,\ \bar{\vartheta}^6_{ij} := (-1)^j v^2_i \otimes \vartheta^1_j,\ \bar{\vartheta}^7_{ij} := v^1_i \otimes \vartheta^4_j,\ \bar{\vartheta}^8_{ij} := (-1)^j v^2_i \otimes \vartheta^3_j\}$, then

$$\begin{aligned} &x \cdot \bar{\vartheta}^1_{ij} = \xi^{i+j}\bar{\vartheta}^1_{ij}, && y \cdot \bar{\vartheta}^1_{ij} = \bar{\vartheta}^3_{ij}, && z \cdot \bar{\vartheta}^1_{ij} = \bar{\vartheta}^2_{ij},\\ &x \cdot \bar{\vartheta}^2_{ij} = \xi^{-(i+j)}\bar{\vartheta}^2_{ij}, && y \cdot \bar{\vartheta}^2_{ij} = \bar{\vartheta}^4_{ij}, && z \cdot \bar{\vartheta}^2_{ij} = (-1)^{i+j}\bar{\vartheta}^1_{ij},\\ &x \cdot \bar{\vartheta}^3_{ij} = -\xi^{i+j}\bar{\vartheta}^3_{ij}, && y \cdot \bar{\vartheta}^3_{ij} = 0, && z \cdot \bar{\vartheta}^3_{ij} = -\omega\bar{\vartheta}^4_{ij},\\ &x \cdot \bar{\vartheta}^4_{ij} = -\xi^{-(i+j)}\bar{\vartheta}^4_{ij}, && y \cdot \bar{\vartheta}^4_{ij} = 0, && z \cdot \bar{\vartheta}^4_{ij} = \omega(-1)^{i+j+1}\bar{\vartheta}^3_{ij},\\ &x \cdot \bar{\vartheta}^5_{ij} = \xi^{i-j}\bar{\vartheta}^5_{ij}, && y \cdot \bar{\vartheta}^5_{ij} = \bar{\vartheta}^7_{ij}, && z \cdot \bar{\vartheta}^5_{ij} = \bar{\vartheta}^6_{ij},\\ &x \cdot \bar{\vartheta}^6_{ij} = \xi^{j-i}\bar{\vartheta}^6_{ij}, && y \cdot \bar{\vartheta}^6_{ij} = \bar{\vartheta}^8_{ij}, && z \cdot \bar{\vartheta}^6_{ij} = (-1)^{i-j}\bar{\vartheta}^5_{ij},\\ &x \cdot \bar{\vartheta}^7_{ij} = -\xi^{i-j}\bar{\vartheta}^7_{ij}, && y \cdot \bar{\vartheta}^7_{ij} = 0, && z \cdot \bar{\vartheta}^7_{ij} = -\omega\bar{\vartheta}^8_{ij},\\ &x \cdot \bar{\vartheta}^8_{ij} = -\xi^{j-i}\bar{\vartheta}^8_{ij}, && y \cdot \bar{\vartheta}^8_{ij} = 0, && z \cdot \bar{\vartheta}^8_{ij} = \omega(-1)^{i-j+1}\bar{\vartheta}^7_{ij}. \end{aligned}$$

Obviously, when $0 < i + j,\ i - j < n$, $M_i \otimes T_j \cong T_{i+j} \oplus T_{i-j}$. It has been shown that in Theorem 3.1 when $n < i + j < 2n$, these modules $\Bbbk\{\bar{\vartheta}^1_{ij},\ \bar{\vartheta}^2_{ij},\ \bar{\vartheta}^3_{ij},\ \bar{\vartheta}^4_{ij}\}$ are isomorphic to the cases of $2n - (i + j)$. When $-n < i - j < 0$, $n < i - j + 2n < 2n$, it's clear to get that the modules are isomorphic to the case of $2n - (i - j + 2n) = j - i$.

For $P_i \otimes P_j$, we have known that $0 \subset S_{i-1(mod\ 4)} \subset P_i$ is the unique composition series of $P_i$ for $0 \leq i \leq 3$. Thus there is an exact sequence

$$0 \longrightarrow S_{i-1(mod\ 4)} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0.$$

By Corollary 3.2, $P_i = P(S_i)$, so there is a split exact sequence

$$0 \longrightarrow S_{i-1(mod\ 4)} \otimes P_j \longrightarrow P_i \otimes P_j \longrightarrow S_i \otimes P_j \longrightarrow 0.$$

Hence

$$P_i \otimes P_j \cong S_{i-1(mod\ 4)} \otimes P_j \oplus S_i \otimes P_j \cong P_{i+j-1(mod\ 4)} \oplus P_{i+j(mod\ 4)}.$$

For $P_i \otimes T_j$, similarly, we have the split exact sequence

$$0 \longrightarrow S_{i-1(mod\ 4)} \otimes T_j \longrightarrow P_i \otimes T_j \longrightarrow S_i \otimes T_j \longrightarrow 0.$$

Hence

$$P_i \otimes T_j \cong S_{i-1(mod\ 4)} \otimes T_j \oplus S_i \otimes T_j \cong T_j \oplus T_{n-j}.$$

For $T_i \otimes T_j,\ i > j$, the unique composition series of $T_i$ is $0 \subset M_{n-i} \subset T_i$ for $1 \leq i \leq n-1$, and there is an exact sequence

$$0 \longrightarrow M_{n-i} \longrightarrow T_i \longrightarrow M_i \longrightarrow 0$$

and the split exact sequence

$$0 \longrightarrow M_{n-i} \otimes T_j \longrightarrow T_i \otimes T_j \longrightarrow M_i \otimes T_j \longrightarrow 0$$

for $T_j = P(M_j)$. Hence

$$T_i \otimes T_j \cong M_{n-i} \otimes T_j \oplus M_i \otimes T_j \cong T_{n-i+j} \oplus T_{n-i-j} \oplus T_{i+j} \oplus T_{i-j}.$$

Then applying the result of (2)(c), we get the result (4). For $i < j$, the result is similar.

The proof is finished. □

**Corollary 3.4.** *The tensor product of any two $D(n)$-modules is commutative.*

## 4. The representation ring of $D(n)$

Let $H$ be a finite dimensional Hopf algebra and $F(H)$ the free abelian group generated by the isomorphic classes $[M]$ of finite dimensional $H$-modules $M$. The abelian group $F(H)$ becomes a ring if we endow $F(H)$ with a multiplication given by the tensor product $[M][N] = [M \otimes N]$. The representation ring (or Green ring) $r(H)$ of the Hopf algebra $H$ is defined to be the quotient ring of $F(H)$ modulo the relations $[M \oplus N] = [M] + [N]$. It follows that the representation ring $r(H)$ is an associative ring with identity given by $[\Bbbk_\varepsilon]$, the trivial 1-dimensional $H$-module. Note that $r(H)$ has a $\mathbb{Z}$-basis consisting of isomorphic classes of finite dimensional indecomposable $H$-modules. In this section we will describe the representation ring $r(D(n))$ of the Hopf algebra $D(n)$ explicitly by the generators and the generating relations.

Let $F_q(y, z)$ be the generalized Fibonacci polynomials defined by

$$F_{q+2}(y, z) = zF_{q+1}(y, z) - yF_q(y, z)$$

for $q \geq 1$, while $F_0(y, z) = 0,\ F_1(y, z) = 1,\ F_2(y, z) = z$. These generalized Fibonacci polynomials appeared in [13, 14].

**Lemma 4.1.** *[13, Lemma 3.11] Let $\mathbb{Z}[y, z]$ be the polynomial algebra over $\mathbb{Z}$ in two variables $y$ and $z$. Then for any $q \geq 1$, we have*

$$F_q(y, z) = \sum_{i=0}^{\left[\frac{q-1}{2}\right]} (-1)^i \binom{q-1-i}{i} y^i z^{q-1-2i},$$

*where $\left[\frac{q-1}{2}\right]$ denotes the biggest integer which is not bigger than $\frac{q-1}{2}$.*

Let $[S_1] = \alpha$, $[M_1] = \beta$ and $[P_0] = \gamma$. In the following the sum $\sum_{i=0}^{m}$ disappears if $m < 0$.

**Lemma 4.2.** *The following statements hold in $r(D(n))$.*

1. $\alpha^4 = 1,\ \alpha^2\beta = \beta,\ \alpha\gamma = \gamma(\gamma - 1)$;
2. $[S_i] = \alpha^i$, $[P_i] = \alpha^i\gamma$ $(0 \le i \le 3)$;
3. *For* $1 \le j \le n-1$,

$$[M_j] = \begin{cases} \sum_{i=0}^{\frac{j-1}{2}}(-1)^i\binom{j-i}{i}\beta^{j-2i} - \sum_{i=0}^{\frac{j-3}{2}}(-1)^i\binom{j-2-i}{i}\beta^{j-2-2i}, & j \text{ is odd},\\ \sum_{i=0}^{\frac{j}{2}}(-1)^i\binom{j-i}{i}\beta^{j-2i} - \sum_{i=0}^{\frac{j-4}{2}}(-1)^i\binom{j-2-i}{i}\beta^{j-2-2i} + (-1)^{\frac{j}{2}}\alpha^2, & j \text{ is even}; \end{cases} \tag{4.1}$$

$$[T_j] = \begin{cases} \sum_{i=0}^{\frac{j-1}{2}}(-1)^i\binom{j-i}{i}\beta^{j-2i}\gamma - \sum_{i=0}^{\frac{j-3}{2}}(-1)^i\binom{j-2-i}{i}\beta^{j-2-2i}\gamma, & j \text{ is odd},\\ \sum_{i=0}^{\frac{j}{2}}(-1)^i\binom{j-i}{i}\beta^{j-2i}\gamma - \sum_{i=0}^{\frac{j-4}{2}}(-1)^i\binom{j-2-i}{i}\beta^{j-2-2i}\gamma + (-1)^{\frac{j}{2}}\alpha^2\gamma, & j \text{ is even}. \end{cases} \tag{4.2}$$

*Proof.* The results of (1), (2) are easy to get from Theorem 3.3(1)(a)–(c) and (3)(a).

We prove (3) by induction. By Theorem 3.3(2)(a), there is

$$[M_2] = \beta^2 - (1 + \alpha^2) = F_3(1,\beta) - \alpha^2$$

and

$$[M_3] = [M_1][M_2] - [M_1] = \beta^3 - 2\beta - \alpha^2\beta = \beta^3 - 3\beta = F_4(1,\beta) - F_2(1,\beta).$$

Suppose that (4.1) holds for $j-1$ being odd and $j$ being even, then for $j+1$ we have

$$\begin{aligned} & [M_{j+1}] \\ = \ & [M_j][M_1] - [M_{j-1}] \\ = \ & (F_{j+1}(1,\beta) - \alpha^2F_{j-1}(1,\beta))\beta - (F_j(1,\beta) - F_{j-2}(1,\beta)) \\ = \ & (F_{j+1}(1,\beta)\beta - F_j(1,\beta)) - \alpha^2(F_{j-1}(1,\beta)\beta - F_{j-2}(1,\beta)) \\ = \ & F_{j+2}(1,\beta) - \alpha^2F_j(1,\beta) \\ = \ & \sum_{i=0}^{\frac{j}{2}}(-1)^i\binom{j+1-i}{i}\beta^{j+1-2i} - \alpha^2\sum_{i=0}^{\frac{j-2}{2}}(-1)^i\binom{j-1-i}{i}\beta^{j-1-2i} \\ = \ & \sum_{i=0}^{\frac{j}{2}}(-1)^i\binom{j+1-i}{i}\beta^{j+1-2i} - \sum_{i=0}^{\frac{j-2}{2}}(-1)^i\binom{j-1-i}{i}\beta^{j-1-2i}. \end{aligned}$$

Similarly, suppose that (4.1) holds for $j-1$ being even and $j$ being odd, then for $j+1$ we directly get that

$$\begin{aligned} [M_{j+1}] &= \sum_{i=0}^{\frac{j+1}{2}}(-1)^i\binom{j+1-i}{i}\beta^{j+1-2i} - \alpha^2\sum_{i=0}^{\frac{j-1}{2}}(-1)^i\binom{j-1-i}{i}\beta^{j-1-2i} \\ &= \sum_{i=0}^{\frac{j+1}{2}}(-1)^i\binom{j+1-i}{i}\beta^{j+1-2i} - \sum_{i=0}^{\frac{j-3}{2}}(-1)^i\binom{j-1-i}{i}\beta^{j-1-2i} - (-1)^{\frac{j-1}{2}}\alpha^2. \end{aligned}$$

By Theorem 3.3(2)(b), we know that $M_j \otimes P_0 \cong T_j$, thus the Eq (4.2) is obvious to be obtained. □

**Corollary 4.3.** *Keep the notations above.*

1. *If $\frac{n-1}{2}$ is odd, then*

$$\begin{aligned}&\sum_{i=0}^{\frac{n+1}{4}}(-1)^i\binom{\frac{n+1}{2}-i}{i}\beta^{\frac{n+1}{2}-2i}-\sum_{i=0}^{\frac{n-7}{4}}(-1)^i\binom{\frac{n-3}{2}-i}{i}\beta^{\frac{n-3}{2}-2i}+(-1)^{\frac{n+1}{4}}\alpha^2\\ =\ &\alpha\sum_{i=0}^{\frac{n-3}{4}}(-1)^i\binom{\frac{n-1}{2}-i}{i}\beta^{\frac{n-1}{2}-2i}-\alpha\sum_{i=0}^{\frac{n-7}{4}}(-1)^i\binom{\frac{n-5}{2}-i}{i}\beta^{\frac{n-5}{2}-2i}.\end{aligned}$$

2. *If $\frac{n-1}{2}$ is even, then*

$$\begin{aligned}&\sum_{i=0}^{\frac{n-1}{4}}(-1)^i\binom{\frac{n+1}{2}-i}{i}\beta^{\frac{n+1}{2}-2i}-\sum_{i=0}^{\frac{n-5}{4}}(-1)^i\binom{\frac{n-3}{2}-i}{i}\beta^{\frac{n-3}{2}-2i}\\ =\ &\alpha\sum_{i=0}^{\frac{n-1}{4}}(-1)^i\binom{\frac{n-1}{2}-i}{i}\beta^{\frac{n-1}{2}-2i}-\alpha\sum_{i=0}^{\frac{n-9}{4}}(-1)^i\binom{\frac{n-5}{2}-i}{i}\beta^{\frac{n-5}{2}-2i}+(-1)^{\frac{n-1}{4}}\alpha^3.\end{aligned}$$

*Proof.* Since $[S_1][M_{\frac{n-1}{2}}] = [M_{\frac{n+1}{2}}]$ by Theorem 3.3(1)(b), and using the equations of (4.1) in Lemma 4.2, we can easily get the results. □

**Corollary 4.4.** *Keep notations as above, then the sets*

$$\{\alpha^i\gamma^k \mid 0\le i\le 3,\ 0\le k\le 1\}\cup\{\alpha^i\beta^j\gamma^k \mid 0\le i\le 1,\ 1\le j\le \frac{n-1}{2},\ 0\le k\le 1\}$$

*form a $\mathbb{Z}$-basis of $r(D(n))$.*

*Proof.* By Lemma 4.1, $\alpha^4 = 1$, and there is a one-to-one correspondence between the set $\{\alpha^i, \alpha^i\gamma \mid 0 \le i \le 3\}$ and the set of $D(n)$-modules $\{[S_i], [P_i] \mid 0 \le i \le 3\}$. By the Eq (4.1), we know that $[M_j]$ is a $\mathbb{Z}$-polynomial with $\alpha$ and $\beta$, and when $j \ge \frac{n+1}{2}$, $[M_j] = [S_1][M_{n-j}]$. By Corollary 4.3, we know that the highest degree of $\beta$ in this polynomial is $\frac{n-1}{2}$, and $\{[M_j] \mid 1 \le j \le n-1\}$ is a $\mathbb{Z}$-linear combination of $\{\alpha^i \mid 0 \le i \le 3\}$, $\{\alpha\beta^j \mid 1 \le j \le \frac{n-1}{2}\}$ and $\{\beta^j \mid 1 \le j \le \frac{n-1}{2}\}$. Consequently, $[T_j]$ is a $\mathbb{Z}$-polynomial with $\alpha$, $\beta$, $\gamma$ and the highest degree of $\gamma$ in this polynomial is 1 since $\gamma^2 = \alpha\gamma + \gamma$. Therefore $[T_j]$ is a $\mathbb{Z}$-linear combination of $\{\alpha^i \mid 0 \le i \le 3\}$, $\{\alpha\beta^j\gamma \mid 1 \le j \le \frac{n-1}{2}\}$ and $\{\beta^j\gamma \mid 1 \le j \le \frac{n-1}{2}\}$.

The result is obtained. □

**Theorem 4.5.** *The representation ring $r(D(n))$ is a commutative ring generated by $a$, $b$, $c$, subject to the following relations*

$$a^4 = 1,\quad a^2b = b,\quad ac = c(c-1)$$

*and*

$$\begin{aligned}b^{\frac{n+1}{2}} = &-\sum_{i=1}^{\frac{n+1}{4}}(-1)^i\binom{\frac{n+1}{2}-i}{i}b^{\frac{n+1}{2}-2i}+\sum_{i=0}^{\frac{n-7}{4}}(-1)^i\binom{\frac{n-3}{2}-i}{i}b^{\frac{n-3}{2}-2i}+(-1)^{\frac{n+5}{4}}a^2\\ &+\ a\sum_{i=0}^{\frac{n-3}{4}}(-1)^i\binom{\frac{n-1}{2}-i}{i}b^{\frac{n-1}{2}-2i}-a\sum_{i=0}^{\frac{n-7}{4}}(-1)^i\binom{\frac{n-5}{2}-i}{i}b^{\frac{n-5}{2}-2i}\end{aligned}$$

*if $\frac{n-1}{2}$ is odd, or*

$$\begin{aligned} b^{\frac{n+1}{2}} = & - \sum_{i=1}^{\frac{n-1}{4}}(-1)^i \binom{\frac{n+1}{2}-i}{i} b^{\frac{n+1}{2}-2i} + \sum_{i=0}^{\frac{n-5}{4}}(-1)^i \binom{\frac{n-3}{2}-i}{i} b^{\frac{n-3}{2}-2i} + (-1)^{\frac{n-1}{4}} a^3 \\ & + a\sum_{i=0}^{\frac{n-1}{4}}(-1)^i \binom{\frac{n-1}{2}-i}{i} b^{\frac{n-1}{2}-2i} - a\sum_{i=0}^{\frac{n-9}{4}}(-1)^i \binom{\frac{n-5}{2}-i}{i} b^{\frac{n-5}{2}-2i} \end{aligned}$$

*if $\frac{n-1}{2}$ is even.*

*Proof.* By Corollary 3.4, we know that the ring $r(D(n))$ is a commutative ring generated by $\alpha, \beta$ and $\gamma$, there is a unique ring epimorphism

$$\Phi : \mathbb{Z}[a, b, c] \to r(D(n))$$

from $\mathbb{Z}[a, b, c]$ to $r(D(n))$ such that

$$\Phi(a) = \alpha,\ \Phi(b) = \beta,\ \Phi(c) = \gamma.$$

By Lemma 4.1, there is

$$\alpha^4 = 1,\ \alpha^2\beta = \beta,\ \alpha\gamma = \gamma(\gamma - 1),$$

thus we have

$$\Phi(a^4 - 1) = 0, \quad \Phi(a^2b - b) = 0, \quad \Phi(ac - c(c - 1)) = 0.$$

Note that by Corollary 4.3,

$$\begin{aligned} \beta^{\frac{n+1}{2}} = & - \sum_{i=1}^{\frac{n+1}{4}}(-1)^i \binom{\frac{n+1}{2}-i}{i} \beta^{\frac{n+1}{2}-2i} + \sum_{i=0}^{\frac{n-7}{4}}(-1)^i \binom{\frac{n-3}{2}-i}{i} \beta^{\frac{n-3}{2}-2i} + (-1)^{\frac{n+5}{4}} \alpha^2 \\ & + \alpha\sum_{i=0}^{\frac{n-3}{4}}(-1)^i \binom{\frac{n-1}{2}-i}{i} \beta^{\frac{n-1}{2}-2i} - \alpha\sum_{i=0}^{\frac{n-7}{4}}(-1)^i \binom{\frac{n-5}{2}-i}{i} \beta^{\frac{n-5}{2}-2i} \end{aligned}$$

when $\frac{n-1}{2}$ is odd, or

$$\begin{aligned} \beta^{\frac{n+1}{2}} = & - \sum_{i=1}^{\frac{n-1}{4}}(-1)^i \binom{\frac{n+1}{2}-i}{i} \beta^{\frac{n+1}{2}-2i} + \sum_{i=0}^{\frac{n-5}{4}}(-1)^i \binom{\frac{n-3}{2}-i}{i} \beta^{\frac{n-3}{2}-2i} + (-1)^{\frac{n-1}{4}} \alpha^3 \\ & + \alpha\sum_{i=0}^{\frac{n-1}{4}}(-1)^i \binom{\frac{n-1}{2}-i}{i} \beta^{\frac{n-1}{2}-2i} - \alpha\sum_{i=0}^{\frac{n-9}{4}}(-1)^i \binom{\frac{n-5}{2}-i}{i} \beta^{\frac{n-5}{2}-2i} \end{aligned}$$

when $\frac{n-1}{2}$ is even. Thus $\Phi$ maps

$$\begin{aligned} b^{\frac{n+1}{2}} &+ \sum_{i=1}^{\frac{n+1}{4}}(-1)^i\binom{\frac{n+1}{2}-i}{i}b^{\frac{n+1}{2}-2i} - \sum_{i=0}^{\frac{n-7}{4}}(-1)^i\binom{\frac{n-3}{2}-i}{i}b^{\frac{n-3}{2}-2i} - (-1)^{\frac{n+5}{4}}a^2 \\ &- a\sum_{i=0}^{[\frac{n-3}{4}]}(-1)^i\binom{\frac{n-1}{2}-i}{i}b^{\frac{n-1}{2}-2i} + a\sum_{i=0}^{\frac{n-7}{4}}(-1)^i\binom{\frac{n-5}{2}-i}{i}b^{\frac{n-5}{2}-2i} \end{aligned}$$

or

$$\begin{aligned} b^{\frac{n+1}{2}} &+ \sum_{i=1}^{\frac{n-1}{4}}(-1)^i\binom{\frac{n+1}{2}-i}{i}b^{\frac{n+1}{2}-2i} - \sum_{i=0}^{\frac{n-5}{4}}(-1)^i\binom{\frac{n-3}{2}-i}{i}b^{\frac{n-3}{2}-2i} - (-1)^{\frac{n-1}{4}}a^3 \\ &- a\sum_{i=0}^{\frac{n-1}{4}}(-1)^i\binom{\frac{n-1}{2}-i}{i}b^{\frac{n-1}{2}-2i} + a\sum_{i=0}^{\frac{n-9}{4}}(-1)^i\binom{\frac{n-5}{2}-i}{i}b^{\frac{n-5}{2}-2i} \end{aligned}$$

to 0. It follows that $\Phi(I) = 0$, and $\Phi$ induces a ring epimorphism

$$\overline{\Phi} : \mathbb{Z}[a, b, c]/I \to r(D(n))$$

such that $\overline{\Phi}(\overline{v}) = \Phi(v)$ for all $v \in \mathbb{Z}[a, b, c]$, where $\overline{v} = \pi(v)$ and $\pi$ is the natural epimorphism $\mathbb{Z}[a, b, c] \to \mathbb{Z}[a, b, c]/I$. Note that the ring $r(D(n))$ is a free $\mathbb{Z}$-module of rank $2n + 6$ with the $\mathbb{Z}$-basis $\{\alpha^i\gamma^k \mid 0 \le i \le 3,\ 0 \le k \le 1\} \cup \{\alpha^i\beta^j\gamma^k \mid 0 \le i \le 1,\ 1 \le j \le \frac{n-1}{2},\ 0 \le k \le 1\}$, so we can define a $\mathbb{Z}$-module homomorphism

$$\begin{aligned} \Psi : r(D(n)) &\longrightarrow \mathbb{Z}[a, b, c]/I \\ \alpha^i\gamma^k &\mapsto \overline{a^ic^k}\ (0 \le i \le 3,\ 0 \le k \le 1), \\ \alpha^i\beta^j\gamma^k &\mapsto \overline{a^ib^jc^k}\ (0 \le i \le 1,\ 1 \le j \le \frac{n-1}{2},\ 0 \le k \le 1). \end{aligned}$$

On the other hand, as a free $\mathbb{Z}$-module, $\mathbb{Z}[a, b, c]/I$ is generated by elements $a^ic^k$ $(0 \le i \le 3,\ 0 \le k \le 1)$ and $a^ib^jc^k$ $(0 \le i \le 1,\ 1 \le j \le \frac{n-1}{2},\ 0 \le k \le 1)$, we have

$$\Psi\overline{\Phi}(\overline{a^ic^k}) = \Psi\Phi(a^ic^k) = \Psi(\alpha^i\gamma^k) = \overline{a^ic^k},$$

$$\Psi\overline{\Phi}(\overline{a^ib^jc^k}) = \Psi\Phi(a^ib^jc^k) = \Psi(\alpha^i\beta^j\gamma^k) = \overline{a^ib^jc^k}.$$

Hence $\Psi\overline{\Phi} = id$, and $\overline{\Phi}$ is injective. Thus, $\overline{\Phi}$ is a ring isomorphism.

The proof is finished. □

**Example 4.6.** *We have the following examples.*

- *The representation ring $r(D(3))$ is a commutative ring generated by $a$, $b$, $c$, subject to the following relations*

$$a^4 = 1, \quad a^2b = b, \quad ac = c(c-1), \quad b^2 = ab + a^2 + 1.$$

- *The representation ring $r(D(5))$ is a commutative ring generated by $a$, $b$, $c$, subject to the following relations*

$$a^4 = 1, \quad a^2b = b, \quad ac = c(c-1), \quad b^3 = ab^2 + 3b - a^3 - a.$$

## 5. Conclusions

We have constructed all the indecomposable modules of the non-pointed Hopf algebra $D(n)$ and established the decomposition formulas of the tensor product of any two indecomposable modules. The representation ring $r(D(n))$ has been characterized by generators and relations. In the further work, we hope to construct all the simple Yetter-Drinfeld modules of $D(n)$ and classify all the finite-dimensional Nichols algebras and finite-dimensional Hopf algebras over $D(n)$.

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## Conflict of interest

The authors declare that they have no competing interests.

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