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Research article

On nonlinear fuzzy set-valued Θ-contractions with applications

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Abstract: Among various improvements in fuzzy set theory, a progressive development has been in process to investigate fuzzy analogues of fixed point theorems of the classical fixed point results. In this direction, taking the ideas of θ-contractions as well as Feng-Liu's approach into account, some new fuzzy fixed point results for nonlinear fuzzy set-valued θ-contractions in the framework of metric-like spaces are introduced in this paper without using the usual Pompeiu-Hausorff distance function. Our established concepts complement, unify and generalize a few important fuzzy and classical fixed point theorems in the corresponding literature. A handful of these special cases of our notions are pointed and analyzed. Some of the main results herein are further applied to derive their analogues in metric-like spaces endowed with partial ordering and binary relations. Comparisons and nontrivial examples are given to authenticate the hypotheses and significance of the obtained ideas.

Keywords: fixed point; fuzzy set; fuzzy set-valued map; metric-like space; multivalued mapping; θ-contraction
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1. Introduction and preliminaries

From onset of learning, man has always been striving towards coping with the natural world and then building a strong link allying life and its requirements. These struggle is made up of three points, namely, understanding the surrounding, acknowledgement of novelty, and arranging for days ahead. In these toggle, a lot of issues such as linguistic interpretation, characterization of linked situations into proper classes, use of restricted concepts, unreliability in data analysis and so on, disturb the accuracy of results. The above noted impediments known with everyday life can be managed by availing the notions of fuzzy sets(fs) due to their suitability in natural world with regards to classical sets. After the coining of fs by Zadeh [33], various areas of mathematics, social sciences and engineering enjoy enormous revolutions. Currently, the crude ideas of fs have been refined and applied in different directions. Along this development, fixed point(fp) theory researches are carried out in two directions in fuzzy mathematics. One is to study fp of point-to-point and multivalued mappings(mvm) defined on fuzzy metric spaces(ms) (e.g., [16, 26, 30]) and the other is to examine fp of fuzzy set-valued maps(fsv) on ms. The latter direction was initiated by Heilpern [13] who used the idea of fs to define a class of fsv and proved a fp theorem(thrm) for fuzzy contraction mappings(fcm) which is a fuzzy similitude of fp thrms due to Nadler [23] and Banach [7]. Later, a host of examiners have come up with the existence of fp of fsv in the sense of Heilpern; for example, see [1, 6, 19, 20, 21, 29].

On the other hand, studies of novel spaces and their axioms have been an alluring focus among the mathematicians. In this context, the view of metric-like spaces(mls), brought up by Amini-Harandi [3] is presently playing out. A refinement of a fp thrm on such spaces have been examined by Hitzler and Seda [14] in the light of logic programming(lprog) semantics. In some applications of lprog, it is needed to have nonzero self distances. To ensure this requirement, different types of modified ms such as partial metric spaces, quasi metric spaces, mls and concerned topologies gained a lot of importance. For some recent fp results in the setting of mls, see [5, 32] and the references therein.

Throughout this paper, the sets $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$, represent the set of real numbers, nonnegative real numbers and the set of natural numbers, respectively. Let $(\heartsuit, \mu)$ be a ms. Denote by $\mathcal{N}(\heartsuit)$, $C(\heartsuit)$, $CB(\heartsuit)$ and $\mathcal{K}(\heartsuit)$, the family of nonempty subsets of $\heartsuit$, the collection of all nonempty closed subsets of $\heartsuit$, the collection of all nonempty closed and bounded subsets of $\heartsuit$ and the class of all nonempty compact subsets of $\heartsuit$, respectively. For $\clubsuit, \spadesuit \in CB(\heartsuit)$, the mapping $\aleph : CB(\heartsuit) \times CB(\heartsuit) \longrightarrow \mathbb{R}$ defined by

$$\aleph(\clubsuit, \spadesuit) = \max\{\sup_{\jmath\in\spadesuit} \mu(\jmath, \clubsuit), \sup_{\ell\in\clubsuit} \mu(\ell, \spadesuit)\},$$

where $\mu(\jmath, \clubsuit) = \inf_{\ell\in\clubsuit} \mu(\jmath, \ell)$, is termed the Pompeiu-Hausdorff metric(PHm).

An element $u \in \heartsuit$ is termed a fp of a mvm $F : \heartsuit \longrightarrow \mathcal{N}(\heartsuit)$ if $u \in Fu$. A mpn $F : \heartsuit \longrightarrow CB(\heartsuit)$ is called a multivalued contraction(mc) if there exists $\lambda \in (0, 1)$ such that $\aleph(F\jmath, F\ell) \leq \lambda\mu(\jmath, \ell)$. In 1969, Nadler [23] established a multi-valued extension of the Banach contraction(Bc) mpn.

Theorem 1.1. *[23] Let $(\heartsuit, \mu)$ be a complete ms(cms) and $F : \heartsuit \longrightarrow CB(\heartsuit)$ be a mc. Then F has at least one fp in $\heartsuit$.*

Following [23], a number of generalizations of fp theorems of mc have been presented, notably, by Berinde-Berinde [8], Du [9], Mizoguchi and Takahashi [22], Pathak [25], Reich [28], to mention a few. The first generalization of Thrm 1.1 without availing the PHm was established by Feng and Liu (F-Liu) [11]. To recall their results, we give the following notation for mvm $F : \heartsuit \longrightarrow C(\heartsuit)$: Let $b \in (0, 1)$ and $\jmath \in \heartsuit$, then define

$$I_b^\jmath = \{\ell \in F\jmath : b\mu(\jmath, \ell) \leq \mu(\jmath, F\jmath)\}.$$

We also recall that a function(fnx) $g : \heartsuit \longrightarrow \mathbb{R}$ is called lower semi-continuous(lsc), if for any sequence(seq) $\{\jmath_n\}_{n\in\mathbb{N}} \subset \heartsuit$ and $u \in \heartsuit$,

$$\jmath_n \longrightarrow u \implies g(u) \leq \lim_{n\longrightarrow\infty} \inf g(\jmath_n).$$

Theorem 1.2. *[11] Let $(\heartsuit, \mu)$ be a cms and $F : \heartsuit \longrightarrow C(\heartsuit)$ be a mvm. If there exist $b, c \in (0, 1)$ with $b < c$ such that for each $\jmath \in \heartsuit$, there exists $\ell \in I_b^\jmath$:*

$$\mu(\ell, F\ell) \le c\mu(\jmath, \ell),$$

then T has at least one fp in $\heartsuit$ provided that the fnx $\jmath \longmapsto \mu(\jmath, F\jmath)$ is lsc.

Klim and Wardoski [17] extended Thrm 1.2 as:

Theorem 1.3. *[17] Let $(\heartsuit, \mu)$ be a cms and $F : \heartsuit \longrightarrow C(\heartsuit)$ be a mvm. If $b \in (0, 1)$ and a fnx $\varphi : \mathbb{R}_+ \longrightarrow [0, b)$ exists:*

$$\lim_{t \longrightarrow s^+} \sup \varphi(t) < b \text{ for all } s \in \mathbb{R}_+,$$

and for each $\jmath \in \heartsuit$, there exists $\ell \in I_b^\jmath$:

$$\mu(\ell, F\ell) \le \varphi(\mu(\jmath, \ell))\mu(\jmath, \ell),$$

then F has at least one fp in $\heartsuit$ provided that the fnx $\jmath \longmapsto \mu(\jmath, F\jmath)$ is lsc.

Theorem 1.4. *[17] Let $(\heartsuit, \mu)$ be a cms and $F : \heartsuit \longrightarrow \mathcal{K}(\heartsuit)$ be a mvm. If $b \in (0, 1)$ and there is a fnx $\varphi : \mathbb{R}_+ \longrightarrow [0, 1)$:*

$$\lim_{t \longrightarrow s^+} \sup \varphi(t) < 1 \text{ for all } s \in \mathbb{R}_+,$$

and for each $\jmath \in \heartsuit$, there exists $\ell \in I_1^\jmath$ such that

$$\mu(\ell, F\ell) \le \varphi(\mu(\jmath, \ell))\mu(\jmath, \ell),$$

then F has at least one fp in $\heartsuit$ provided that the fnx $\jmath \longmapsto \mu(\jmath, F\jmath)$ is lsc.

In the literature, there are several variants of Thrms 1.2 and 1.3 (see, e.g. [2, 4, 10]).

Alternatively, one of the well-known improvements of Banach contraction(Bc) was introduced by Jleli and Samet [15] under the name θ-contraction. We recall a few of this concept as follows. Let Ω be the set of fnx $\theta : (0, \infty) \longrightarrow (1, \infty)$:

(θ_1) θ is nondecreasing;
(θ_2) for each seq $\{t_n\}_{n\in\mathbb{N}} \subset (0, \infty)$, $\lim_{n\longrightarrow\infty} \theta(t_n) = 1$ if and only if $\lim_{n\longrightarrow\infty} t_n = 0^+$;
(θ_3) $\eta \in (0, 1)$ and $l \in (0, \infty]$: $\lim_{t\longrightarrow 0^+} \frac{\theta(t)-1}{t^\eta} = l$.

Let $(\heartsuit, \mu)$ be a ms and $\theta \in \Omega$. A mpn $g : \heartsuit \longrightarrow \heartsuit$ is called θ-contraction(θ-con) if $\tau \in (0, 1)$: For all $\jmath, \ell \in \heartsuit$ with $\mu(g\jmath, g\ell) > 0$,

$$\theta(\mu(g\jmath, g\ell)) \le [\theta(\mu(\jmath, \ell))]^\tau. \tag{1.1}$$

If we take varying forms of $\theta \in \Omega$, we get some types of contractions. For example, let $\theta(t) = e^{\sqrt{t}}$, then $\theta \in \Omega$ and (1.1) becomes

$$\mu(g\jmath, g\ell) \le \tau^2\mu(\jmath, \ell),$$

for all $\jmath, \ell \in \heartsuit$ with $\mu(g\jmath, g\ell) > 0$. Similarly, taking $\theta(t) = e^{\sqrt{te^t}}$, $\theta \in \Omega$, (1.1) changes to

$$\frac{\mu(g\jmath, g\ell)}{\mu(\jmath, \ell)} e^{\mu(g\jmath, g\ell) - \mu(\jmath, \ell)} \le \tau^2, \tag{1.2}$$

for all $\jmath, \ell \in \heartsuit$ with $\mu(g\jmath, g\ell) > 0$. Clearly, if a mpn g is a Bc, then it satisfies (1.2). However, the reverse of this statement is not true in general (see [15]). Moreover, it is easy to deduce that if g is a θ-con, then g is a contractive mpn, that is, $\mu(g\jmath, g\ell) < \mu(\jmath, \ell)$ for all $\jmath, \ell \in \heartsuit$ with $\jmath \neq \ell$. It follows that every θ-con on a ms is continuous.

Theorem 1.5. *[15, Cor.2.1] Let $(\heartsuit, \mu)$ be a cms and $g : \heartsuit \longrightarrow \heartsuit$ be a single-valued mpn. If g is a θ-con, then g has a unique fp in $\heartsuit$.*

Along the line, the notion of θ-con was moved to mvm by Hancer et al. [12]. Let $(\heartsuit, \mu)$ be a ms, $F : \heartsuit \longrightarrow CB(\heartsuit)$ be a mvm and $\theta \in \Omega$. Then F is called a multivalued(mv) θ-cont if there exists $\tau \in (0, 1)$:

$$\theta(\aleph(F\jmath, F\ell)) \leq [\theta(\mu(\jmath, \ell))]^{\tau} \tag{1.3}$$

for all $\jmath, \ell \in \heartsuit$ with $\aleph(F\jmath, F\ell) > 0$.

Theorem 1.6. *[12] Let $(\heartsuit, \mu)$ be a cms and $F : \heartsuit \longrightarrow \mathcal{K}(\heartsuit)$ be a mv θ-cont. Then F has at least one fp in $\heartsuit$.*

In [12, Ex.1], it has been shown that we cannot replace $\mathcal{K}(\heartsuit)$ with $CB(\heartsuit)$ in Thrm 1.6. However, we can consider $CB(\heartsuit)$ instead of $\mathcal{K}(\heartsuit)$ by appending on θ:
(θ_4) $\theta(\inf \clubsuit) = \inf \theta(\clubsuit)$ for all $\clubsuit \subset (0, \infty)$ with $\inf \clubsuit > 0$.
Observe that if θ satisfies (θ_1), then it satisfies (θ_4) if and only if it is r-cont. Let

$$\mho = \{\theta | \theta : (0, \infty) \longrightarrow (1, \infty) \text{ satisfying } (\theta_1) - (\theta_4)\}.$$

Theorem 1.7. *[12] Let $(\heartsuit, \mu)$ be a cms and $F : \heartsuit \longrightarrow CB(\heartsuit)$ be a mv θ-cont. If $\theta \in \mho$, then F has at least one fp in $\heartsuit$.*

We know that a non-fs $\clubsuit$ of $\heartsuit$ is completely determined by its characteristic fnx $\chi_{\clubsuit}$, defined by $\chi_{\clubsuit} : \clubsuit \longrightarrow \{0, 1\}$:

$$\chi_{\clubsuit}(\jmath) = \begin{cases} 1, & \text{if } \jmath \in \clubsuit \\ 0, & \text{if } \jmath \notin \clubsuit. \end{cases}$$

The value $\chi_{\clubsuit}(\jmath)$ points out whether an element belongs to $\clubsuit$ or not. Clearly, this correspondence between a set and its characteristic fnx is one-to-one. This view is employed to define fs by allowing an element $\jmath \in \clubsuit$ to have any possible value in the $[0, 1]$. Thus, a fs in $\heartsuit$ is a fnx with domain $\heartsuit$ and values in $[0, 1] = I$. The collection of all fs in $\heartsuit$ is denoted by $I^{\heartsuit}$. If $\clubsuit$ is a fs in $\heartsuit$, then the fnx value $\clubsuit(\jmath)$ is named the grade of membership of $\jmath$ in $\clubsuit$. The $\flat$-level set of a fs $\clubsuit$ is designed by $[\clubsuit]_{\flat}$ and is defined as follows:

$$[\clubsuit]_{\flat} = \begin{cases} \overline{\{\jmath \in \heartsuit : \clubsuit(\jmath) > 0\}}, & \text{if } \flat = 0 \\ \{\jmath \in \heartsuit : \clubsuit(\jmath) \geq \flat\}, & \text{if } \flat \in (0, 1]. \end{cases}$$

A fs $\clubsuit$ in $\heartsuit$ is said to be convex if for all $\jmath, \ell \in \heartsuit$ and $t \in (0, 1)$, $\clubsuit(t\jmath + (1 - t)\ell) \geq \min\{\clubsuit(\jmath), \clubsuit(\ell)\}$. A fs $\clubsuit$ in a ms $\heartsuit$ is said to be an approximate quantity(aq) if and only if $[\clubsuit]_{\flat}$ is compact and convex in $\heartsuit$ and $\sup_{\jmath \in \heartsuit} \clubsuit(\jmath) = 1$ (see [33]). We design the collection of all aq in $\heartsuit$ by $W(\heartsuit)$. If $\flat \in [0, 1]$: $[\clubsuit]_{\flat}, [\spadesuit]_{\flat} \in \mathcal{K}(\heartsuit)$, then define

$$D_\flat(\clubsuit,\spadesuit) = \aleph([\clubsuit]_\flat, [\spadesuit]_\flat).$$

$$\mu_\infty(\clubsuit,\spadesuit) = \sup_\flat D_\flat(\clubsuit,\spadesuit).$$

Note that μ_∞ is a metric on $\mathcal{K}(\heartsuit)$ (induced by the PHm $\aleph$) and the completeness of $(\heartsuit,\mu)$ implies the completeness of the corresponding ms $(I_{\mathcal{K}(\heartsuit)},\mu_\infty)$ (see [13]). Furthermore, $(\heartsuit,\mu) \longmapsto (\mathcal{K}(\heartsuit),\aleph) \longmapsto (I_{\mathcal{K}(\heartsuit)},\mu_\infty)$, are isometric embeddings via the relations $\jmath \longrightarrow \{\jmath\}$ (crisp set) and $M \longrightarrow \chi_M$, respectively; where

$$I_{\mathcal{K}(\heartsuit)} = \{\clubsuit \in I^\heartsuit : [\clubsuit]_\flat \in \mathcal{K}(\heartsuit), \text{ for each } \flat \in [0,1]\}.$$

Definition 1.8. [13] Let $\heartsuit$ be a nonempty set. A mpn $¥ : \heartsuit \longrightarrow I^\heartsuit$ is called a fsv. A fsv $¥$ is a fuzzy subset of $\heartsuit \times \heartsuit$. The fnx value $¥(\jmath)(\ell)$ is named the grade of membership of ℓ in the fs $¥(\jmath)$. A point $u \in \heartsuit$ is termed a fuzzy fp of $¥$ if $\flat \in (0,1]$: $u \in [¥u]_\flat$.

Definition 1.9. [13] Let $(\heartsuit,\mu)$ be a ms. A mpn $¥ : \heartsuit \longrightarrow W(\heartsuit)$ is called fuzzy λ-contraction if there exists $\lambda \in (0,1)$ such that for all $\jmath,\ell \in \heartsuit$,

$$\mu_\infty(¥(\jmath),¥(\ell)) \le \lambda\mu(\jmath,\ell).$$

Theorem 1.10. *[13, Th.3.1] Every fuzzy λ-contraction on a cms has at least one fuzzy fp.*

Definition 1.11. [3] Let $\heartsuit$ be a nonempty set and $\sigma : \heartsuit \times \heartsuit \longrightarrow \mathbb{R}_+$ be a mpn:

(i) If $\sigma(\jmath,\ell) = 0$, then $\jmath = \ell$;
(ii) $\sigma(\jmath,\ell) = \sigma(\ell,\jmath)$;
(iii) $\sigma(\jmath,z) \le \sigma(\jmath,\ell) + \sigma(\ell,z)$,

for all $\jmath,\ell,z \in \heartsuit$. Then, σ is named a metric-like(ml) on $\heartsuit$ and $(\heartsuit,\sigma)$ is called a mls.

Definition 1.12. [3] Let $(\heartsuit,\sigma)$ be a mls. Then, a seq $\{\jmath_n\}_{n\in\mathbb{N}}$ in $\heartsuit$ is said to be:

(i) σ-convergent to a limit u in $\heartsuit$, if

$$\lim_{n\longrightarrow\infty} \sigma(\jmath_n,u) = \sigma(u,u).$$

(ii) σ-Cauchy(C), if $\lim_{n,m\longrightarrow\infty} \sigma(\jmath_n,\jmath_m)$ exists and is finite.
(iv) σ-complete if for every σ-C seq $\{\jmath_n\}_{n\in\mathbb{N}}$, there exists $u \in \heartsuit$:

$$\lim_{n,m\longrightarrow\infty} \sigma(\jmath_n,\jmath_m) = \sigma(u,u) = \lim_{n\longrightarrow\infty} \sigma(\jmath_n,u).$$

A subset A of a mls $(\heartsuit,\sigma)$ is said to be bounded if there is a point $p \in \heartsuit$ and a positive constant ϖ such that $\sigma(a,p) \le \varpi$ for all $a \in A$.

For two mls $(\heartsuit,\sigma)$ and (Y,μ), a fnx $g : \heartsuit \longrightarrow Y$ is continuous if

$$\lim_{n\longrightarrow\infty} \sigma(\jmath_n,u) = \lim_{n\longrightarrow\infty} \mu(g\jmath_n,gu).$$

Example 1.13. [3, 5]

(i) Let $\heartsuit = \mathbb{R}$. The mpn $\sigma : \heartsuit \times \heartsuit \longrightarrow \mathbb{R}_+$ defined as

$$\sigma(\jmath, \ell) = |\jmath - \ell| + |\jmath| + |\ell|,$$

for all $\jmath, \ell \in \heartsuit$, is a ml. The mls $(\mathbb{R}, \sigma)$ is σcomplete.

(ii) Let $\heartsuit = C([a, b])$ be the space of all continuous functions on the interval $[a, b]$. The mpn $\sigma : \heartsuit \times \heartsuit \longrightarrow \mathbb{R}_+$ defined as

$$\sigma(\jmath, \ell) = |\jmath| + |\ell| + k, \ k \geq 0,$$

for all $\jmath, \ell \in \heartsuit$, is a ml. The mls $(C([a, b]), \sigma)$ is σ-complete.

(iii) Let $\heartsuit = [0, 1]$. The mpn $\sigma : \heartsuit \times \heartsuit \longrightarrow \mathbb{R}_+$ defined by $\sigma(\jmath, \ell) = \jmath + \ell - \jmath\ell$ is a ml on $\heartsuit$.

(iv) Let $\heartsuit = \mathbb{R}$ and $\sigma(\jmath, \ell) = \max\{|\jmath|, |\ell|\}$ for all $\jmath, \ell \in \heartsuit$. Then σ is a ml on $\heartsuit$.

Among several improvements in fs theory, a huge effort has been in process to examine fuzzy analogues of fp thrm of the non-fuzzy fp results. In this pursuit, taking the idea of Jleli and Samet [15] as well as Feng and Liu [11] approach into account in this paper, some new fuzzy fp results for nonlinear fsv θ-cont in the framework of mls are launched. Our notions complement and generalize a few important fuzzy and classical fp thrms including the results of Altun and Minak [2], F-Liu [11], Klim and Wardowski [17], and others in the corresponding literature. Some of our main results are further applied to deduce their analogues in mls equipped with partial ordering and binary relations(brel).

2. Main results

Let $(\heartsuit, \sigma)$ be a mls and $¥ : \heartsuit \longrightarrow I^{\heartsuit}$ be a fsv, where $I^{\heartsuit}$ is the family of fs in $\heartsuit$. For each $\jmath \in \heartsuit$ and $s \in (0, 1]$, define the set $\theta_s^{\jmath} \subseteq \heartsuit$ and subcollections $I_{\mathcal{K}(\heartsuit)}$, $I_{C(\heartsuit)}$ of $I^{\heartsuit}$ as follows:

$$\theta_s^{\jmath} = \{\ell \in [¥\jmath]_\flat : [\theta(\sigma(\jmath, \ell))]^s \leq \theta(\sigma(\jmath, [¥\jmath]_\flat)), \text{ for each } \flat \in (0, 1]\}.$$

$$I_{\mathcal{K}(\heartsuit)} = \{A \in I^{\heartsuit} : [A]_\flat \in \mathcal{K}(\heartsuit), \text{ for each } \flat \in (0, 1]\}.$$

$$I_{C(\heartsuit)} = \{A \in I^{\heartsuit} : [A]_\flat \in C(\heartsuit), \text{ for each } \flat \in (0, 1]\}.$$

Remark 1. For the set $\theta_s^{\jmath}$, we note that

Case 1. If $¥ : \heartsuit \longrightarrow I_{\mathcal{K}(\heartsuit)}$, then we have $\theta_s^{\jmath} \neq \emptyset$ for all $s \in (0, 1]$ and $\jmath \in \heartsuit$ with $\sigma(\jmath, [¥\jmath]_\flat) > 0$ for some $\flat \in (0, 1]$. In fact, since $[¥\jmath]_\flat \in \mathcal{K}(\heartsuit)$, we have $\ell \in [¥\jmath]_\flat$ with $\sigma(\jmath, \ell) = \sigma(\jmath, [¥\jmath]_\flat)$ for each $\jmath \in \heartsuit$. Hence, $\theta(\sigma(\jmath, \ell)) = \theta(\sigma(\jmath, [¥\jmath]_\flat))$. Therefore, $\ell \in \theta_s^{\jmath}$ for all $s \in (0, 1]$.

Case 2. If $¥ : \heartsuit \longrightarrow I_{C(\heartsuit)}$, then $\theta_s^{\jmath}$ may be empty for some $\jmath \in \heartsuit$ and $s \in (0, 1]$. To see this, let $\heartsuit = [0, 5]$ and

$$\sigma(\jmath, \ell) = \begin{cases} 3, & \text{if } \jmath, \ell \in [0, 2]. \\ \jmath^2 + \ell^2 + 1, & \text{if one of } \jmath, \ell \notin [2, 5]. \end{cases}$$

Then $(\heartsuit, \sigma)$ is a mls. Notice that σ is not a metric on $\heartsuit$, since $\sigma(0, 0) = 3 > 0$. Now, let $\theta(t) = e^{\sqrt{\frac{t}{3}}}$ for $0 < t \leq 3$ and $\theta(t) = 25t$ for $t > 3$. Obviously, $\theta \in \Omega$. Take $\jmath = 0$ and define a fsv ¥ as follows:

$$¥(0)(t) = \begin{cases} \frac{\flat}{15}, & \text{if } 0 \leq t < 2. \\ \frac{\flat}{6}, & \text{if } 2 \leq t \leq 3. \\ 0, & \text{if } 3 < t \leq 5. \end{cases}$$

Then, there exists $\frac{\flat}{7} \in (0,1]$: $[¥\jmath]_{\frac{\flat}{7}} = [2,3]$. Clearly, $0 \notin [¥0]_{\frac{\flat}{7}}$. Indeed,

$$\begin{aligned}\sigma(0,[¥0]_{\frac{\flat}{7}}) &= \inf\{\sigma(0,\ell) : \ell \in [2,3]\}\\ &= \sigma(0,2) = 3 > 0.\end{aligned}$$

Thus, we see that

$$\begin{aligned}\theta^0_{\frac{1}{2}} &= \{\ell \in [¥0]_{\frac{\flat}{7}} : [\theta(\sigma(0,\ell))]^{\frac{1}{2}} \leq \theta(\sigma(0,[¥0]_{\frac{\flat}{7}}))\}\\ &= \{\ell \in [2,3] : [\theta(\ell^2+1)]^{\frac{1}{2}} \leq \theta(3)\}\\ &= \{\ell \in [2,3] : 5\sqrt{\ell^2+1} \leq e\}\\ &= \emptyset.\end{aligned}$$

Case 3. If $¥ : \heartsuit \longrightarrow I_{C(\heartsuit)}$ and $\theta \in \mho$, then $\theta^{\jmath}_s \neq \emptyset$ for all $s \in (0,1)$ and $\jmath \in \heartsuit$ with $\sigma(\jmath,[¥\jmath]_\flat) > 0$. To see this, note that since θ is right continuous(r-cont), there exists $\varsigma > 1$:

$$\theta(\varsigma(\sigma(\jmath,[¥\jmath]_\flat))) \leq [\theta(\sigma(\jmath,[¥\jmath]_\flat))]^{\frac{1}{s}}.$$

Since > 1, we can find $\ell \in [¥\jmath]_\flat$ such that $\sigma(\jmath,\ell) \leq \varsigma\sigma(\jmath,[¥\jmath]_\flat)$. Hence, from (θ_1), we get

$$\begin{aligned}\theta(\sigma(\jmath,\ell)) &\leq \theta(\varsigma\sigma(\jmath,[¥\jmath]_\flat))\\ &\leq [\theta(\sigma(\jmath,[¥\jmath]_\flat))]^{\frac{1}{s}},\end{aligned}$$

from which it comes up that $[\theta(\sigma(\jmath,\ell))]^s \leq \theta(\sigma(\jmath,[¥\jmath]_\flat))$; that is, $\ell \in \theta^{\jmath}_s$.

Definition 2.1. Let $(\heartsuit,\sigma)$ be a mls, $¥ : \heartsuit \longrightarrow I_{C(\heartsuit)}$ be a fsv and $\theta \in \mho$. Then ¥ is called a nonlinear fsv θ-cont of type(A), if there exist $s \in (0,1)$ and a fnx $\xi : \mathbb{R}_+ \longrightarrow [0,s)$:

$$\lim_{t\longrightarrow\varsigma^+}\sup\xi(t) < s \text{ for all } \varsigma \in \mathbb{R}_+ \tag{2.1}$$

and for any $\jmath \in \heartsuit$ with $\sigma(\jmath,[¥\jmath]_\flat) > 0$, there exists $\ell \in \theta^{\jmath}_s$:

$$\theta(\sigma(\ell,[¥\ell]_\flat)) \leq [\theta(\sigma(\jmath,\ell))]^{\xi(\sigma(\jmath,\ell))}. \tag{2.2}$$

Definition 2.2. Let $(\heartsuit,\sigma)$ be a mls, $¥ : \heartsuit \longrightarrow I_{\mathcal{K}(\heartsuit)}$ be a fsv and $\theta \in \Omega$. Then ¥ is called a nonlinear fsv θ-cont of type(B), if there exists a fnx $\xi : \mathbb{R}_+ \longrightarrow [0,1)$:

$$\lim_{t\longrightarrow\varsigma^+}\sup\xi(t) < 1 \text{ for all } \varsigma \in \mathbb{R}_+, \tag{2.3}$$

and for any $\jmath \in \heartsuit$ with $\sigma(\jmath,[¥\jmath]_\flat) > 0$, there exists $\ell \in \theta^{\jmath}_1$ such that

$$\theta(\sigma(\ell,[¥\ell]_\flat)) \leq [\theta(\sigma(\jmath,\ell))]^{\xi(\sigma(\jmath,\ell))}. \tag{2.4}$$

Taking the above preliminaries into account, we present the following thrms.

Theorem 2.3. *Let $(\heartsuit,\sigma)$ be a complete metric-like space(cmls) and $¥ : \heartsuit \longrightarrow I_{C(\heartsuit)}$ be a fsv. If ¥ is a nonlinear fsv θ-cont of type(A), then ¥ has at least one fuzzy fp in $\heartsuit$ provided that the fnx $\jmath \longmapsto \sigma(\jmath,[¥\jmath]_\flat)$ is lsc.*

Proof. Assume that ¥ has no fuzzy fp in ♡. Then, for all $\jmath \in \heartsuit$ and $\flat \in (0,1]$, $\sigma(\jmath, [¥\jmath]_\flat) > 0$. Given that $[¥\jmath]_\flat \in C(\heartsuit)$ for each $\jmath \in \heartsuit$ and $\theta \in \mho$, by Case 3 of Remark 1, $\theta_s^\jmath$ is nonempty for all $s \in (0,1)$. Now, for any initial point $\jmath_0$, we get $\jmath_1 \in \theta_s^{\jmath_0}$:

$$\theta(\sigma(\jmath_1, [¥\jmath_1]_\flat)) \leq [\theta(\sigma(\jmath_0, \jmath_1))]^{\xi(\sigma(\jmath_0,\jmath_1))},$$

and for $\jmath_1 \in \heartsuit$, we get $\jmath_2 \in \theta_s^{\jmath_1}$:

$$\theta(\sigma(\jmath_2, [¥\jmath_2]_\flat)) \leq [\theta(\sigma(\jmath_1, \jmath_2))]^{\xi(\sigma(\jmath_1,\jmath_2))}.$$

On this lane, we get a seq $\{\jmath_n\}_{n\in\mathbb{N}}$ in $(\heartsuit, \sigma)$ with $\jmath_{n+1} \in \theta_s^{\jmath_n}$:

$$\theta(\sigma(\jmath_{n+1}, [¥\jmath_{n+1}]_\flat)) \leq [\theta(\sigma(\jmath_n, \jmath_{n+1}))]^{\xi(\sigma(\jmath_n,\jmath_{n+1}))}, n = 0, 1, 2, \cdots \tag{2.5}$$

Next, we will demonstrate that $\{\jmath_n\}_{n\in\mathbb{N}}$ is a C-seq in ♡. Since $\jmath_{n+1} \in \theta_s^{\jmath_n}$, we get

$$[\theta(\sigma(\jmath_n, \jmath_{n+1}))]^s \leq \theta(\sigma(\jmath_n, [¥\jmath_n]_\flat)). \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$\theta(\sigma(\jmath_{n+1}, [¥\jmath_{n+1}]_\flat)) \leq [\theta(\sigma(\jmath_n, [¥\jmath_n]_\flat))]^{\frac{\xi(\sigma(\jmath_n,\jmath_{n+1}))}{s}}, \tag{2.7}$$

and

$$\theta(\sigma(\jmath_{n+1}, \jmath_{n+2})) \leq [\theta(\sigma(\jmath_n, \jmath_{n+1}))]^{\frac{\xi(\sigma(\jmath_n,\jmath_{n+1}))}{s}}. \tag{2.8}$$

Using (2.7), (2.8) and (θ_1), we see that $\{\sigma(\jmath_n, [¥\jmath_n]_\flat)\}_{n\in\mathbb{N}}$ and $\{\sigma(\jmath_n, \jmath_{n+1})\}_{n\in\mathbb{N}}$ are nonincreasing seq and thus convergent. From (2.1), we get $\gamma \in [0, s)$: $\lim_{n\longrightarrow\infty} \xi(\sigma(\jmath_n, \jmath_{n+1})) = \gamma$. Hence, there exists $b \in (\gamma, s)$ and $n_0 \in \mathbb{N}$: $\xi(\sigma(\jmath_n, \jmath_{n+1})) < b$ for all $n \geq n_0$. Therefore, using (2.8), we find that for all $n \geq n_0$,

$$\begin{aligned}
1 <\ & \theta(\sigma(\jmath_n, \jmath_{n+1})) \\
\leq\ & [\theta(\sigma(\jmath_{n-1}, \jmath_n))]^{\frac{\xi(\sigma(\jmath_{n-1},\jmath_n))}{s}} \\
\leq\ & [\theta(\sigma(\jmath_{n-2}, \jmath_{n-1}))]^{\frac{\xi(\sigma(\jmath_{n-2},\jmath_{n-1}))}{s}\frac{\xi(\sigma(\jmath_{n-1},\jmath_n))}{s}} \\
\vdots\ & \\
\leq\ & [\theta(\sigma(\jmath_0, \jmath_1))]^{\frac{\xi(\sigma(\jmath_0,\jmath_1))}{s}\cdots\frac{\xi(\sigma(\jmath_{n-2},\jmath_{n-1}))}{s}\frac{\xi(\sigma(\jmath_{n-1},\jmath_n))}{s}} \\
\leq\ & [\theta(\sigma(\jmath_0, \jmath_1))]^{\frac{\xi(\sigma(\jmath_0,\jmath_1))}{s}\cdots\frac{\xi(\sigma(\jmath_{n_0-1},\jmath_{n_0}))}{s}\frac{\xi(\sigma(\jmath_{n_0},\jmath_{n_0+1}))}{s}\cdots\frac{\xi(\sigma(\jmath_{n-2},\jmath_{n-1}))}{s}\frac{\xi(\sigma(\jmath_{n-1},\jmath_n))}{s}} \\
\leq\ & [\theta(\sigma(\jmath_0, \jmath_1))]^{\frac{\xi(\sigma(\jmath_{n_0},\jmath_{n_0+1}))}{s}\cdots\frac{\xi(\sigma(\jmath_{n-2},\jmath_{n-1}))}{s}\frac{\xi(\sigma(\jmath_{n-1},\jmath_n))}{s}} \\
\leq\ & [\theta(\sigma(\jmath_0, \jmath_1))]^{\frac{b^{(n-n_0)}}{s^{(n-n_0)}}}.
\end{aligned}$$

Hence, for all $n \geq n_0$,

$$1 < \theta(\sigma(\jmath_n, \jmath_{n+1})) \leq [\theta(\sigma(\jmath_0, \jmath_1))]^{\left(\frac{b}{s}\right)^{(n-n_0)}}. \tag{2.9}$$

Since $\lim_{n\longrightarrow\infty} \left(\frac{b}{s}\right)^{(n-n_0)} = 0$, then, as $n \longrightarrow \infty$ in (2.9), we have

$$\lim_{n\longrightarrow\infty} \theta(\sigma(\jmath_n, \jmath_{n+1})) = 1. \tag{2.10}$$

Therefore, from (θ_2), $\lim_{n\longrightarrow\infty}\sigma(\jmath_n,\jmath_{n+1})=0^+$, and from (θ_3), it follows that there exists $\eta\in(0,1)$ and $l\in(0,\infty]$:

$$\lim_{n\longrightarrow\infty}\frac{\theta(\sigma(\jmath_n,\jmath_{n+1}))}{[\sigma(\jmath_n,\jmath_{n+1})]^\eta}=l. \tag{2.11}$$

From (2.11), we consider:
Case 1. $l=\infty$. For this, let $\delta=\frac{l}{2}>0$. Whence, we get $n_1\in\mathbb{N}$ with $n\geq n_1$,

$$\left|\frac{\theta(\sigma(\jmath_n,\jmath_{n+1}))-1}{[\sigma(\jmath_n,\jmath_{n+1})]^\eta}-l\right|\leq\delta,$$

from which we have $\frac{\theta(\sigma(\jmath_n,\jmath_{n+1}))-1}{[\sigma(\jmath_n,\jmath_{n+1})]^\eta}-l\geq l-\delta=\delta$. Then, for all $n\geq n_1$ and $\rho=\frac{1}{\delta}$,

$$n[\sigma(\jmath_n,\jmath_{n+1})]^\eta\leq\rho n[\theta(\sigma(\jmath_n,\jmath_{n+1}))]-1.$$

Case 2. $l=\infty$. Let $\delta>0$ be an arbitrary positive number. In this case, there exists $n_1\in\mathbb{N}$ such that for all $n\geq n_1$, $\frac{\theta(\sigma(\jmath_n,\jmath_{n+1}))}{[\sigma(\jmath_n,\jmath_{n+1})]^\eta}\geq\delta$. That is, for all $n\geq n_1$,

$$n[\sigma(\jmath_n,\jmath_{n+1})]^\eta\leq\rho n[\theta(\sigma(\jmath_n,\jmath_{n+1}))-1].$$

Hence, from Cases 1 and 2, there exists $\rho>0$ and $n_1\in\mathbb{N}$ such that for all $n\geq n_1$,

$$n[\sigma(\jmath_n,\jmath_{n+1})]^\eta\leq\rho n[\theta(\sigma(\jmath_n,\jmath_{n+1}))-1].$$

By (2.9), we have, for all $n\geq n_2=\max\{n_0,n_1\}$,

$$n[\sigma(\jmath_n,\jmath_{n+1})]^\eta\leq\rho n[[\theta(\sigma(\jmath_0,\jmath_1))]^{\left(\frac{b}{s}\right)^{(n-n_0)}}-1]. \tag{2.12}$$

As $n\longrightarrow\infty$ in (2.12), we have $\lim_{n\longrightarrow\infty}n[\sigma(\jmath_n,\jmath_{n+1})]^\eta=0$. Whence, $n_3\in\mathbb{N}$: $n[\sigma(\jmath_n,\jmath_{n+1})]^\eta\leq 1$ for all $n\geq n_3$, which implies that

$$\sigma(\jmath_n,\jmath_{n+1})\leq\frac{1}{n^{\frac{1}{\eta}}}. \tag{2.13}$$

Now, let $m,n\in\mathbb{N}$ with $m>n\geq n_3$. Then, it comes from (2.13) that

$$\begin{aligned}\sigma(\jmath_n,\jmath_m)\leq\ &\sigma(\jmath_n,\jmath_{n+1})+\sigma(\jmath_{n+1},\jmath_{n+2})+\cdots+\sigma(\jmath_{m-1},\jmath_m)\\ =\ &\sum_{i=n}^{m-1}\sigma(\jmath_i,\jmath_{i+1})\leq\sum_{i=n}^{\infty}\sigma(\jmath_i,\jmath_{i+1})\\ \leq\ &\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{\eta}}}.\end{aligned} \tag{2.14}$$

Obviously, the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{\eta}}}$ is c-gent by C-root test. Thus, taking limit as $n\longrightarrow\infty$ in (2.14), gives $\sigma(\jmath_n,\jmath_m)\longrightarrow 0$. This shows that $\{\jmath_n\}_{n\in\mathbb{N}}$ is a C-seq in $(\heartsuit,\sigma)$. The completeness of this space produces $u\in\heartsuit$ such that $\jmath_n\longrightarrow u$ as $n\longrightarrow\infty$. To see that u is a fuzzy fp of ¥, assume that $u\notin[¥u]_\flat$ for all $\flat\in(0,1]$ and $\sigma(u,[¥u]_\flat)>0$. Since $\sigma(\jmath_n,[¥\jmath_n]_\flat)\longrightarrow 0$ as $n\longrightarrow\infty$ and the fnx $\jmath\longmapsto\sigma(\jmath,[¥\jmath]_\flat)$ is lsc, we obtain

$$0\leq\sigma(u,[¥u]_\flat)\leq\lim_{n\longrightarrow\infty}\inf\sigma(\jmath_n,[¥\jmath_n]_\flat)=0,$$

a contradiction. Consequently, there exists $u\in(0,1]$ such that $u\in[¥u]_\flat$. □

Remark 2. If we consider $\mathcal{K}(\heartsuit)$ instead of $CB(\heartsuit)$ in Thrm 2.3, we can take off the assumption (θ_4) on θ. Moreover, by considering Case 1 of Remark 1, we can let $s = 1$ and easily obtain the next result.

Theorem 2.4. *Let $(\heartsuit, \sigma)$ be a cmls and ¥ : $\heartsuit \longrightarrow I_{\mathcal{K}(\heartsuit)}$ be a fsv. If ¥ is a nonlinear fsv θ-cont of type(B), then ¥ has at least one fuzzy fp in $\heartsuit$ provided that the fnx $\jmath \longmapsto \sigma(\jmath, [¥\jmath]_\flat)$ is lsc.*

Proof. Assume that ¥ has no fuzzy fp in $\heartsuit$. Then, for all $\jmath \in \heartsuit$ and $\flat \in (0, 1]$, $\sigma(\jmath, [¥\jmath]_\flat) > 0$. Since $[¥\jmath]_\flat \in \mathcal{K}(\heartsuit)$ for every $\jmath \in \heartsuit$, then by Case 1 of Remark 1, the set $\theta_s^\jmath$ is nonempty. Thus, there exists $\ell \in \theta_s^\jmath$ such that $\sigma(\jmath, \ell) = \sigma(\jmath, [¥\jmath]_\flat)$. Let $\jmath_0 \in \heartsuit$ be an initial point. Then, from (2.4) and following the proof of Thrm 2.3, we have that there exists a C-seq $\{\jmath_n\}_{n\in\mathbb{N}}$ in $\heartsuit$ with $\jmath_{n+1} \in [¥\jmath_n]_\flat$, $\jmath_n \neq \jmath_{n+1}$ such that $\sigma(\jmath_n, \jmath_{n+1}) = \sigma(\jmath_n, [¥\jmath_n]_\flat)$, $\theta(\sigma(\jmath_{n+1}, [¥\jmath_{n+1}]_\flat)) \leq [\theta(\sigma(\jmath_n, \jmath_{n+1}))]^{\xi(\sigma(\jmath_n,\jmath_{n+1}))}$ and $\jmath_n \longrightarrow u$ as $n \longrightarrow \infty$. Since $\jmath \longmapsto \sigma(\jmath, [¥\jmath]_\flat)$ is lsc, we obtain

$$0 \leq \sigma(u, [¥u]_\flat) \leq \lim_{n\longrightarrow\infty} \inf \sigma(\jmath_n, [¥\jmath_n]_\flat) = 0,$$

a contradiction. Therefore, ¥ has at least one fuzzy fp in $\heartsuit$. □

Example 2.5. Let $\heartsuit = [0, \infty)$ and $\sigma(\jmath, \ell) = |\jmath| + |\ell|$ for all $\jmath, \ell \in \heartsuit$. Then $(\heartsuit, \sigma)$ is a cmls. Observe that σ is not a metric on $\heartsuit$, since $\sigma(1, 1) = 2 > 0$. Now, define a fsv ¥ : $\heartsuit \longrightarrow I_{C(\heartsuit)}$ as follows:
For $\jmath \in [0, 1)$,

$$¥(\jmath)(t) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq t \leq \frac{\jmath}{9}, \\ 0, & \text{otherwise}, \end{cases}$$

For $\jmath \in [1, \infty)$,

$$¥(\jmath)(t) = \begin{cases} \frac{1}{14}, & \text{if } 0 \leq t < \frac{\jmath}{6}, \\ \frac{1}{3}, & \text{if } \frac{\jmath}{6} \leq t \leq \frac{\jmath}{2}, \\ \frac{1}{10}, & \text{if } \frac{\jmath}{2} \leq t < \infty. \end{cases}$$

Taking $\flat = \frac{1}{6}$, we have

$$[¥\jmath]_\flat = \begin{cases} \left[0, \frac{\jmath}{9}\right], & \text{if } \jmath \in [0, 1), \\ \left[\frac{\jmath}{6}, \frac{\jmath}{2}\right], & \text{if } \jmath \in [1, \infty). \end{cases}$$

Since

$$\sigma(\jmath, [¥\jmath]_\flat) = \begin{cases} 0, & \text{if } \jmath \in [0, 1), \\ \frac{3\jmath}{2}, & \text{if } \jmath \in [1, \infty), \end{cases}$$

the fnx $\jmath \longmapsto \sigma(\jmath, [¥\jmath]_\flat)$ is lsc. Next, we will show that the contraction condition (2.2) holds. Let $\theta(t) = e^{\sqrt{te^t}}$, $s \in \left(\frac{1}{e}, 1\right)$ and define $\xi : \mathbb{R}_+ \longrightarrow [0, s)$ as $\xi(t) = \frac{1}{e}$, then (2.2) is converted to

$$\frac{\sigma(\ell, [¥\ell]_\flat)}{\sigma(\jmath, \ell)} e^{\sigma(\ell,[¥\ell]_\flat)-\sigma(\jmath,\ell)} \leq [\xi(\sigma(\jmath, \ell))]^2. \tag{2.15}$$

So, we will check alternatively that ¥ satisfies (2.15). Note that if $\sigma(\jmath, [¥\jmath]_\flat) > 0$, then $\jmath \neq 0$. Thus, for $\jmath \in (0, \infty)$, we have $\ell = \frac{\jmath}{9} \in \theta_s^\jmath$ for all $s \in \left(\frac{1}{e}, 1\right)$ and

$$
\begin{aligned}
\frac{\sigma(\ell,[\text{¥}\ell]_\flat)}{\sigma(\jmath,\ell)}e^{\sigma(\ell,[\text{¥}\ell]_\flat)-\sigma(\jmath,\ell)} &= \frac{\frac{10\jmath}{81}e^{-\frac{80\jmath}{81}}}{\frac{10\jmath}{9}} \\
&\leq \frac{1}{9}\leq\frac{1}{e^2} \\
&= \left[\xi\left(\frac{10\jmath}{9}\right)\right]^2=[\xi(\sigma(\jmath,\ell))]^2.
\end{aligned}
$$

Hence, all the hypotheses of Thrms 2.3 and 2.4 are satisfied. Thus, ¥ has at least one fuzzy fp in ♡.

3. Consequences

In this section, we deduce a few inferences of Thrms 2.3 and 2.4.

Corollary 1. *Let* $(\heartsuit,\sigma)$ *be a cmls,* $\text{¥}:\heartsuit\longrightarrow I_{C(\heartsuit)}$ *be a fsv and* $\theta\in\mho$*. If there exists* $\xi,s\in(0,1)$ *with* $\xi<s$ *and* $\ell\in\theta_s^\jmath$*:*

$$\theta(\sigma(\ell,[\text{¥}\ell]_\flat))\leq[\theta(\sigma(\jmath,[\text{¥}\jmath]_\flat))]^\xi,$$

for each $\jmath\in\heartsuit$ *with* $\sigma(\jmath,[\text{¥}\jmath]_\flat)>0$*, then ¥ has at least one fuzzy fp in ♡ provided that the fnx* $\jmath\longmapsto\sigma(\jmath,[\text{¥}\jmath]_\flat)$ *is lsc.*

Corollary 2. *Let* $(\heartsuit,\sigma)$ *be a cmls,* $\text{¥}:\heartsuit\longrightarrow I_{\mathcal{K}(\heartsuit)}$ *be a fsv and* $\theta\in\Omega$*. If* $\xi\in(0,1)$ *and* $\ell\in\theta_s^\jmath$ *with*

$$\theta(\sigma(\ell,[\text{¥}\ell]_\flat))\leq[\theta(\sigma(\jmath,[\text{¥}\jmath]_\flat))]^\xi,$$

for each $\jmath\in\heartsuit$ *with* $\sigma(\jmath,[\text{¥}\jmath]_\flat)>0$*, then ¥ has at least one fuzzy fp in ♡ provided that the fnx* $\jmath\longmapsto\sigma(\jmath,[\text{¥}\jmath]_\flat)$ *is lsc.*

Since $\theta_s^\jmath\subset[\text{¥}\jmath]_\flat$, we can infer more consequences; for instance, from Corollary 1, we have the next result.

Corollary 3. *Let* $(\heartsuit,\sigma)$ *be a cmls,* $\text{¥}:\heartsuit\longrightarrow I_{C(\heartsuit)}$ *be a fsv. If* $\xi\in(0,1)$ *and* $\ell\in[\text{¥}\jmath]_\flat$ *with*

$$\theta(\sigma(\ell,[\text{¥}\ell]_\flat))\leq[\theta(\sigma(\jmath,[\text{¥}\jmath]_\flat))]^\xi,$$

for each $\jmath\in\heartsuit$ *with* $\sigma(\jmath,[\text{¥}\jmath]_\flat)>0$*, then ¥ has at least one fuzzy fp in ♡ provided that the fnx* $\jmath\longmapsto\sigma(\jmath,[\text{¥}\jmath]_\flat)$ *is lsc.*

Consistent with Feng and Liu [11], for $a\in(0,1)$ and each $\jmath\in\heartsuit$, define the set $I_a^\jmath\subset\heartsuit$ as follows:

$$I_a^\jmath=\{\ell\in[\text{¥}\jmath]_\flat|\ a\sigma(\jmath,\ell)\leq\sigma(\jmath,[\text{¥}\jmath]_\flat)\text{ for some }\flat\in(0,1]\}.$$

Corollary 4. *Let* $(\heartsuit,\sigma)$ *be a cmls,* $\text{¥}:\heartsuit\longrightarrow I_{C(\heartsuit)}$ *be a fsv. If* $c\in(0,1)$ *: for each* $\jmath\in\heartsuit$*, there is* $\ell\in I_a^\jmath$*:*

$$\sigma(\ell,[\text{¥}\ell]_\flat)\leq c\sigma(\jmath,\ell),$$

then ¥ has at least one fuzzy fp in ♡ provided that the fnx $\jmath\longmapsto\sigma(\jmath,[\text{¥}\jmath]_\flat)$ *is lsc.*

Proof. Put $\theta(t)=e^{\sqrt{t}}$, $\xi=\sqrt{c}$ and $s=\sqrt{a}$ in Corollary 1. □

4. Application in ordered metric-like spaces

The study of existence of fp on ms equipped with a partial order is one of the very interesting progress in the area of fp theory. This trend was introduced by Turinici [31] in 1986, but it became one of the core research subject after the results of Ran and Reurings in [27] and Nieto and Rodriguez [24].

In this section, we consider mls endowed with a partial order. Accordingly, $(\heartsuit, \sigma, \preceq)$ is termed an ordered mls, if:

(i) $(\heartsuit, \sigma)$ is a mls, and;
(ii) $(\heartsuit, \preceq)$ is a partially ordered set.

Any two elements $\jmath, \ell \in \heartsuit$ are said to be comparable if either $\jmath \preceq \ell$ or $\ell \preceq \jmath$ holds. Let $¥ : \heartsuit \longrightarrow I^{\heartsuit}$ be a fuzzy set-valued map. For each $\jmath \in \heartsuit$ with $\sigma(\jmath, [¥\jmath]_\flat) > 0$ and some $\flat \in (0, 1]$, define the set $\theta_s^{\jmath,\preceq} \subseteq \heartsuit,\ s \in (0, 1]$ as

$$\theta_s^{\jmath,\preceq} = \{\ell \in [¥\jmath]_\flat : [\theta(\sigma(\jmath, \ell))]^s \leq \theta(\sigma(\jmath, \ell)),\ \jmath \preceq \ell\}.$$

Definition 4.1. Let $\heartsuit$ be a ns. We say that a fsv $¥ : \heartsuit \longrightarrow I^{\heartsuit}$ is $\flat$-comparative if there exists $\flat \in (0, 1]$ such that for each $\jmath \in \heartsuit$ and $\ell \in [¥\jmath]_\flat$ with $\jmath \preceq \ell$, we have $\ell \preceq u$ for all $u \in [¥\ell]_\flat$.

Theorem 4.2. *Let $(\heartsuit, \sigma, \preceq)$ be a complete ordered mls, $¥ : \heartsuit \longrightarrow I_{C(\heartsuit)}$ be a fsv and $\theta \in \mho$. Assume that*

$(C1)$ *the mpn $\jmath \longmapsto \sigma(\jmath, [¥\jmath]_\flat)$ is ordered lsc;*
$(C2)$ *$s \in (0, 1)$ and a fnx $\xi : \mathbb{R}_+ \longrightarrow [0, s)$ exists:*

$$\lim_{t \longrightarrow \varsigma^+} \sup \xi(t) < s \text{ for all } \varsigma \in \mathbb{R}_+;$$

$(C3)$ *for each $\jmath \in \heartsuit$, we have $\ell \in \theta_s^{\jmath,\preceq}$ with $\jmath \preceq \ell$ such that*

$$\theta(\sigma(\jmath, [¥\ell]_\flat)) \leq [\theta(\sigma(\jmath, \ell))]^{\xi(\sigma(\jmath,\ell))};$$

$(C4)$ *$¥$ is $\flat$-comparative;*
$(C5)$ *if $\{\jmath_n\}_{n\in\mathbb{N}} \subset \heartsuit$ with $\jmath_{n+1} \in [¥\jmath_n]_\flat$, $\jmath_n \longrightarrow u \in \heartsuit$ as $n \longrightarrow \infty$, then $\jmath_n \preceq u$ for all $n \in \mathbb{N}$.*

Proof. In line with the proof of Thrm 2.3 and the fact that $\theta_s^{\jmath,\preceq} \subseteq \heartsuit$, we can show that $\{\jmath_n\}_{n\in\mathbb{N}}$ is a C-seq in $(\heartsuit, \sigma, \preceq)$ with $\jmath_{n-1} \preceq \jmath_n$ for all $\in \mathbb{N}$. The completeness of this space produces $u \in \heartsuit$ with $\jmath \longrightarrow u$ as $n \longrightarrow \infty$. By Condition $(C5)$, $\jmath_n \preceq u$ for all $n \in \mathbb{N}$. From this point, Thrm 2.3 can be employed to find $u \in \heartsuit$ such that $u \in [¥u]_\flat$. □

5. Application in metric-like spaces endowed with a binary relation

Let $(\heartsuit, \sigma, \mathcal{R})$ be a binary mls, where $\mathcal{R}$ is a brel on $\heartsuit$. Define $\mathfrak{B} = \mathcal{R} \cup \mathcal{R}^{-1}$. It is easy to notice that for all $\jmath, \ell \in \heartsuit$, $\jmath\mathfrak{B}\ell$ if and only if $\jmath\mathcal{R}\ell$ or $\ell\mathcal{R}\jmath$.

Definition 5.1. Let $\heartsuit$ be a nonempty set. We say that a fsv $¥ : \heartsuit \longrightarrow I^{\heartsuit}$ is $(\flat, \mathcal{R})$-comparative, if $\flat \in (0, 1]$ and a brel $\mathcal{R}$ on $\heartsuit$ such that for each $\jmath \in \heartsuit$ and $\ell \in [¥\jmath]_\flat$ with $\jmath\mathfrak{B}\ell$,we have $\ell\mathfrak{B}u$ for all $u \in [¥\ell]_\flat$.

Definition 5.2. Let $(\heartsuit, \sigma, \mathcal{R})$ be a mls endowed with a brel $\mathcal{R}$ and $¥ : \heartsuit \longrightarrow I^{\heartsuit}$ be a fsv. A fnx $g : (\heartsuit, \sigma, \mathcal{R}) \longrightarrow \mathbb{R}$ is called binary lsc, if $g(u) \leq \lim_{n\longrightarrow\infty} \inf g(\jmath_n)$ for all seq $\{\jmath_n\}_{\mathbb{N}}$ in $\heartsuit$ with $[¥\jmath_n]_\flat \mathfrak{B} [¥\jmath_{n+1}]_\flat$ for all $n \in \mathbb{N}$ and $\jmath_n \longrightarrow u \in \heartsuit$ as $n \longrightarrow \infty$.

For each $\jmath \in \heartsuit$ with $\sigma(\jmath, [¥\jmath]_\flat) > 0$, and a brel $\mathcal{R}$ on $\heartsuit$, define $\theta_s^{\jmath,\mathcal{R}}$, $s \in (0, 1]$:

$$\theta_s^{\jmath,\mathcal{R}} = \{\ell \in [¥\jmath]_\flat : [\theta(\sigma(\jmath, \ell))]^s \leq \theta(\sigma(\jmath, \ell)),\ \jmath\mathfrak{B}\ell\}.$$

Theorem 5.3. *Let $(\heartsuit, \sigma, \mathcal{R})$ be a binary complete mls, $¥ : \heartsuit \longrightarrow I_{C(\heartsuit)}$ be a fsv and $\theta \in \mho$. Assume that*

$(C1)$ *the mpn $\jmath \longmapsto \sigma(\jmath, [¥\jmath]_\flat)$ is binary lsc;*
$(C2)$ *there exist $s \in (0, 1)$ and a fnx $\xi : \mathbb{R}_+ \longrightarrow [0, s)$:*

$$\lim_{t\longrightarrow\varsigma^+} \sup \xi(t) < s \text{ for all } \varsigma \in \mathbb{R}_+;$$

$(C3)$ *for each $\jmath \in \heartsuit$, there exists $\ell \in \theta_s^{\jmath,\mathcal{R}}$ with $\jmath\mathfrak{B}\ell$ such that*

$$\theta(\sigma(\jmath, [¥\ell]_\flat)) \leq [\theta(\sigma(\jmath, \ell))]^{\xi(\sigma(\jmath,\ell))};$$

$(C4)$ *$¥$ is $(\flat, \mathcal{R})$-comparative;*
$(C5)$ *if $\{\jmath_n\}_{n\in\mathbb{N}} \subset \heartsuit$ with $\jmath_{n+1} \in [¥\jmath_n]_\flat$, $\jmath_n \longrightarrow u \in \heartsuit$ as $n \longrightarrow \infty$, then $\jmath_n\mathfrak{B}u$ for all $n \in \mathbb{N}$.*

Proof. The proof follows similar ideas of Thrm 4.2. □

6. Application in multivalued mappings on metric-like spaces

In this section, we employ some results from the former section to deduce their classical multivalued analogues in the framework of mls. It is well-known that mls cannot be Hasudorff (for details, see [18]), making it impossible for the usual studies of fp of set-valued mappings via the Pompeiu-Hausdorff metric. However, using the Feng and Liu's technique (see [11]), this shortcoming can be overcome.

Let $F : \heartsuit \longrightarrow \mathcal{N}(\heartsuit)$ be a mvm. Denote by $\Lambda_s^\jmath$, the set

$$\Lambda_s^\jmath = \{\ell \in F\jmath : [\theta(\sigma(\jmath, \ell))]^s \leq \theta(\sigma(\jmath, F\jmath))\}.$$

Theorem 6.1. *Let $(\heartsuit, \sigma)$ be a cmls, $F : \heartsuit \longrightarrow C(\heartsuit)$ be a mvm and $\theta \in \mho$. If there exist $s \in (0, 1)$ and a fnx $\xi : \mathbb{R}_+ \longrightarrow [0, s)$:*

$$\lim_{t\longrightarrow\varsigma^+} \sup \xi(t) < s \text{ for all } \varsigma \in \mathbb{R}_+,$$

and for each $\jmath \in \heartsuit$ with $\sigma(\jmath, F\jmath) > 0$, there is $\ell \in \Lambda_s^\jmath$:

$$\theta(\sigma(\ell, F\ell)) \leq [\theta(\sigma(\jmath, \ell))]^{\xi(\sigma(\jmath,\ell))}, \tag{6.1}$$

then F has at least one fp in $\heartsuit$ provided that the fnx $\jmath \longmapsto \sigma(\jmath, F\jmath)$ is lsc.

Proof. Let $\flat : \heartsuit \longrightarrow (0, 1]$ be a mpn and consider a fsv $¥ : \heartsuit \longrightarrow I_{C(\heartsuit)}$ defined by

$$¥(\jmath)(t) = \begin{cases} \flat(\jmath), & \text{if } t \in F\jmath. \\ 0, & \text{if } t \notin F\jmath. \end{cases}$$

Then, setting $\flat(\jmath) := \flat$ for each $\jmath \in \heartsuit$, we have

$$[¥\jmath]_\flat = \{t \in \heartsuit : ¥(\jmath)(t) \geq \flat\} = F\jmath.$$

Consequently, Thrm 2.3 can be applied to find $u \in \heartsuit$ such that $u \in [¥u]_\flat = Fu$. □

On the same line of deriving Thrm 6.1, the next two results follow from Thrm 2.4 and Corollary 4, respectively.

Theorem 6.2. *Let $(\heartsuit, \sigma)$ be a cmls, $F : \heartsuit \longrightarrow \mathcal{K}(\heartsuit)$ be a mvm and $\theta \in \Omega$. If there is a fnx $\xi : \mathbb{R}_+ \longrightarrow [0, 1)$:*

$$\lim_{t \longrightarrow \varsigma^+} \sup \xi(t) < 1 \text{ for all } \varsigma \in \mathbb{R}_+,$$

and for each $\jmath \in \heartsuit$ with $\sigma(\jmath, F\jmath) > 0$, there is $\ell \in \Lambda_1^\jmath$:

$$\theta(\sigma(\ell, F\ell)) \leq [\theta(\sigma(\jmath, \ell))]^{\xi(\sigma(\jmath,\ell))},$$

then F has at least one fp in $\heartsuit$ provided that the fnx $\jmath \longmapsto \sigma(\jmath, F\jmath)$ is lsc.

Theorem 6.3. *Let $(\heartsuit, \sigma)$ be a cmls, $F : \heartsuit \longrightarrow C(\heartsuit)$ be a mvm. If there exists $c \in (0, 1)$ such that for each $\jmath \in \heartsuit$, there is $\ell \in I_a^\jmath$:*

$$\sigma(\ell, F\ell) \leq c\sigma(\jmath, \ell),$$

then F has at least one fp in $\heartsuit$ provided that $c < a$ and the fnx $\jmath \longmapsto \sigma(\jmath, F\jmath)$ is lsc.

We construct the next examples to verify the hypotheses of Thrms 6.1–6.3.

Example 6.4. Let $\heartsuit = [-1, 0] \cup \{2n : n \in \mathbb{N}\}$ and $\sigma : \heartsuit \times \heartsuit \longrightarrow \mathbb{R}_+$ be defined as follows:

$$\sigma(\jmath, \ell) = \begin{cases} 1, & \text{if } \jmath = \ell \text{ and one of } \jmath, \ell \notin [-1, 0]. \\ 40, & \text{if } \jmath \neq \ell \text{ and one of } \jmath, \ell \notin [-1, 0]. \\ |\jmath| + |\ell|, & \text{if } \jmath, \ell \in [-1, 0]. \end{cases}$$

Then $(\heartsuit, \sigma)$ is a cmls. Notice that σ is not a metric on $\heartsuit$, since $\sigma(2, 2) = 1 > 0$. Now, define a mvm $F : \heartsuit \longrightarrow C(\heartsuit)$ by

$$F\jmath = \begin{cases} \left\{\frac{\jmath}{50}\right\}, & \text{if } \jmath \in [-1, 0]. \\ \{\jmath, \jmath + 1, \jmath + 2, \cdots\}, & \text{if } \jmath \in \{2n : n \in \mathbb{N}\}. \end{cases}$$

It is easy to see that

$$\sigma(\jmath, F\jmath) = \begin{cases} \frac{51|\jmath|}{50}, & \text{if } \jmath \in [-1, 0]. \\ 1, & \text{if } \jmath \notin [-1, 0], \end{cases}$$

and the fnx $\jmath \longmapsto \sigma(\jmath, F\jmath)$ is lsc. Next, we show that the contraction condition (6.1) holds. Taking $\theta(t) = e^{\sqrt{te^t}}$, $s \in \left(\frac{1}{e}, 1\right)$ and $\xi(t) = \frac{1}{e}$, (6.1) becomes

$$\frac{\sigma(\ell, F\ell)}{\sigma(\jmath, \ell)} e^{\sigma(\ell,F\ell) - \sigma(\jmath,\ell)} \leq [\xi(\sigma(\jmath, \ell))]^2. \tag{6.2}$$

Thus, it is enough to show that F satisfies (6.2). Observe that if $\sigma(\jmath, F\jmath) > 0$, then $\jmath \neq 0$. Hence, for $\jmath \in [-1, 0)$, there is $\ell = \frac{\jmath}{50} \in \theta_s^{\jmath}$ for all $s \in \left(\frac{1}{e}, 1\right)$ and

$$\begin{aligned}\frac{\sigma(\ell, F\ell)}{\sigma(\jmath, \ell)} e^{\sigma(\ell,F\ell)-\sigma(\jmath,\ell)} &= \frac{\frac{101|\jmath|}{2500}}{\frac{51|\jmath|}{50}} e^{-\frac{2449|\jmath|}{2500}} \\ &= \frac{101}{2550} e^{-\frac{2449|\jmath|}{2500}} \\ &\leq \frac{101}{2550} < \frac{1}{e^2} \\ &= \left[\xi\left(\frac{51|\jmath|}{50}\right)\right]^2 = [\xi(\sigma(\jmath, \ell))]^2.\end{aligned}$$

Moreover, for $\jmath \notin [-1, 0]$, we have $\ell = \jmath + 1 \in \theta_s^{\jmath}$ for all $s \in \left(\frac{1}{e}, 1\right)$ and

$$\begin{aligned}\frac{\sigma(\ell, F\ell)}{\sigma(\jmath, \ell)} e^{\sigma(\ell,F\ell)-\sigma(\jmath,\ell)} &= \frac{1}{40} e^{1-40} \\ &= \frac{1}{40} e^{-39} \leq \frac{1}{40} \\ &< \frac{1}{e^2} = [\xi(40)]^2 \\ &= [\xi(\sigma(\jmath, \ell))]^2.\end{aligned}$$

Hence, all the hypotheses of Thrms 6.1 and 6.2 are satisfied. It follows that F has a fp in $\heartsuit$.

On the other hand, since $(\heartsuit, \sigma)$ is not a ms, then all the results of Altun and Durmaz [10] and Thrms 1.3 and 1.4 due to Klim and Wardowski [17] are not applicable to this example.

Example 6.5. Let $\heartsuit = \left\{\frac{1}{3^n} : n \in \mathbb{N}\right\} \cup \{0, 1\}$ and $\sigma(\jmath, \ell) = |\jmath| + |\ell|$ for all $\jmath, \ell \in \heartsuit$. Then $(\heartsuit, \sigma)$ is a cmls. Note that σ is not a metric on $\heartsuit$, since $\sigma(1, 1) = 2 > 0$. Now, define the mvm $F : \heartsuit \longrightarrow C(\heartsuit)$ as follows:

$$F\jmath = \begin{cases}\left\{\frac{1}{3^{n+1}}, 1\right\}, & \text{if } \jmath = \frac{1}{3^n},\ n \in \mathbb{N} \cup \{0\}. \\ \left\{0, \frac{1}{3}\right\}, & \text{if } \jmath = 0.\end{cases}$$

By elementary computation, we see that

$$\sigma(\jmath, F\jmath) = \begin{cases}\frac{4}{3^{n+1}}, & \text{if } \jmath = \frac{1}{3^n},\ n \in \mathbb{N}. \\ 0, & \text{if } \jmath = 0. \\ 2, & \text{if } \jmath = 1.\end{cases}$$

Therefore, the fnx $\jmath \longmapsto \sigma(\jmath, F\jmath)$ is lsc. Moreover, there exists $\ell = 0 \in I_{0.8}^{\jmath}$ such that for each $\jmath \in \heartsuit$,

$$\sigma(\ell, F\ell) = 0 \leq c\sigma(\jmath, \ell),$$

for all $c < a = 0.8$. Hence, all the assumptions of Thrm 6.3 are satisfied. Consequently, F has a fp in $\heartsuit$.

However, note that since $(\heartsuit, \sigma)$ is not a ms, Thrm 1.2 of Feng and Liu [11] cannot be applied in this example to obtain a fp of F.

Remark 3.

(i) Thrms 2.3 and 2.4 are fuzzy generalizations of the main results of Altun and Minak [2] as well as Durmaz and Altun [10] in the setting of mls.
(ii) Thrms 6.1 and 6.2 are extensions of the results in [10] and Thrms 1.3 and 1.4 due to Klim and Wardowski [17] in the bodywork of mls.
(iii) Thrm 6.3 is a proper generalization of Thrm 1.2 due to Feng and Liu [11].

7. Conclusions

In this note, the concept of nonlinear fsv θ-con in the setting of mls has been introduced. By using the techniques of Feng and Liu [11] along with θ-con due to Jleli and Samet [15], we established the existence of fuzzy fp for the new contractions. As some consequences of our main theorems, a few fp results of mls endowed with partial ordering and brel as well as mvm are deduced. All the fp results established in this work are also valid for ms.

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Conflict of interest

The authors declare that they have no competing interests.

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