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Research article

Boundary value problems for the Lamé-Navier system in fractal domains

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Abstract: The aim of this paper is to establish a representation formula for the solutions of the Lamé-Navier system in linear elasticity theory. We also study boundary value problems for such a system in a bounded domain $\Omega \subset \mathbb{R}^3$, allowing a very general geometric behavior of its boundary. Our method exploits the connections between this system and some classes of second order partial differential equations arising in Clifford analysis.

Keywords: Lamé-Navier system; linear elasticity; fractal boundaries; Clifford analysis
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1. Introduction

The displacement vector $\vec{u}$ of the points of a three-dimensional isotropic elastic body in the absence of body forces is described by the Lamé-Navier system

$$\mu\triangle\vec{u} + (\mu+\lambda)\mathrm{grad}(\mathrm{div}\vec{u}) = 0. \tag{1.1}$$

Here, the quantities $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$ are the basic constants characterizing the elastic properties of the body (constants usually referred to as Lamé parameters [1, 2]). For more details we refer to [3–7].

It has recently been shown in [8] that the Lamé equation (1.1) admits the form

$$\left(\frac{\mu+\lambda}{2}\right)\partial_{\underline{x}}\vec{u}\partial_{\underline{x}} + \left(\frac{3\mu+\lambda}{2}\right)\partial_{\underline{x}}^2\vec{u} = 0, \tag{1.2}$$

where

$$\partial_{\underline{x}} := e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}$$

stands for the Dirac operator constructed with the basis of the real Clifford algebra $\mathbb{R}_{0,3}$. The elements in the kernel of $\partial_{\underline{x}}$ are called monogenic functions [9, 10], which represent the main object of the so-called Clifford analysis.

Applications of this function theory to elastic materials are remarkable and have already been developed in [11–18]. More on these interesting topics the reader can find in the books [19, 20]. It should be pointed out, however, that the study of boundary value problems for such physical models has been confined to smoothly bounded domains, since there exist enough obstacles to a rigorous treatment of such problems in the more challenging case of domains with fractal boundaries. From the point of view of engineering applications considerable interest attaches to the solution of more general problems when the body under consideration admits a boundary of more general character.

Fractals are not only relevant from a mathematical point of view, but also have important applications and are widely used in physics, biology, pharmaceutical sciences and chemistry [21–25]. It is for these reasons that it is not unreasonable to consider the above problems under such a general geometric conditions.

In this paper we make essential use of the methods introduced in [26, 27] to derive a representation formula for the solutions (1.1) and its applications to boundary value problems for such a system in a very wide classes of regions. We stress that our approach allows domains with fractal boundaries, a question that as far as we know has not been considered before. The present work represents a three-dimensional generalization of the recently published paper [28], where the Lamé system is considered on plane domains with fractal boundary, using classical complex analysis techniques.

2. Preliminaries

Let e_1, e_2, e_3 be an orthonormal basis of $\mathbb{R}^3$, with the multiplication rules

$$e_i^2 = -1,\ e_ie_j = -e_je_i,\ i, j = 1, 2, 3,\ i < j.$$

In this way, the Euclidean space

$$\mathbb{R}^3 = \{\underline{x} = x_1e_1 + x_2e_2 + x_3e_3,\ x_i \in \mathbb{R}, i = 1, 2, 3\}$$

is embedded in the real Clifford algebra $\mathbb{R}_{0,3}$ generated by e_1, e_2, e_3 over the field of real numbers $\mathbb{R}$.

An element $a \in \mathbb{R}_{0,3}$ may be written as $a = \sum_A a_Ae_A$, where a_A are real constants and A runs over all the possible ordered sets

$$A = \{1 \leq i_1 < \cdots < i_k \leq 3\}, \text{ or } A = \emptyset,$$

and

$$e_A := e_{i_1}e_{i_2}e_{i_k},\ e_0 = e_\emptyset = 1.$$

The scalar part of a is defined by$Sc[a] := a_0$.

The product of two Clifford vectors admits the splitting

$$\underline{x}\,\underline{y} = \underline{x} \bullet \underline{y} + \underline{x} \wedge \underline{y}\,,$$

where

$$\underline{x} \bullet \underline{y} = -\sum_{j=1}^{3} x_jy_j$$

is a scalar, while

$$\underline{x} \wedge \underline{y} = \sum_{j<k} e_j e_k (x_j y_k - x_k y_j)$$

is a 2-vector.

In general, we will consider functions defined on subsets of $\mathbb{R}^3$ and taking values in $\mathbb{R}_{0,3}$, which can be written as $f = \sum_A f_A e_A$, the f_A's being $\mathbb{R}$-valued functions.

The spaces of all k-time continuous differentiable and p-integrable functions are component-wise defined and denoted by $C^k(\mathbf{E})$ and $L^p(\mathbf{E})$ respectively, where $\mathbf{E} \subset \mathbb{R}^3$.

The Dirac operator $\partial_{\underline{x}}$ in $\mathbb{R}^3$ is defined for C^1-functions as

$$\partial_{\underline{x}} = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \partial_{x_3} e_3.$$

This operator allows a factorization of the Laplacian $\triangle$ in $\mathbb{R}^3$, namely

$$\partial_{\underline{x}}^2 = -\triangle.$$

The fundamental solution of $\triangle$ is given by

$$E_1(\underline{x}) = \frac{1}{\sigma_3 |\underline{x}|}, \ \underline{x} \neq 0,$$

where σ_3 denotes the surface area of the unit sphere in $\mathbb{R}^3$.

The so-called Clifford-Cauchy kernel is then constructed as

$$E_0(\underline{x}) := \partial_{\underline{x}} E_1(\underline{x}) = -\frac{1}{\sigma_3} \frac{\underline{x}}{|\underline{x}|^3},$$

which satisfies the equations $\partial_{\underline{x}} E_0 = E_0 \partial_{\underline{x}} = 0$ in $\mathbb{R}^3 \setminus \{0\}$.

The $\mathbb{R}_{0,3}$-valued solutions of $\partial_{\underline{x}} f = 0$ ($f \partial_{\underline{x}} = 0$) are called left monogenic (right monogenic) functions. Those functions which simultaneously satisfy both equations are referred as two-sided monogenic.

Unless stated otherwise, we always suppose that Ω is a smoothly bounded Jordan domain of $\mathbb{R}^3$. Later, the above smoothness assumption will be completely relaxed including the general case of a fractal boundary. In the sequel, the following notation will be used for the interior and exterior domains: $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$.

The Cliffordian-Stokes theorem [9] leads to the Borel-Pompeiu integral representation formula for $\mathbb{R}_{0,3}$-valued functions $f \in C^1(\Omega \cup \Gamma)$. Namely,

$$f(\underline{x}) = C^l_\Gamma f(\underline{x}) + \mathcal{T}^l_\Omega \partial_{\underline{x}} f(\underline{x}) \ \ for \ \ \underline{x} \in \Omega, \tag{2.1}$$

where

$$(C^l_\Gamma \varphi)(\underline{x}) := \int_\Gamma E_0(\underline{y} - \underline{x}) \underline{n}(\underline{y}) \varphi(\underline{y}) dS(\underline{y}), \ \underline{x} \notin \Gamma,$$

and

$$\mathcal{T}^l_\Omega \varphi(\underline{x}) = -\int_\Omega E_0(\underline{y} - \underline{x}) \varphi(\underline{y}) dV(\underline{y})$$

are, respectively, the Cauchy and Teodorescu transforms of φ.

Hereby $\underline{n}(\underline{y})$ is the outward normal at $\underline{y} \in \Gamma$, and dS (dV) denotes the surface (volume) measure.

In particular, for left monogenic functions one has in Ω

$$f(\underline{x}) = \int_{\Gamma} E_0(\underline{y} - \underline{x})\underline{n}(\underline{y})f(\underline{y})dS(\underline{y}). \tag{2.2}$$

Right-handed versions of formulas (2.1) and (2.2) are similarly obtained by using the integral transforms

$$[C^r_{\Gamma}\varphi](\underline{x}) = \int_{\Gamma} \varphi(\underline{y})\underline{n}(\underline{y})E_0(\underline{y} - \underline{x})dS(\underline{y}), \ \underline{x} \notin \Gamma,$$

and

$$\mathcal{T}^r_{\Omega}\varphi(\underline{x}) = -\int_{\Omega} \varphi(\underline{y})E_0(\underline{y} - \underline{x})dV(\underline{y}).$$

The inframonogenic functions have been introduced in [29, 30] (see also [8, 26, 27]) as the $\mathbb{R}_{0,3}$-valued solutions of the sandwich equation

$$\partial_{\underline{x}} f \partial_{\underline{x}} = 0. \tag{2.3}$$

Such functions represent a refinement of the more traditional biharmonic functions, see for instance [31, 32].

As proved in [26] any function f in $C^1(\Omega)$, inframonogenic in Ω, can be represented by

$$\begin{aligned} f(\underline{x}) = C^{\textbf{infra}}_{\Gamma} f(\underline{x}) := & \int_{\Gamma} f(\underline{y})\underline{n}(\underline{y})E_0(\underline{y} - \underline{x})dS(\underline{y}) \\ & + \frac{1}{2}\int_{\Gamma} E_0(\underline{y} - \underline{x})\underline{n}(\underline{y})(f(\underline{y})\partial_{\underline{y}})(\underline{y} - \underline{x})dS(\underline{y}) \\ & + \frac{1}{2}\sum_{i=1}^{3} e_i \int_{\Gamma} E_1(\underline{y} - \underline{x})\underline{n}(\underline{y})(f(\underline{y})\partial_{\underline{y}})dS(\underline{y})e_i \end{aligned} \tag{2.4}$$

and the operator

$$\begin{aligned} \mathcal{T}^{\textbf{infra}}_{\Omega} f(\underline{x}) := & -\frac{1}{2}\Bigg[\int_{\Omega} E_0(\underline{y} - \underline{x})f(\underline{y})(\underline{y} - \underline{x})dV(\underline{y}) \\ & + \sum_{i=1}^{3} e_i \int_{\Omega} E_1(\underline{y} - \underline{x})f(\underline{y})dV(\underline{y})e_i\Bigg] \end{aligned}$$

runs as a right inverse of the sandwich operator $\partial_{\underline{x}} \cdot \partial_{\underline{x}}$, i.e. $\partial_{\underline{x}}\mathcal{T}^{\textbf{infra}}_{\Omega} f \partial_{\underline{x}} = f$.

3. Boundary value problems for the Lamé-Navier system

We start this section by considering a Clifford reformulation of the Lamé system (1.1) obtained in [8].

By the use of the identities

$$\partial_{\underline{x}}^2 \vec{u} = -\mathrm{grad}(\mathrm{div}\vec{u}) + \mathrm{rot}(\mathrm{rot}\vec{u})$$

and

$$\partial_{\underline{x}} \vec{u} \partial_{\underline{x}} = -\mathrm{grad}(\mathrm{div}\vec{u}) - \mathrm{rot}(\mathrm{rot}\vec{u}),$$

the system (1.1) becomes

$$\mathcal{L}_{\alpha,\beta}\vec{u} := \alpha \partial_{\underline{x}} \vec{u} \partial_{\underline{x}} + \beta \partial_{\underline{x}}^2 \vec{u} = 0, \tag{3.1}$$

where $\alpha = \frac{\mu+\lambda}{2}, \beta = \frac{3\mu+\lambda}{2}$.

This implies immediately the factorization $\mathcal{L}_{\alpha,\beta}\vec{u} = \partial_{\underline{x}} \partial_{\underline{x}}^{\alpha,\beta} \vec{u}$, where

$$\partial_{\underline{x}}^{\alpha,\beta} \vec{u} = \alpha \partial_{\underline{x}} \vec{u} + \beta \vec{u} \partial_{\underline{x}}$$

is a first-order Dirac type operator introduced and studied recently in [33].

As already mentioned in [33] an analogous factorization for the Lamé system is derived in [19, p. 85], where use has been made of the operator $M^{-1} f = \frac{\lambda+2\mu}{\mu} f_0 + \underline{f}$. Indeed, we have

$$\mathcal{L}_{\alpha,\beta} \underline{f} = \mu \partial_{\underline{x}} \mathcal{M}^{-1} \partial_{\underline{x}} \underline{f}.$$

This approach allows the entry of quaternionic analysis techniques in obtaining integral representation formula for the solution of (1.1) as the composition of Teodorescu and Cauchy transforms.

The idea of the present paper is more in the direction of [26, 33], where explicit integral representation formulas are obtained in terms of properly defined Cauchy and Teodorescu transforms, this time closely related to the Lamé-Navier operator $\mathcal{L}_{\alpha,\beta}$. Our method can be extended without difficulty to the multidimensional elasticity theory [1, 34].

It is easily seen that $\alpha \neq 0$ and $\beta \neq 0$. This follows from the conditions $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$.

The Dirichlet problem for the system of elastostatics in a Lipschitz bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ:

$$\begin{cases} \mathcal{L}_{\alpha,\beta}\vec{u} = 0 \text{ in } \Omega \\ \vec{u} = \vec{f} \text{ in } \Gamma \end{cases} \tag{3.2}$$

was considered, for example, in [2].

It will be seen now how the above Clifford reformulation offers the possibility of proving in a very simple manner the following uniqueness theorem. Compare with [19, Theorem 4.3.3].

Theorem 1. *Let be $\vec{f} \in C(\Gamma)$. If a solution of the Dirichlet problem (3.2) exists in $C^2(\Omega) \cap C^1(\overline{\Omega})$, the solution is unique.*

Proof.
As usually, we are reduced to prove that the problem

$$\begin{cases} \mathcal{L}_{\alpha,\beta}\vec{u} = 0 \text{ in } \Omega \\ \vec{u} = 0 \text{ in } \Gamma \end{cases} \tag{3.3}$$

allows only the null solution $\vec{u} \equiv 0$.

By means of the Stokes formula we have

$$\int_{\Omega} (\underline{u}(\underline{y})\partial_{\underline{y}})(\underline{u}(\underline{y})\partial_{\underline{y}})dV(\underline{y}) + \int_{\Omega} \underline{u}(\underline{y})(\partial_{\underline{y}}\underline{u}(\underline{y})\partial_{\underline{y}})dV(\underline{y}) = \\ = \int_{\Gamma} \underline{u}(\underline{y})\underline{n}(y)\underline{u}(\underline{y})\partial_{\underline{y}}dS(\underline{y})$$

$$\int_{\Omega} (\underline{u}(\underline{y})\partial_{\underline{y}})(\partial_{\underline{y}}\underline{u}(\underline{y}))dV(\underline{y}) + \int_{\Omega} \underline{u}(\underline{y})\partial_{\underline{y}}^2\underline{u}(\underline{y})dV(\underline{y}) = \\ = \int_{\Gamma} \underline{u}(\underline{y})\underline{n}(y)\partial_{\underline{y}}\underline{u}(\underline{y})dS(\underline{y}),$$

which gives

$$\int_{\Omega} [\alpha(\underline{u}(\underline{y})\partial_{\underline{y}})^2 + \beta(\underline{u}(\underline{y})\partial_{\underline{y}})(\partial_{\underline{y}}\underline{u}(\underline{y}))]dV(\underline{y}) + \int_{\Omega} [\alpha\partial_{\underline{y}}\underline{u}(\underline{y})\partial_{\underline{y}} + \\ + \beta\partial_{\underline{y}}^2\underline{u}(\underline{y})]dV(\underline{y}) = \int_{\Gamma} \alpha\underline{u}(\underline{y})\underline{n}(\underline{y})(\underline{u}(\underline{y})\partial_{\underline{y}})dS(\underline{y}) + \\ + \int_{\Gamma} \beta\underline{u}(\underline{y})\underline{n}(\underline{y})\partial_{\underline{y}}(\underline{u}(\underline{y}))dS(\underline{y}).$$

and finally

$$\int_{\Omega} [\alpha(\underline{u}(\underline{y})\partial_{\underline{y}})^2 + \beta(\underline{u}(\underline{y})\partial_{\underline{y}})(\partial_{\underline{y}}\underline{u}(\underline{y}))]dV(\underline{y}) = 0. \tag{3.4}$$

Since $\underline{u}(\underline{y})\partial_{\underline{y}} = -\text{div}\underline{u}(\underline{y}) - \text{rot}\underline{u}(\underline{y})$, it follows that

$$(\underline{u}(\underline{y})\partial_{\underline{y}})^2 = (\text{div}\underline{u}(\underline{y}))^2 - |\text{rot}\underline{u}(\underline{y})|^2 + 2\text{div}\underline{u}(\underline{y})\text{rot}\underline{u}(\underline{y})$$

and moreover

$$(\underline{u}(\underline{y})\partial_{\underline{y}})(\partial_{\underline{y}}\underline{u}(\underline{y})) = (-\text{div}\underline{u}(\underline{y}) - \text{rot}\underline{u}(\underline{y}))(-\text{div}\underline{u}(\underline{y}) + \text{rot}\underline{u}(\underline{y})) = \\ = (\text{div}\underline{u}(\underline{y}))^2 + |\text{rot}\underline{u}(\underline{y})|^2 - \text{div}(\text{rot}\underline{u}(\underline{y})) + \text{div}(\text{rot}\underline{u}(\underline{y})) = \\ = (\text{div}\underline{u}(\underline{y}))^2 + |\text{rot}\underline{u}(\underline{y})|^2.$$

Taking the scalar part in (3.4) yields

$$\int_{\Omega} (\alpha + \beta)(\text{div}\underline{u}(\underline{y}))^2 + (\beta - \alpha)|\text{rot}\underline{u}(\underline{y})|^2 dV(\underline{y})) = 0. \tag{3.5}$$

If $\beta \leq \alpha$, then $3\mu + \lambda \leq \mu + \lambda$ and $2\mu \leq 0$, the later being false, since $\mu > 0$. Similarly, the assumption $\beta \leq -\alpha$ leads to a contradiction with $\frac{\lambda}{\mu} > -\frac{2}{3}$.

Therefore

$$(\alpha + \beta)(\operatorname{div} \underline{u})^2 + (\beta - \alpha)|\operatorname{rot} \underline{u}|^2 = 0,$$

and hence

$$\begin{aligned} \operatorname{div} \underline{u} &= 0 \\ \operatorname{rot} \underline{u} &= 0, \end{aligned}$$

which together with the boundary condition $\underline{u} = 0$, completes the proof. □

We are rather interested in the investigation of the jump problem for the Lamé system (1.1):

$$\begin{array}{ll} \mathcal{L}_{\alpha,\beta}\vec{u}(\underline{x}) = 0, & \underline{x} \in \mathbb{R}^3 \setminus \Gamma, \\ \vec{u}^+(\underline{x}) - \vec{u}^-(\underline{x}) = \vec{f}(\underline{x}), & \underline{x} \in \Gamma, \\ [\vec{u}\partial_{\underline{x}}]^+(\underline{x}) - [\vec{u}\partial_{\underline{x}}]^-(\underline{x}) = \vec{f}(\underline{x})\partial_{\underline{x}}, & \underline{x} \in \Gamma, \end{array} \tag{3.6}$$

where $\vec{u}^\pm(\underline{x})$ are the limit values of $\vec{u}$ at the point $\underline{x} \in \Gamma$ as this point is approached from $\Omega_\pm$, respectively.

The vector valued function $\vec{f}$ is assumed to be in the higher order Lipschitz class $\mathrm{Lip}(1 + \alpha, \Gamma)$, $0 < \alpha < 1$; i.e. for each real component f_i, $i = 1, 2, 3$, of $\vec{f}$ there exists a collection $\{f_i^{(j)},\ 0 \leq |j| \leq 1\}$ of real uniformly bounded functions on Γ, with $f_i^{(0)} = f_i$, and so that

$$R_j(x, y) = f_i^{(j)}(x) - \sum_{|j+l| \leq 1} \frac{f_i^{(j+l)}(y)}{l!}(x - y)^l, \quad x, y \in \Gamma \tag{3.7}$$

satisfies

$$|R_j(x, y)| = O(|x - y|^{1+\alpha-|j|}), \quad x, y \in \Gamma, |j| \leq 1. \tag{3.8}$$

Following [35, Theorem 4, page 177], any function $\vec{f}$ in $\mathrm{Lip}(1 + \alpha, \Gamma)$ can be extended to the whole $\mathbb{R}^3$ as a continuously differentiable function, with the abuse of notation again denoted by $\vec{f}$. The Whitney extension $\vec{f}$ has α-Hölder continuous partial derivatives, and moreover

$$|\partial^{(j)}\vec{f}(\underline{x})| \leqslant c\, dist(\underline{x}, \Gamma)^{\alpha-1}, \tag{3.9}$$

for $|(j)| = 2$ and $\underline{x} \in \mathbb{R}^3 \setminus \Gamma$. It will be assumed in the sequel that c is a positive constant, which may have different values at different occurrences.

The above problem will be studied both for the more standard case of sufficiently smooth surfaces as well as for the pathological situation of considering domains with fractal boundary.

4. Smooth boundary case

Before further development of problem (3.6) let us state and prove a sort of Borel-Pompeiu representation formula in terms of the Lamé operator $\mathcal{L}_{\alpha,\beta}$. Introduce the notation

$$C_\Gamma^{\mathcal{L}}\vec{f}(\underline{x}) := \frac{\alpha^*}{2} \int_\Gamma E_0(\underline{y} - \underline{x})\underline{n}(\underline{y})\vec{f}(\underline{y})(\underline{y} - \underline{x})dS(\underline{y})$$

$$+\frac{\alpha^*}{2}\sum_{i=1}^{3} e_i \int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\vec{f}(\underline{y})dS(\underline{y})e_i - \beta^* \int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\vec{f}(\underline{y})dS(\underline{y})$$

$$\mathcal{T}^{\mathcal{L}}_\Omega\vec{\varphi}(\underline{x}) := \alpha^*\mathcal{T}^{\mathbf{infra}}_\Omega\vec{\varphi} + \beta^*\int_\Omega E_1(\underline{y}-\underline{x})\vec{\varphi}(\underline{y})dV(\underline{y}) =$$

$$-\alpha^*\int_\Omega E_0(\underline{y}-\underline{x})\langle\underline{y}-\underline{x},\vec{\varphi}(\underline{y})\rangle dV(\underline{y}) + \beta^*\int_\Omega E_1(\underline{y}-\underline{x})\vec{\varphi}(\underline{y})dV(\underline{y}),$$

where

$$\alpha^* = \frac{1}{2}\left[\frac{1}{2\mu+\lambda} - \frac{1}{\mu}\right], \beta^* = \frac{1}{2}\left[\frac{1}{2\mu+\lambda} + \frac{1}{\mu}\right].$$

Let us first prove that $\mathcal{T}^{\mathcal{L}}_\Omega$ works as an inverse operator for $\mathcal{L}_{\alpha,\beta}$. For that there is no restriction on Ω other than the requirement of being open and bounded.

Theorem 2. *Let be $\vec{f} \in C^2(\Omega)$, then*

$$\mathcal{L}_{\alpha,\beta}[\mathcal{T}^{\mathcal{L}}_\Omega\vec{f}](\underline{x}) = \begin{cases} \vec{f}(\underline{x}), & \underline{x}\in\Omega_+ \\ 0, & \underline{x}\in\Omega_-. \end{cases}$$

Proof.
We restrict our consideration to $\underline{x} \in \Omega_+$. The case $\underline{x} \in \Omega_-$ does not meet with any essentially new difficulties.

Taking into account that

$$\partial_{\underline{x}}[\mathcal{T}^l_\Omega\vec{f}] = \vec{f},\ \ \partial_{\underline{x}}[\mathcal{T}^{\mathbf{infra}}_\Omega\vec{f}]\partial_{\underline{x}} = \vec{f},$$

we have

$$\mathcal{L}_{\alpha,\beta}[\mathcal{T}^{\mathcal{L}}_\Omega\vec{f}] = \mathcal{L}_{\alpha,\beta}[\alpha^*\mathcal{T}^{\mathbf{infra}}_\Omega\vec{f} + \beta^*\int_\Omega E_1(\underline{y}-\underline{x})\vec{f}(\underline{y})dV(\underline{y})] =$$

$$= (\alpha\alpha^* + \beta\beta^*)\vec{f} + (\alpha\beta^* + \beta\alpha^*)(\mathcal{T}^l_\Omega\vec{f})\partial_{\underline{x}} = \vec{f},$$

where we used the identities $\beta^*\beta + \alpha^*\alpha = 1$ and $\alpha\beta^* + \beta\alpha^* = 0$. □

Theorem 3. *Let $\vec{f} \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then, for $\underline{x} \in \Omega$ we have*

$$\vec{f}(\underline{x}) = \beta^*\beta C^l_\Gamma\vec{f}(\underline{x}) + \alpha^*\alpha C^r_\Gamma\vec{f}(\underline{x})$$

$$-\alpha^*\beta\left\{\int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\vec{f}(\underline{y})\partial_{\underline{y}}dS(\underline{y}) - \int_\Gamma \partial_{\underline{y}}\vec{f}(\underline{y})\underline{n}(\underline{y})E_1(\underline{y}-\underline{x})dS(\underline{y})\right\}$$

$$+C^{\mathcal{L}}_\Gamma[\alpha\vec{f}(\underline{x})\partial_{\underline{x}} + \beta\partial_{\underline{x}}\vec{f}(\underline{x})] + \mathcal{T}^{\mathcal{L}}_\Omega[\mathcal{L}_{\alpha,\beta}\vec{f}(\underline{x})].$$

We need first the following auxiliary result.

Lemma 4.

$$\mathcal{T}_{\Omega}^{\mathbf{infra}}\partial_{\underline{x}}^{2}\vec{f}(x)=-C_{\Gamma}^{\mathbf{infra}}\partial_{\underline{x}}\vec{f}(x)-\int_{\Omega}\partial_{\underline{y}}\vec{f}(\underline{y})E_0(\underline{y}-\underline{x})dV(\underline{y}).$$

Proof.
The proof is quite similar to that of [26, Theorem 3.1] and requires the use of the identity

$$\partial_{\underline{y}}[\partial_{\underline{y}}\vec{f}(y)(y-x)]=(\partial_{\underline{y}}^{2}\vec{f}(y))(y-x)+\sum_{i=1}^{3}e_i\partial_{\underline{y}}\vec{f}(y)e_i.$$

Proof of Theorem 3
It follows that

$$\mathcal{T}_{\Omega}^{\mathcal{L}}\mathcal{L}_{\alpha,\beta}\vec{f}(\underline{x})=\alpha^{*}\alpha\mathcal{T}_{\Omega}^{\mathbf{infra}}\partial_{\underline{x}}\vec{f}(\underline{x})\partial_{\underline{x}}+\alpha^{*}\beta\mathcal{T}_{\Omega}^{\mathbf{infra}}\partial_{\underline{y}}^{2}\vec{f}(\underline{y})+$$
$$+\beta^{*}\alpha\int_{\Omega}E_1(\underline{y}-\underline{x})\partial_{\underline{y}}\vec{f}(\underline{y})\partial_{\underline{y}}dV(\underline{y})+\beta^{*}\beta\int_{\Omega}E_1(\underline{y}-\underline{x})\partial_{\underline{y}}^{2}\vec{f}(\underline{y})dV(\underline{y}).$$

Now we make use of both, Lemma 4, [26, Theorem 3.1] and the iterated Borel-Pompeiu formula associated to $\partial_{\underline{x}}^{2}\vec{f}$ (see [36, 37])

$$\vec{f}(\underline{x})=C_{\Gamma}^{l}\vec{f}(\underline{x})-\int_{\Gamma}E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\partial_{\underline{y}}\vec{f}(\underline{y})dS(\underline{y})+\int_{\Omega}E_1(\underline{y}-\underline{x})\partial_{\underline{x}}^{2}\vec{f}(\underline{y})dV(\underline{y}).$$

Consequently,

$$\mathcal{T}_{\Omega}^{\mathcal{L}}\mathcal{L}_{\lambda,\mu}\vec{f}(\underline{x})=\alpha^{*}\alpha\vec{f}(\underline{x})-\alpha^{*}\alpha C_{\Gamma}^{r}\vec{f}(\underline{x})-\alpha^{*}\alpha C_{\Gamma}^{\mathbf{infra}}\vec{f}(\underline{x})\partial_{\underline{x}}-$$
$$-\alpha^{*}\beta C_{\Gamma}^{\mathbf{infra}}\partial_{\underline{x}}\vec{f}(\underline{x})-\alpha^{*}\beta\int_{\Omega}\partial_{\underline{y}}\vec{f}(\underline{y})E_0(\underline{y}-\underline{x})dV(\underline{y})+$$
$$+\beta^{*}\beta\vec{f}(\underline{x})-\beta^{*}\beta C_{\Gamma}^{l}\vec{f}(\underline{x})+\beta^{*}\beta\int_{\Gamma}E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\partial_{\underline{y}}\vec{f}(\underline{y})dS(\underline{y})-$$
$$-\beta^{*}\alpha\int_{\Omega}E_0(\underline{y}-\underline{x})\vec{f}(\underline{y})\partial_{\underline{y}}dV(\underline{y})+\beta^{*}\alpha\int_{\Gamma}E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\vec{f}(\underline{y})\partial_{\underline{y}}dS(\underline{y}).$$

Since

$$\int_{\Omega}E_1(\underline{y}-\underline{x})\partial_{\underline{y}}\vec{f}(\underline{y})\partial_{\underline{y}}dV(\underline{y})+\int_{\Omega}E_0(\underline{y}-\underline{x})\vec{f}(\underline{y})\partial_{\underline{y}}dV(\underline{y})=$$
$$=\int_{\Gamma}E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\vec{f}(\underline{y})\partial_{\underline{y}}dS(\underline{y})$$

and

$$\int_{\Omega}\partial_{\underline{y}}\vec{f}(\underline{y})\partial_{\underline{y}}E_1(\underline{y}-\underline{x})dV(\underline{y})+\int_{\Omega}\partial_{\underline{y}}\vec{f}(\underline{y})E_0(\underline{y}-\underline{x})dV(\underline{y})=$$

$$= \int_{\Gamma} \partial_{\underline{y}} \vec{f}(\underline{y}) \underline{n}(\underline{y}) E_1(\underline{y} - \underline{x}) dS(\underline{y}),$$

after subtracting them we obtain

$$\int_{\Gamma} E_1(\underline{y} - \underline{x}) \underline{n}(\underline{y}) \vec{f}(\underline{y}) \partial_{\underline{y}} dS(\underline{y}) - \int_{\Gamma} \partial_{\underline{y}} \vec{f}(\underline{y}) \underline{n}(\underline{y}) E_1(\underline{y} - \underline{x}) dS(\underline{y}) =$$
$$= \int_{\Omega} E_0(\underline{y} - \underline{x}) \vec{f}(\underline{y}) \partial_{\underline{y}} dV(\underline{y}) - \int_{\Omega} \partial_{\underline{y}} \vec{f}(\underline{y}) E_0(\underline{y} - \underline{x}) dV(\underline{y})$$

and finally

$$\begin{aligned} \vec{f}(\underline{x}) &= \beta^* \beta C^l_{\Gamma} \vec{f}(\underline{x}) + \alpha^* \alpha C^r_{\Gamma} \vec{f}(\underline{x}) \\ &-\alpha^* \beta \left\{ \int_{\Gamma} E_1(\underline{y} - \underline{x}) \mathbf{n}(\underline{y}) \vec{f}(\underline{y}) \partial_{\underline{y}} dS(\underline{y}) - \int_{\Gamma} \partial_{\underline{y}} \vec{f}(\underline{y}) \underline{n}(\underline{y}) E_1(\underline{y} - \underline{x}) dS(\underline{y}) \right\} \\ &+ C^{\mathcal{L}}_{\Gamma}[\alpha \vec{f}(\underline{x}) \partial_{\underline{x}} + \beta \partial_{\underline{x}} \vec{f}(\underline{x})] + \mathcal{T}^{\mathcal{L}}_{\Omega}[\mathcal{L}_{\alpha,\beta} \vec{f}(\underline{x})]. \end{aligned}$$

Corollary 5. *Let $\vec{f} \in C^2(\Omega) \cap C^1(\overline{\Omega})$. If, moreover, $\vec{f}$ satisfies (3.1) then in Ω we have*

$$\begin{aligned} \vec{f}(\underline{x}) &= \beta^* \beta C^l_{\Gamma} \vec{f}(\underline{x}) + \alpha^* \alpha C^r_{\Gamma} \vec{f}(\underline{x}) \\ &-\alpha^* \beta \left\{ \int_{\Gamma} E_1(\underline{y} - \underline{x}) \underline{n}(\underline{y}) \vec{f}(\underline{y}) \partial_{\underline{y}} dS(\underline{y}) - \int_{\Gamma} \partial_{\underline{y}} \vec{f}(\underline{y}) \underline{n}(\underline{y}) E_1(\underline{y} - \underline{x}) dS(\underline{y}) \right\} \\ &+ C^{\mathcal{L}}_{\Gamma}[\alpha \vec{f}(\underline{x}) \partial_{\underline{x}} + \beta \partial_{\underline{x}} \vec{f}(\underline{x})]. \end{aligned}$$

Let now $\vec{f}$ be intrinsically defined as a C^1-smooth function on Γ. A direct but non-trivial calculation shows that a function given by

$$\begin{aligned} \vec{F}(\underline{x}) = \vec{f}(\underline{x}) &= \beta^* \beta C^l_{\Gamma} \vec{f}(\underline{x}) + \alpha^* \alpha C^r_{\Gamma} \vec{f}(\underline{x}) \\ &-\alpha^* \beta \left\{ \int_{\Gamma} E_1(\underline{y} - \underline{x}) \underline{n}(\underline{y}) \vec{f}(\underline{y}) \partial_{\underline{y}} dS(\underline{y}) - \int_{\Gamma} \partial_{\underline{y}} \vec{f}(\underline{y}) \underline{n}(\underline{y}) E_1(\underline{y} - \underline{x}) dS(\underline{y}) \right\} \\ &+ C^{\mathcal{L}}_{\Gamma}[\alpha \vec{f}(\underline{x}) \partial_{\underline{x}} + \beta \partial_{\underline{x}} \vec{f}(\underline{x})] \end{aligned}$$

satisfies the Lamé system in $\mathbb{R}^3 \setminus \Gamma$.

Now we are able to characterize the solvability of Problem (3.6).

Theorem 6. *Let $\vec{f} \in Lip(1 + \alpha, \Gamma)$. Then a solution of (3.6) is given by*

$$\begin{aligned} \vec{u}(\underline{x}) &= \beta^* \beta C^l_{\Gamma} \vec{f}(\underline{x}) + \alpha^* \alpha C^r_{\Gamma} \vec{f}(\underline{x}) \\ &-\alpha^* \beta \left\{ \int_{\Gamma} E_1(\underline{y} - \underline{x}) \underline{n}(\underline{y}) \vec{f}(\underline{y}) \partial_{\underline{y}} dS(\underline{y}) - \int_{\Gamma} \partial_{\underline{y}} \vec{f}(\underline{y}) \underline{n}(\underline{y}) E_1(\underline{y} - \underline{x}) dS(\underline{y}) \right\} \\ &+ C^{\mathcal{L}}_{\Gamma}[\alpha \vec{f}(\underline{x}) \partial_{\underline{x}} + \beta \partial_{\underline{x}} \vec{f}(\underline{x})]. \end{aligned} \tag{4.1}$$

Moreover, it is unique under the vanishing conditions $\vec{u}(\infty) = \vec{u}\partial_{\underline{x}}(\infty) = 0$.

Proof.
Using the formulae of Plemelj–Sokhotski [9] we conclude that the first two summands in (4.1) have by passage through Γ the jump $\beta^*\beta f(\underline{x})$ and $\alpha^*\alpha f(\underline{x})$, respectively. Since the remaining terms are weekly–singular parametric integrals, they have no jumps through Γ. Consequently,

$$\vec{u}^+(\underline{x}) - \vec{u}^-(\underline{x}) = (\beta^*\beta + \alpha^*\alpha)\vec{f}(\underline{x}) = \vec{f}(\underline{x}), \ \ \underline{x} \in \Gamma.$$

The proof of the second jump condition needs some more calculations. Indeed, we have in $\mathbb{R}^3 \setminus \Gamma$ that

$$\begin{aligned} \vec{u}\partial_{\underline{x}} &= \beta^*\beta[C^l_\Gamma \vec{f}(\underline{x})]\partial_{\underline{x}} \\ &-\alpha^*\beta\left\{\left[\int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\vec{f}(\underline{y})\partial_{\underline{y}} dS(\underline{y})\right]\partial_{\underline{x}} + C^r_\Gamma \partial_{\underline{x}}\vec{f}(\underline{x})\right\} \\ &+\alpha^* C^l_\Gamma[\alpha\vec{f}(\underline{x})\partial_{\underline{x}} + \beta\partial_{\underline{x}}\vec{f}(\underline{x})] \\ &-\beta^*\left[\int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})(\alpha\vec{f}(\underline{y})\partial_{\underline{y}} + \beta\partial_{\underline{y}}\vec{f}(\underline{y}))dS(\underline{y})\right]\partial_{\underline{x}}. \end{aligned}$$

After using $\alpha^*\beta = -\beta^*\alpha$, we obtain

$$\begin{aligned} \vec{u}\partial_{\underline{x}} &= \beta^*\beta\left[C^l_\Gamma\vec{f}(\underline{x}) - \int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\partial_{\underline{y}}\vec{f}(\underline{y})dS(\underline{y})\right]\partial_{\underline{x}} \\ &+\alpha^*\beta\left[C^l_\Gamma\partial_{\underline{x}}\vec{f}(\underline{x}) - C^r_\Gamma\partial_{\underline{x}}\vec{f}(\underline{x})\right] + \alpha\alpha^* C^l_\Gamma[\vec{f}\partial_{\underline{x}}](\underline{x}). \end{aligned}$$

As usual let χ_Ω be the characteristic function of Ω. Using again the iterated Borel-Pompeiu formula associated to $\partial^2_{\underline{x}}\vec{f}$ (see [36, 37]),

$$\begin{aligned} \chi_\Omega(\underline{x})\vec{f}(\underline{x}) &= C^l_\Gamma\vec{f}(\underline{x}) - \int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\partial_{\underline{y}}\vec{f}(\underline{y})dS(\underline{y}) \\ &+ \int_\Omega E_1(\underline{y}-\underline{x})\partial^2_{\underline{x}}\vec{f}(\underline{y})dV(\underline{y}), \end{aligned}$$

we get

$$\begin{aligned} [C^l_\Gamma\vec{f}(\underline{x}) - \int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\partial_{\underline{y}}\vec{f}(\underline{y})dS(\underline{y})]\partial_{\underline{x}} &= \chi_\Omega(\underline{x})\vec{f}(\underline{x})\partial_{\underline{x}} \\ &+ \int_\Omega E_0(\underline{y}-\underline{x})\partial^2_{\underline{x}}\vec{f}(\underline{y})dV(\underline{y}), \end{aligned}$$

which, after applying in the right-hand side the Borel-Pompeiu formula associated to $\partial_{\underline{x}}$, gives

$$[C^l_\Gamma\vec{f}(\underline{x}) - \int_\Gamma E_1(\underline{y}-\underline{x})\underline{n}(\underline{y})\partial_{\underline{y}}\vec{f}(\underline{y})dS(\underline{y})]\partial_{\underline{x}} = C^r_\Gamma[\vec{f}\partial_{\underline{x}}](\underline{x}).$$

Finally we have

$$\vec{u}\partial_{\underline{x}} = \beta^*\beta C_\Gamma^r[\vec{f}\partial_{\underline{x}}](\underline{x}) + \alpha^*\beta[C_\Gamma^l\partial_{\underline{x}}\vec{f}(\underline{x}) - C_\Gamma^r\partial_{\underline{x}}\vec{f}(\underline{x})] + \alpha\alpha^* C_\Gamma^l[\vec{f}\partial_{\underline{x}}](\underline{x}),$$

from which the second boundary condition in (3.6) is again a direct consequence of the Plemelj-Sokhotski formulae.

The proof of the uniqueness can be done indirectly. Assume that $\vec{u}_1, \vec{u}_2$ are two solutions of (3.6), then it implies that $\vec{\omega} = \vec{u}_1 - \vec{u}_2$ fulfills

$$\begin{aligned} \mathcal{L}_{\alpha,\beta}\vec{\omega}(\underline{x}) &= 0, \quad \underline{x} \in \mathbb{R}^3 \setminus \Gamma \\ \vec{\omega}^+(\underline{x}) &= \vec{\omega}^-(\underline{x}), \quad \underline{x} \in \Gamma, \\ [\vec{\omega}\partial_{\underline{x}}]^+(\underline{x}) &= [\vec{\omega}\partial_{\underline{x}}]^-(\underline{x}), \quad \underline{x} \in \Gamma, \\ \vec{\omega}(\infty) &= \vec{\omega}\partial_{\underline{x}}(\infty) = 0. \end{aligned} \tag{4.2}$$

Let us prove that $\vec{\omega} \equiv 0$.

Since $\vec{\omega}$ satisfies $\mathcal{L}_{\alpha,\beta}\vec{\omega}(\underline{x}) = 0$ in $\mathbb{R}^3\setminus\Gamma$, the function $\varpi = \vec{\omega}\partial_{\underline{x}}$ satisfies the equation $\alpha\partial_{\underline{x}}\varpi + \beta\varpi\partial_{\underline{x}} = 0$ there. From the previous statement it is easy to show that the $\mathbb{R}_{0,3}$-valued function $\varpi_* = \varpi_0 + \frac{\beta-\alpha}{\beta+\alpha}\underline{\varpi}$ is (left) monogenic in $\mathbb{R}^3 \setminus \Gamma$.

Hence, the auxiliary function ϖ_* is a solution of the boundary value problem

$$\begin{aligned} \partial_{\underline{x}}\phi(\underline{x}) &= 0, \quad \underline{x} \in \mathbb{R}^3 \setminus \Gamma \\ \phi^+(\underline{x}) &= \phi^-(\underline{x}), \quad \underline{x} \in \Gamma, \\ \phi(\infty) &= 0. \end{aligned} \tag{4.3}$$

From the Painlevé and Liouville theorems in Clifford analysis [9] it follows that the above problem has the unique trivial solution $\phi \equiv 0$, so we have $\varpi_* \equiv 0$ and hence $\varpi \equiv 0$ in $\mathbb{R}^3$.

Consequently, $\vec{\omega}$ is (right) monogenic in $\mathbb{R}^3 \setminus \Gamma$, vanishes at ∞ and has no jump through Γ. Finally, a repeated use of the Painlevé and Liouville theorems yields $\vec{\omega} \equiv 0$ and we are done. □

5. Fractal boundary case

The main new ingredient of this section is the extension of our previous considerations to the case of domains Ω admitting a fractal boundary. We follow [38] in assuming that Γ is d-summable for $2 < d < 3$, which means that the integral

$$\int_0^1 N_\Gamma(\tau)\, \tau^{d-1}\, d\tau$$

exists in the improper sense. Here$N_\Gamma(\tau)$ denotes the least number of balls of radius τ needed to cover Γ.

As was early remarked in [38] any surface Γ with fractal box dimension $\mathrm{D}(\Gamma)$ is d-summable for any $d = \mathrm{D}(\Gamma) + \epsilon$, $\epsilon > 0$.

The following result was proved in [38] and it is really the heart of the proof of the main theorem of this section.

Lemma 7. *[38] If Ω is a Jordan domain of $\mathbb{R}^3$ and its boundary Γ is d-summable, then the expression $\sum_{Q\in\mathcal{W}} |Q|^d$, called the d-sum of the Whitney decomposition $\mathcal{W}$ of Ω, is finite.*

Recall that the Whitney decomposition of Ω involves a collection of disjoint cubes, whose lengths are proportional to their distance from Γ. For details we refer the reader to [35].

To solve the problem (3.6) in the fractal setting, we need first a few results.

Lemma 8. *Let $\vec{f} \in Lip(1+\alpha, \Gamma)$, then $\mathcal{L}_{\alpha,\beta}[\vec{f}] \in L^p(\Omega)$ for any $p \leq \frac{3-d}{1-\alpha}$.*

Proof.
From (3.9), we have $|\mathcal{L}_{\alpha,\beta}\vec{f}(\underline{x})| \leqslant c\, dist(\underline{x}, \Gamma)^{\alpha-1}$ for $\underline{x} \in \Omega$. After such estimate, the statement can be proved quite analogously to [39, Lemma 4.1]. □

Lemma 9. *Let $\vec{f} \in Lip(1+\alpha, \Gamma)$ with $\alpha > \frac{d}{3}$. Then the functions $\mathcal{T}^{\mathcal{L}}_{\Omega}(\mathcal{L}_{\alpha,\beta}\vec{f})$ and $[\mathcal{T}^{\mathcal{L}}_{\Omega}(\mathcal{L}_{\alpha,\beta}\vec{f})]\partial_{\underline{x}}$ are continuous in $\mathbb{R}^3$.*

Proof.
Let $\vec{\varphi} := \mathcal{L}_{\alpha,\beta}[\vec{f}]$ and prove first the continuity of $\mathcal{T}^{\mathcal{L}}_{\Omega}\vec{\varphi}$. Indeed, take $\underline{x}, \underline{z} \in \mathbb{R}^3$, then

$$\mathcal{T}^{\mathcal{L}}_{\Omega}(\varphi)(\underline{x}) - \mathcal{T}^{\mathcal{L}}_{\Omega}(\varphi)(\underline{z}) = \beta^* \int_{\Omega} [E_1(\underline{y}-\underline{x}) - E_1(\underline{y}-\underline{z})]\vec{\varphi}(\underline{y})dV(\underline{y}) -$$

$$-\alpha^* \int_{\Omega} [E_0(\underline{y}-\underline{x})\langle \underline{y}-\underline{x}, \vec{\varphi}(\underline{y})\rangle - E_0(\underline{y}-\underline{z})\langle \underline{y}-\underline{z}, \vec{\varphi}(\underline{y})\rangle]dV(\underline{y}).$$

If follows that

$$|E_1(\underline{y}-\underline{x}) - E_1(\underline{y}-\underline{z})| \leq c\left|\frac{1}{|\underline{y}-\underline{x}|^{m-2}} - \frac{1}{|\underline{y}-\underline{z}|^{m-2}}\right| =$$

$$c\Big||\underline{y}-\underline{z}| - |\underline{y}-\underline{x}|\Big| \sum_{k=1}^{m-2} \frac{1}{|\underline{y}-\underline{z}|^{m-1-k}|\underline{y}-\underline{x}|^k} \leq$$

$$c|\underline{x}-\underline{z}| \sum_{k=1}^{m-2} \frac{1}{|\underline{y}-\underline{z}|^{m-1-k}|\underline{y}-\underline{x}|^k}.$$

Then

$$\left|\int_{\Omega} [E_1(\underline{y}-\underline{x}) - E_1(\underline{y}-\underline{z})]\vec{\varphi}(\underline{y})dV(\underline{y})\right| \leq$$

$$c|\underline{x}-\underline{z}| \sum_{k=1}^{m-2} \int_{\Omega} \frac{1}{|\underline{y}-\underline{z}|^{m-1-k}|\underline{y}-\underline{x}|^k} |\vec{\varphi}(\underline{y})| dV(\underline{y}). \tag{5.1}$$

It follows from $\alpha > \frac{d}{3}$ and Lemma 8 that $\vec{\varphi}$ is integrable in Ω. Consequently, every integral in (5.1) is finite and hence

$$\left|\int_{\Omega} [E_1(\underline{y}-\underline{x}) - E_1(\underline{y}-\underline{z})]\vec{\varphi}(\underline{y})dV(\underline{y})\right|$$

goes to 0 as $\underline{x} \to \underline{z}$.

On the other hand,

$$\begin{aligned}
&\Big|E_0(\underline{y}-\underline{x})\langle \underline{y}-\underline{x},\vec{\varphi}(\underline{y})\rangle - E_0(\underline{y}-\underline{z})\langle \underline{y}-\underline{z},\vec{\varphi}(\underline{y})\rangle\Big| \leq \\
&\Big|E_0(\underline{y}-\underline{x})\langle \underline{y}-\underline{x},\vec{\varphi}(\underline{y})\rangle - E_0(\underline{y}-\underline{x})\langle \underline{y}-\underline{z},\vec{\varphi}(\underline{y})\rangle\Big| + \\
&\Big|E_0(\underline{y}-\underline{x})\langle \underline{y}-\underline{z},\vec{\varphi}(\underline{y})\rangle - E_0(\underline{y}-\underline{z})\langle \underline{y}-\underline{z},\vec{\varphi}(\underline{y})\rangle\Big| \leq \\
&\Big|E_0(\underline{y}-\underline{x})\langle \underline{z}-\underline{x},\vec{\varphi}(\underline{y})\rangle\Big| + \Big|[E_0(\underline{y}-\underline{x}) - E_0(\underline{y}-\underline{z})]\langle \underline{y}-\underline{z},\vec{\varphi}(\underline{y})\rangle\Big| \leq \\
&|\underline{x}-\underline{z}||E_0(\underline{y}-\underline{x})||\vec{\varphi}(\underline{y})| + |E_0(\underline{y}-\underline{x}) - E_0(\underline{y}-\underline{z})||\underline{y}-\underline{z}||\vec{\varphi}(\underline{y})|.
\end{aligned}$$

Because of

$$|E_0(\underline{y}-\underline{x}) - E_0(\underline{y}-\underline{z})| \leq c|\underline{x}-\underline{z}| \sum_{i=1}^{m-1} \frac{1}{|\underline{y}-\underline{x}|^i |\underline{y}-\underline{z}|^{m-i}}$$

it follows that

$$\begin{aligned}
&\Big|E_0(\underline{y}-\underline{x})\langle \underline{y}-\underline{x},\vec{\varphi}(\underline{y})\rangle - E_0(\underline{y}-\underline{z})\langle \underline{y}-\underline{z},\vec{\varphi}(\underline{y})\rangle\Big| \leq \\
&c|\underline{x}-\underline{z}|\Big[|E_0(\underline{y}-\underline{x})||\vec{\varphi}(\underline{y})| + \sum_{i=1}^{m-1} \frac{1}{|\underline{y}-\underline{x}|^i |\underline{y}-\underline{z}|^{m-1-i}}|\vec{\varphi}(\underline{y})|\Big].
\end{aligned}$$

At this point we use again the integrability of $\vec{\varphi}$ together with the above inequality to see that

$$\int_\Omega [E_0(\underline{y}-\underline{x})\langle \underline{y}-\underline{x},\vec{\varphi}(\underline{y})\rangle - E_0(\underline{y}-\underline{z})\langle \underline{y}-\underline{z},\vec{\varphi}(\underline{y})\rangle]$$

goes to 0 as $\underline{x} \to \underline{z}$, which summarizing proves the continuity of $\mathcal{T}^{\mathcal{L}}_\Omega \vec{\varphi}$.

To prove the continuity of $[\mathcal{T}^{\mathcal{L}}_\Omega(\mathcal{L}_{\alpha,\beta}\vec{f})]\partial_{\underline{x}}$ we use the identity

$$\Big[\int_\Omega E_1(\underline{y}-\underline{x})\vec{\varphi}(\underline{y})dV(\underline{y})\Big]\partial_{\underline{x}} = \mathcal{T}^r_\Omega \vec{\varphi}.$$

and the following one proved in [26, Theorem 4.1]:

$$\Big[\mathcal{T}^{\textbf{infra}}_\Omega \vec{\varphi}\Big]\partial_{\underline{x}} = \mathcal{T}^l_\Omega \vec{\varphi}.$$

Therefore, we have

$$[\mathcal{T}^{\mathcal{L}}_\Omega \vec{\varphi}]\partial_{\underline{x}} = \alpha^* \mathcal{T}^l_\Omega \vec{\varphi} + \beta^* \mathcal{T}^r_\Omega \vec{\varphi}.$$

According to the condition $\alpha > \frac{d}{3}$ and Lemma 8, we conclude that $\vec{\varphi} \in L^p(\Omega)$ with $p = \frac{3-d}{1-\alpha} > 3$. Now the assertion is proved by appealing to [10, Proposition 8.1]. □

Let us come back to the task of finding a solution of (3.6) in our general geometric context.

Theorem 10. *Let $\vec{f} \in Lip(1+\alpha,\Gamma)$. Under the assumption $\alpha > \frac{d}{3}$ the problem (3.6) has a solution given by*

$$\vec{u}(\underline{x}) = \chi_\Omega(\underline{x})\vec{f}(\underline{x}) - \mathcal{T}^{\mathcal{L}}_\Omega(\mathcal{L}_{\alpha,\beta}\vec{f})(\underline{x}), \quad \underline{x} \in \mathbb{R}^3 \setminus \Gamma. \tag{5.2}$$

Proof.
It is easy to verify that $\vec{u}$ satisfies $\mathcal{L}_{\alpha,\beta}\vec{u} = 0$ in $\mathbb{R}^3 \setminus \Gamma$, which follows from Theorem 2. Moreover, the validity of the boundary conditions in (3.6) is straightforwardly implied by Lemma 9. □

Remark 11. *As we have seen already with the case of sufficiently smooth boundaries, the uniqueness of the solution of (3.6) is directly related with the removability of* Γ *for continuous monogenic functions. Although this result is no longer available in general, nevertheless a Dolzhenko theorem proved in [40] is instead more appropriate to deal with the picture of uniqueness in the case of a d-summable boundary* Γ*, see [39, Theorem 4.2]. Due to the deep similarity we will omit the details.*

6. Conclusions

Consider a vector field $\vec{u} \in C^2(\Omega) \cap C^1(\overline{\Omega})$, which is a solution of the Lamé-Navier system $\mathcal{L}_{\alpha,\beta}\vec{u} = 0$ in Ω. We have shown that $\vec{u}$ admits in Ω an integral representation formula in terms of its boundary values and those of their first order partial derivatives. We also provide a particular solution of the inhomogeneous Lamé-Navier system $\mathcal{L}_{\alpha,\beta}\vec{u} = f$ by means of the generalized Teodorescu transform $\mathcal{T}^{\mathcal{L}}_{\Omega}\vec{f}$. The above results are applied to obtain an explicit solution of boundary value problems for such a system in a very wide class of bounded domains in $\mathbb{R}^3$, .

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