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Research article

Edge irregular reflexive labeling for the r-th power of the path

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Abstract: Let $G(V, E)$ be a graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. Let k be a natural number, a total k-labeling $\varphi : V(G) \bigcup E(G) \to \{0, 1, 2, 3, ..., k\}$ is called an edge irregular reflexive k-labeling if the vertices of G are labeled with the set of even numbers from $\{0, 1, 2, 3, ..., k\}$ and the edges of G are labeled with numbers from $\{1, 2, 3, ..., k\}$ in such a way for every two different edges xy and $x^{'}y^{'}$ their weights $\varphi(x) + \varphi(xy) + \varphi(y)$ and $\varphi(x^{'}) + \varphi(x^{'}y^{'}) + \varphi(y^{'})$ are distinct. The reflexive edge strength of G, $res(G)$, is defined as the minimum k for which G has an edge irregular reflexive k-labeling. In this paper, we determine the exact value of the reflexive edge strength for the r-th power of the path P_n, where $r \geq 2$, $n \geq r + 4$.

Keywords: edge irregular reflexive labeling; reflexive edge strength; r-th power graph
Mathematics Subject Classification: 05C12, 05C78, 05C90

1. Introduction

Throughout this paper we consider G a connected, simple, and undirected graph, where V and E are denote to sets of vertices and edges of G with cardinalities $|V|$ and $|E|$, respectively. Chartrand et al. presented in [12] the edge k-labeling of graph G, $\varphi : E(G) \to \{1, 2, 3, ..., k\}$ such that the aggregate of the labels of edges incident with a vertex is different for all the vertices of G. Such labelings were referred to as irregular assignments and irregular strength, $s(G)$, of a graph G is known as the smallest k for which G has an irregular assignment using labels at most k. In [16] Lahel gives a comprehensive over view of the strength of irregularity. For some studies on irregularity strength see papers by Amar and Tongi [5], Dimitiz et al. [13], Gyárfás [14], and Nierhoff [17]. In [7] Bača et al. motivated the concept of irregular strength and started to investigate the total edge irregularity strength of a graph. An edge irregular total k-labeling of the graph G is a labeling $\varphi : V(G) \bigcup E(G) \to \{1, 2, 3, ..., k\}$ such that for every two distinct edges xy, $x^{'}y^{'}$ of G the edge weights $wt_{\varphi}(xy) = \varphi(x) + \varphi(xy) + \varphi(y) \neq wt_{\varphi}(x^{'}y^{'}) = \varphi(x^{'}) + \varphi(x^{'}y^{'}) + \varphi(y^{'})$. The total edge irregularity strength, $tes(G)$ is defined as the smallest k for which G has an edge irregular total k-labeling. Some interesting studies on the total edge irregularity strength can be seen in [1–4, 8, 9]. Further, Ryan, Munasinghe, and Tanna in [18] introduced the concept of the edge irregular reflexive. For a graph $G(V, E)$ they define an edge labeling $\varphi_e : E(G) \rightarrow \{1, 2, 3, ..., k_e\}$ and a vertex labeling $\varphi_v : V(G) \rightarrow \{0, 2, 4, ..., 2k_v\}$, then defined the labeling φ by:

$$\varphi(x) = \begin{cases} \varphi_v(x), & \text{if } x \in V(G) \\ \varphi_e(x), & \text{if } x \in E(G) \end{cases}$$

is a total k-labeling , where $k = max\{k_e, 2k_v\}$. Moreover, if for every two different edges xy and $x'y'$ of G one has $wt_\varphi(xy) \neq wt_\varphi(x'y')$, where $wt_\varphi(xy) = \varphi_e(xy) + \varphi_v(x) + \varphi_v(y)$, then the total k-labeling φ is called an edge irregular reflexive labeling of graph G. The reflexive edge strength, $res(G)$, is defined as the smallest k for which G has an edge irregular reflexive k-labeling. For more research on reflexive edge strength see [6, 10, 11, 15, 20–22]. In this paper, we estimate the exact value of the reflexive edge strength for the r-th power of the path P_n, where $r \geq 2$, $n \geq r + 4$.

Definition 1.1. *([19]) The r-th power of a graph G, denoted by G^r, is a graph with the same vertex set of G such that adding edges between the vertices which are at distance at most r, see Figure 1.*

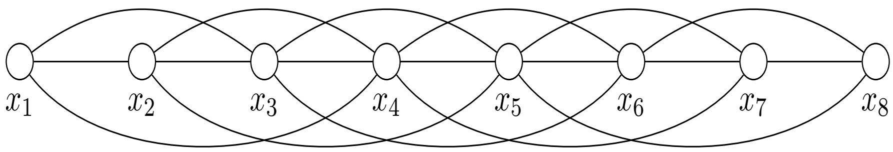

Figure 1. The 3-th power of P_8.

When we prove the result, we will often use the following lemma, which has been proved in [18].

Lemma 1.1. *([18]). For every graph G,*

$$res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } |E(G)| \not\equiv 2, 3 \ (mod\ 6), \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } |E(G)| \equiv 2, 3 \ (mod\ 6)\ . \end{cases}$$

Furthermore, Bača et al. [10] proposed the following conjecture:

Conjecture 1.1. *([10]) Consider the graph G, which has a maximum degree $\triangle = \triangle(G)$. Hence:*

$$res(G) = max\{\lfloor \frac{\triangle + 2}{2} \rfloor, \lceil \frac{|E(G)|}{3} \rceil + r\}$$

where $r = 1$ for $|E(G)| \equiv 2, 3$ (mod 6), and zero otherwise.

2. The r-th power of the path

The r-th power of a path P_n denoted by P_n^r, $n \geq 3, r \geq 2$. Let us denote to the vertex set and edge set of P_n^r by $V(P_n^r) = \{x_i, 1 \leq i \leq n\}$ and $E(P_n^r) = \bigcup_{j=1}^{r} \{x_i x_{i+j}, 1 \leq i \leq n - j\}$. In the next theorem, we determine the reflexive edge strength of various powers of a path P_n.

Theorem 1. *For the r-th power of a path P_n, $r \geq 2$, $n \geq r+4$.*

$$res(P_n^r) = \begin{cases} \lceil \frac{r(2n-r-1)}{6} \rceil, & \textit{if } \frac{r(2n-r-1)}{2} \not\equiv 2,3 \textit{ (mod 6)}, \\ \lceil \frac{r(2n-r-1)}{6} \rceil + 1, & \textit{if } \frac{r(2n-r-1)}{2} \equiv 2,3 \textit{ (mod 6)} . \end{cases}$$

Proof. Note that the r-th power of P_n has $\frac{r(2n-r-1)}{2}$ edges. The lower bound for res of the r-th power of P_n is as follow from the Lemma 1. $res(P_n^r) \geq k = \lceil \frac{r(2n-r-1)}{6} \rceil + 1$ if $\frac{r(2n-r-1)}{2} \equiv 2,3$ (mod 6) and $res(P_n^r) \geq k = \lceil \frac{r(2n-r-1)}{6} \rceil$ if $\frac{r(2n-r-1)}{2} \not\equiv 2,3$ (mod 6). Moreover, we prove that:

$$res(P_n^r) \leq \begin{cases} \lceil \frac{r(2n-r-1)}{6} \rceil, & \text{if } \frac{r(2n-r-1)}{2} \not\equiv 2,3 \text{ (mod 6)}, \\ \lceil \frac{r(2n-r-1)}{6} \rceil + 1, & \text{if } \frac{r(2n-r-1)}{2} \equiv 2,3 \text{ (mod 6)} . \end{cases}$$

Let $k^* = max\{\lfloor \frac{2k+r(r+1)-4}{2r} \rfloor, 3\}$ and for $n \geq r+4$, we recognise two cases.
Case 1. When $\lceil \frac{n-k^*}{2} \rceil \geq r$.
Construct the total k-labeling φ of P_n^r in the following way:
The corresponding labeling for P_8^3 is illustrated in Figure 2. Otherwise we have the following labeling:

$$\varphi(x_i) = \begin{cases} 0, & 1 \leq i \leq k^* \\ 2\lfloor \frac{rk^*}{4} \rfloor, & k^*+1 \leq i \leq k^* + \lceil \frac{n-k^*}{2} \rceil \\ k, & k^* + \lceil \frac{n-k^*}{2} \rceil + 1 \leq i \leq n \end{cases}$$

Furthermore, the labels of edges are defined as the following :

$$\varphi(x_i x_{i+1}) = \begin{cases} i, & 1 \leq i \leq k^* - 1 \\ \frac{r(2k^*-r-1)}{2} - 2\lfloor \frac{rk^*}{4} \rfloor + 1, & i = k^* \\ (r-1)k^* - 4\lfloor \frac{rk^*}{4} \rfloor + i, & k^*+1 \leq i \leq k^* + \lceil \frac{n-k^*}{2} \rceil - 1 \\ \frac{r(2k^*-r-1)}{2} + r\lceil \frac{n-k^*}{2} \rceil - \\ -k - 2\lfloor \frac{rk^*}{4} \rfloor + 1, & i = k^* + \lceil \frac{n-k^*}{2} \rceil \\ \frac{r(2n-r-1)}{2} - \frac{\lfloor \frac{n-k^*}{2} \rfloor (\lfloor \frac{n-k^*}{2} \rfloor - 3)}{2} - \\ -n - 2k + i, & k^* + \lceil \frac{n-k^*}{2} \rceil + 1 \leq i \leq n-1 \end{cases}$$

$$\varphi(x_i x_{i+2}) = \begin{cases} k^* + i - 1, & 1 \leq i \leq k^* - 2 \\ (r-1)k^* - \frac{r(r+1)}{2} - \\ -2\lfloor \frac{rk^*}{4} \rfloor + i + 3, & k^* - 1 \leq i \leq k^* \\ (r-1)k^* - 4\lfloor \frac{rk^*}{4} \rfloor + \\ +\lceil \frac{n-k^*}{2} \rceil + i - 1, & k^*+1 \leq i \leq k^* + \lceil \frac{n-k^*}{2} \rceil - 2 \\ (r-1)(k^* + \lceil \frac{n-k^*}{2} \rceil) - \frac{r(r+1)}{2} - \\ -2\lfloor \frac{rk^*}{4} \rfloor - k + i + 3, & k^* + \lceil \frac{n-k^*}{2} \rceil - 1 \leq i \leq k^* + \lceil \frac{n-k^*}{2} \rceil \\ \frac{r(2n-r-1)}{2} - \frac{\lfloor \frac{n-k^*}{2} \rfloor (\lfloor \frac{n-k^*}{2} \rfloor - 5)}{2} - \\ -n - 2k + i - 1, & k^* + \lceil \frac{n-k^*}{2} \rceil + 1 \leq i \leq n-2 \end{cases}$$

For $3 \leq j \leq r$, there exist three subcases:

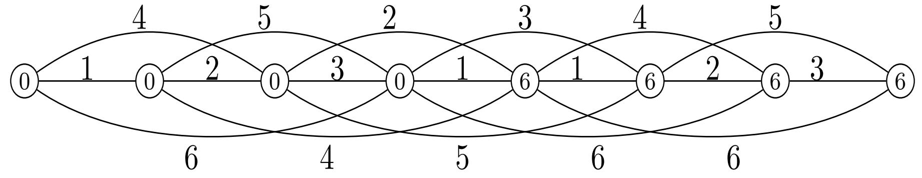

Figure 2. A reflexive irregular 6-labeling of P_8^3.

Subcase 1.1. If $\lfloor \frac{n-k^*}{2} \rfloor > r$, hence the edges label as following:

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{k^*-j+1}x_{k^*}) + i, & 1 \le i \le k^* - j \\ \varphi(x_{k^*}x_{k^*+j-1}) - k^* + j + i, & k^* - j + 1 \le i \le k^* \\ \varphi(x_{k^*+\lceil \frac{n-k^*}{2} \rceil - j+1} x_{k^*+\lceil \frac{n-k^*}{2} \rceil}) - \\ -k^* + i, & k^* + 1 \le i \le k^* + \lceil \frac{n-k^*}{2} \rceil - j \\ \varphi(x_{k^*+\lceil \frac{n-k^*}{2} \rceil} x_{k^*+\lceil \frac{n-k^*}{2} \rceil + j-1}) - \\ -\lceil \frac{n-k^*}{2} \rceil - k^* + j + i, & k^* + \lceil \frac{n-k^*}{2} \rceil - j + 1 \le i \le k^* + \lceil \frac{n-k^*}{2} \rceil \\ \varphi(x_{n-j+1}x_n) - \lceil \frac{n-k^*}{2} \rceil - k^* + i, & k^* + \lceil \frac{n-k^*}{2} \rceil + 1 \le i \le n - j \end{cases}$$

Subcase 1.2. If $\lfloor \frac{n-k^*}{2} \rfloor = r$, here for $3 \le j \le r-1$ the edge labels are common the pervious subcase, and for $j = r$ we define edge labels as follows:

$$\varphi(x_i x_{i+r}) = \begin{cases} \varphi(x_{k^*-r+1}x_{k^*}) + i, & 1 \le i \le k^* - r \\ \varphi(x_{k^*}x_{k^*+r-1}) - k^* + r + i, & k^* - r + 1 \le i \le k^* \\ \varphi(x_{k^*+\lceil \frac{n-k^*}{2} \rceil} x_{k^*+\lceil \frac{n-k^*}{2} \rceil + r-1}) - k^* + i, & k^* + 1 \le i \le k^* + \lceil \frac{n-k^*}{2} \rceil . \end{cases}$$

Subcase 1.3. If $\lfloor \frac{n-k^*}{2} \rfloor = r-1$, for $3 \le j \le r-2$ the edge labels are common the Subcase 1.1 now, we construct edge labels only for $j = r-1$ and $j = r$ as follows:

$$\varphi(x_i x_{i+r-1}) = \begin{cases} \varphi(x_{k^*-r+2}x_{k^*}) + i, & 1 \le i \le k^* - r + 1 \\ \varphi(x_{k^*}x_{k^*+r-2}) - k^* + r + i - 1, & k^* - r + 2 \le i \le k^* \\ \varphi(x_{k^*+\lceil \frac{n-k^*}{2} \rceil - r+2} x_{k^*+\lceil \frac{n-k^*}{2} \rceil}) - \\ -k^* + i, & k^* + 1 \le i \le k^* + \lceil \frac{n-k^*}{2} \rceil - r + 1 \\ \varphi(x_{k^*+\lceil \frac{n-k^*}{2} \rceil} x_{k^*+\lceil \frac{n-k^*}{2} \rceil + r-2}) - \\ -k^* - \lceil \frac{n-k^*}{2} \rceil + r + i - 1, & k^* + \lceil \frac{n-k^*}{2} \rceil - r + 2 \le i \le k^* + \lceil \frac{n-k^*}{2} \rceil - 1. \end{cases}$$

$$\varphi(x_i x_{i+r}) = \begin{cases} \varphi(x_{k^*-r+1}x_{k^*}) + i, & 1 \le i \le k^* - r \\ \varphi(x_{k^*}x_{k^*+r-1}) - k^* + r + i, & k^* - r + 1 \le i \le k^* \\ \varphi(x_{k^*+\lceil \frac{n-k^*}{2} \rceil} x_{k^*+\lceil \frac{n-k^*}{2} \rceil + r-1}) - k^* + i, & k^* + 1 \le i \le k^* + \lceil \frac{n-k^*}{2} \rceil - 1. \end{cases}$$

An explanation of above labeling is depicted in Figure 3. Evidently the vertices of P_n^r labeled with even numbers. Hence we will compute the weights of edges under the labeling φ:
The edge set of P_n^r can be divided into five mutually separated subsets, $A_s, 1 \le s \le 5$ as follows:
For $1 \le j \le r$,

- $A_1 = \{x_i x_{i+j} : 1 \le i \le k^* - j\}$: The set of all edges which have endpoints tagged with 0,

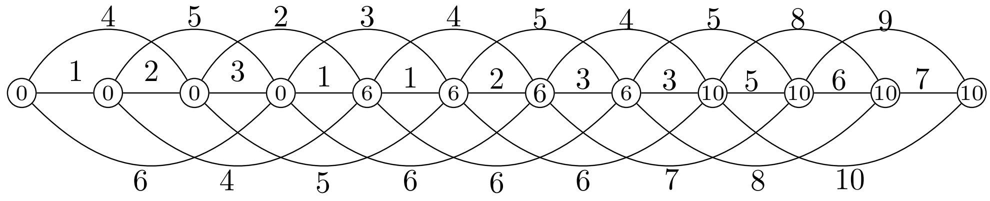

Figure 3. A reflexive irregular 10-labeling of P_{12}^3.

- $A_2 = \{x_i x_{i+j} : k^* - j \leq i \leq k^*\}$: The set of all edges which have endpoints tagged with 0 and $2\lfloor \frac{rk^*}{4} \rfloor$,
- $A_3 = \{x_i x_{i+j} : k^* \leq i \leq k^* + \lceil \frac{n-k^*}{2} \rceil - j\}$: The set of all edges which have endpoints tagged with $2\lfloor \frac{rk^*}{4} \rfloor$,
- $A_4 = \{x_i x_{i+j} : k^* + \lceil \frac{n-k^*}{2} \rceil - r \leq i \leq k^* + \lceil \frac{n-k^*}{2} \rceil\}$: The set of all edges which have endpoints tagged with $2\lfloor \frac{rk^*}{4} \rfloor$ and k,
- $A_5 = \{x_i x_{i+j} : k^* + \lceil \frac{n-k^*}{2} \rceil + 1 \leq i \leq n - j\}$: The set of all edges which have endpoints tagged with k.

Therefore, the edge weights of P_n^r under the labeling φ are the following:

1. The edge weights of the set A_1, get the consecutive numbers from the set $\{1, 2, ..., \frac{r(2k^*-r-1)}{2}\}$,
2. The edge weights of the set A_2, receive the consecutive numbers from the set $\{\frac{r(2k^*-r-1)}{2} + 1, ..., rk^*\}$,
3. The edge weights of the set A_3, get the consecutive numbers from the set $\{rk^* + 1, ..., \frac{r(2k^*-r-1)}{2} + r\lceil \frac{n-k^*}{2} \rceil\}$,
4. The edge weights of the set A_4, get the consecutive numbers from the set $\{\frac{r(2k^*-r-1)}{2} + r\lceil \frac{n-k^*}{2} \rceil + 1, ..., \frac{r(2n-r-1)}{2} - \frac{\lfloor \frac{n-k^*}{2} \rfloor (\lfloor \frac{n-k^*}{2} \rfloor - 1)}{2}\}$,
5. Finally, the edge weights of the set A_5, receive the consecutive numbers from the set $\{\frac{r(2n-r-1)}{2} - \frac{\lfloor \frac{n-k^*}{2} \rfloor (\lfloor \frac{n-k^*}{2} \rfloor - 1)}{2} + 1, ..., \frac{r(2n-r-1)}{2}\}$.

An explanation of above corresponding weights is depicted in Figure 4. It is easy to check that the

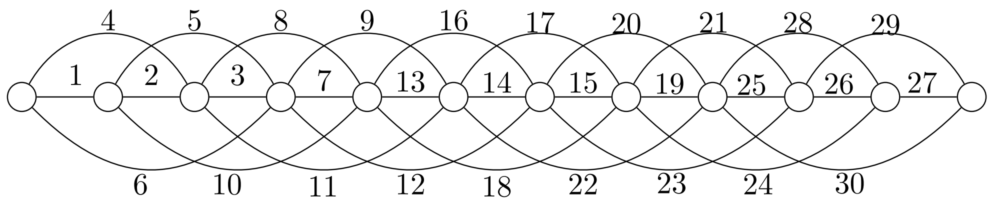

Figure 4. The edge weights of P_{12}^3.

weights of the edges are different numbers from the set $\{1, 2, 3, ..., \frac{r(2n-r-1)}{2}\}$.
Case 2. When $\lceil \frac{n-k^*}{2} \rceil < r$.
In this case we have three subcases.
Subcase 2.1. If $k^* < \frac{n}{2}$. We can define the total k-labeling φ of P_n^r as follows:
The corresponding labelings for P_7^3 , P_9^5 and P_{10}^4 are illustrated in Figure 7, 8 and 9 respectively.

Otherwise we have following labeling:

$$\varphi(x_i) = \begin{cases} 0, & 1 \le i \le k^* \\ 2\lceil \frac{k}{4} \rceil, & k^*+1 \le i \le n-k^*+1 \\ k, & n-k^*+2 \le i \le n \end{cases}$$

Moreover, to define edge labels for P_n^r we have two subcases:
Subcase 2.1.1 If $k^* \ge r$, then we construct the edge labels as follows:

$$\varphi(x_i x_{i+1}) = \begin{cases} i, & 1 \le i \le k^*-1 \\ \frac{r(2k^*-r-1)}{2} - 2\lceil \frac{k}{4} \rceil + 1, & i = k^* \\ (r-1)k^* - 4\lceil \frac{k}{4} \rceil + i, & k^*+1 \le i \le n-k^* \\ rk^* + \frac{(n-2k^*+1)(n-2k^*)}{2} - & \\ -2\lceil \frac{k}{4} \rceil - k + 1, & i = n-k^*+1 \\ \frac{r(2n-r-1)}{2} - \frac{(k^*-1)(k^*-4)}{2} - & \\ -n - 2k + i, & n-k^*+2 \le i \le n-1 \end{cases}$$

For $1 < j < n - 2k^* + 1$,

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{k^*-j+1} x_{k^*}) + i, & 1 \le i \le k^*-j \\ \varphi(x_{k^*} x_{k^*+j-1}) - k^* + i + j, & k^*-j+1 \le i \le k^* \\ \varphi(x_{n-k^*-j+1} x_{n-k^*}) - k^* + i, & k^*+1 \le i \le n-k^*-j \\ \varphi(x_{n-k^*} x_{n-k^*+j-1} + k^* - n - k^* + i, & n-k^*-j+1 \le i \le n-k^* \\ \varphi(x_{n-j+1} x_n) - k^* - n + i, & n-k^*+1 \le i \le n-j\ . \end{cases}$$

$$\varphi(x_i x_{i+n-2k^*+1}) = \begin{cases} \varphi(x_{3k^*-n} x_{k^*}) + i, & 1 \le i \le 3k^*-n-1 \\ \varphi(x_{k^*} x_{n-k^*}) + n - 3k^* + i + 1, & 3k^*-n \le i \le k^* \\ \varphi(x_{n-k^*} x_{2n-3k^*}) - k^* + i, & k^*+1 \le i \le n-k^*+1 \\ \varphi(x_{2k^*} x_n) - n + k^* + i - 1, & n-k^*+2 \le i \le 2k^*-1\ . \end{cases} \tag{2.1}$$

$$\varphi(x_i x_{i+n-2k^*+2}) = \begin{cases} \varphi(x_{3k^*-n-1} x_{k^*}) + i, & 1 \le i \le 3k^*-n-2 \\ \varphi(x_{k^*} x_{n-k^*-1}) + n - 3k^* + i + 2, & 3k^*-n-1 \le i \le k^*-1 \\ \frac{r(2k^*-r-1)}{2} + \frac{(n-2k^*+1)(2r-n+2k^*)}{2} - & \\ -k + 1, & i = k^* \\ \varphi(x_{n-k^*} x_{2n+3k^*}) - k^* + i, & k^*+1 \le i \le n-k^*+1 \\ \varphi(x_{2k^*-1} x_n) - n + k^* + i, & n-k^*+2 \le i \le 2k^*-2 \end{cases}$$

Now, for $n - 2k^* + 3 \le i \le r$ we have three subcases:
Subcase 2.1.1.1 If $k^* \ge r + 2$, hence the labels of edges are defined as follows:

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{k^*-j+1} x_{k^*}) + i, & 1 \le i \le k^*-j \\ \varphi(x_{n-k^*-j+2} x_{n-k^*+1}) - k^* + j + i, & k^*-j+1 \le i \le n-k^*-j+1 \\ \varphi(x_{k^*} x_{k^*+j-1}) - n + k^* + j + i - 1, & n-k^*-j+2 \le i \le k^* \\ \varphi(x_{n-k^*+1} x_{n-k^*+j}) - k^* + i, & k^*+1 \le i \le n-k^*+1 \\ \varphi(x_{n-j+1} x_n) - n + k^* + i - 1, & n-k^*+2 \le i \le n-j\ . \end{cases} \tag{2.2}$$

Subcase 2.1.1.2 If $k^* = r + 1$, thus the edges $n - 2k^* + 3 \leq j \leq r - 1$, $x_i x_{i+j}$, $1 \leq i \leq n - j$ are labeled by Eq (2.2) and for $j = r$ the edges are labeled as follows:

$$\varphi(x_i x_{i+r}) = \begin{cases} \varphi(x_{k^*-r+1} x_{k^*}) + i, & 1 \leq i \leq k^* - r \\ \varphi(x_{n-k^*-r+2} x_{n-k^*+1}) - k^* + r + i, & k^* - r + 1 \leq i \leq n - k^* - r + 1 \\ \varphi(x_{k^*} x_{k^*+r-1}) - n + k^* + r + i - 1, & n - k^* - r + 2 \leq i \leq k^* \\ \varphi(x_{n-k^*+1} x_{n-k^*+r}) - k^* + i, & k^* + 1 \leq i \leq n - k^* + 1 . \end{cases}$$

Subcase 2.1.1.3 If $k^* = r$, then the edges $x_i x_{i+j}$, $n - 2k^* + 3 \leq j \leq r - 2$, $1 \leq i \leq n - j$ are labeled by Eq (2.2) and for $j = r - 1$ and $j = r$ the edges are labeled as follows:

$$\varphi(x_i x_{i+r-1}) = \begin{cases} \varphi(x_2 x_{k^*}) + 1, & i = 1 \\ \varphi(x_{n-2k^*+3} x_{n-k^*+1}) + i - 1, & 2 \leq i \leq n - 2k^* + 2 \\ \varphi(x_{k^*} x_{2k^*-2}) - n + 2k^* + i - 2, & n - 2k^* + 3 \leq i \leq k^* \\ \varphi(x_{n-k^*+1} x_{n-1}) - k^* + i, & k^* + 1 \leq i \leq n - k^* + 1 . \end{cases}$$

$$\varphi(x_i x_{i+r}) = \begin{cases} \varphi(x_{n-2k^*+2} x_{n-k^*+1}) + i, & 1 \leq i \leq n - 2k^* + 1 \\ \varphi(x_{k^*} x_{2k^*-1}) - n + 2k^* + i - 1, & n - 2k^* + 2 \leq i \leq k^* \\ \varphi(x_{n-k^*+1} x_n) - k^* + i, & k^* + 1 \leq i \leq n - k^* . \end{cases}$$

Subcase 2.1.2 If $k^* = r - 1$, hence we can defined the edge labels as follows:

$$\varphi(x_i x_{i+1}) = \begin{cases} i, & 1 \leq i \leq k^* - 1 \\ \frac{k^*(k^*-1)}{2} - 2\lceil \frac{k}{4} \rceil + 1, & i = k^* \\ k^{*2} - 4\lceil \frac{k}{4} \rceil + i, & k^* + 1 \leq i \leq n - k^* \\ \frac{n(n+1)+2k^*(3k^*-2n)+2}{2} - 2\lceil \frac{k}{4} \rceil - k, & i = n - k^* + 1 \\ k^*(n - k^* + 1) - 2k + i - 3, & n - k^* + 2 \leq i \leq n - 1 . \end{cases}$$

For $j = n - 2k^* + 1$, we used the Eq (2.1) to label the edges and for $j = n - 2k^* + 2$ the edge labels given by:

$$\varphi(x_i x_{i+n-2k^*+2}) = \begin{cases} \varphi(x_{3k^*-n-1} x_{k^*}) + i, & 1 \leq i \leq 3k^* - n - 2 \\ \varphi(x_{k^*} x_{n-k^*-1}) + n - 3k^* + i + 2, & 3k^* - n - 1 \leq i \leq k^* - 1 \\ \frac{k^*(6n-7k^*-1)-n(n-1)+2}{2} - k, & i = k^* \\ \varphi(x_{n-k^*} x_{2n+3k^*}) - k^* + i, & k^* + 1 \leq i \leq n - k^* + 1 \\ \varphi(x_{2k^*-1} x_n) - n + k^* + i - 1, & n - k^* + 2 \leq i \leq 2k^* - 2 . \end{cases}$$

Further, the edges for $n - 2k^* + 3 \leq j \leq r - 3$ are labeled by Eq (2.2) and for $j = r - 2$, $j = r - 1$, and $j = r$ the edges are labeled as follows:

$$\varphi(x_i x_{i+r-2}) = \begin{cases} \varphi(x_2 x_{k^*}) + 1, & i = 1 \\ \varphi(x_{n-2k^*+3} x_{n-k^*+1}) + i - 1, & 2 \leq i \leq n - 2k^* + 2 \\ \varphi(x_{k^*} x_{2k^*-2}) - n + 2k^* + i - 2, & n - 2k^* + 3 \leq i \leq k^* \\ \varphi(x_{n-k^*+1} x_{n-1}) - k^* + i, & k^* + 1 \leq i \leq n - k^* + 1 . \end{cases}$$

$$\varphi(x_i x_{i+r-1}) = \begin{cases} \varphi(x_{n-k^*-r+3} x_{n-k^*+1}) + i, & 1 \le i \le n - 2k^* + 1 \\ \varphi(x_{k^*} x_{2k^*-1}) - n + 2k^* + i - 1, & n - 2k^* + 2 \le i \le k^* \\ \varphi(x_{n-k^*+1} x_n) - k^* + i, & k^* + 1 \le i \le n - k^* . \end{cases}$$

$$\varphi(x_i x_{i+r}) = \begin{cases} \varphi(x_{n-2k^*+1} x_{n-k^*+1}) + i, & 1 \le i \le n - 2k^* \\ \varphi(x_{k^*} x_{2k^*}) + i, & n - 2k^* + 1 \le i \le k^* . \end{cases}$$

An explanation of above labeling is depicted in Figure 5.

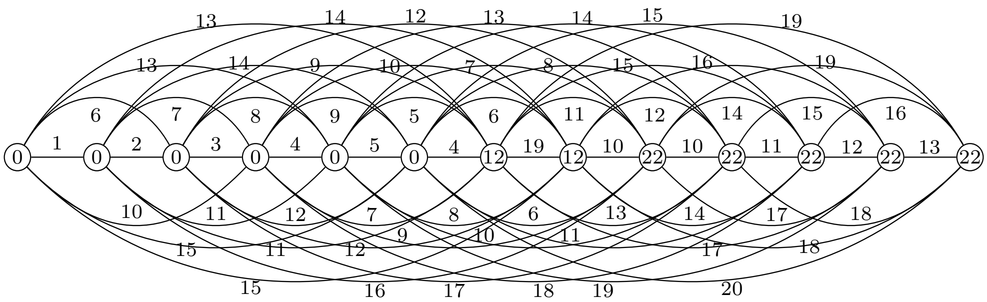

Figure 5. A reflexive irregular 22-labeling of P_{13}^{7}.

Also in this case the vertices are labeled with even number. Now we will estimate the weights of edges under the labeling φ:

We split the edge set of P_n^r into six mutually separated subsets, A_s, $1 \le s \le 6$ as follows: In Subcase 2.1.1.1, and Subcase 2.1.1.2 we have:

- $A_1 = \{x_i x_{i+j} : 1 \le j \le r, 1 \le i \le k^* - j\}$: The set of all edges which have endpoints tagged with 0,
- $A_2 = \{x_i x_{i+j} : 1 \le j \le n - 2k^* + 1, k^* - j + 1 \le i \le k^*\} \bigcup \{x_i x_{i+j} : n - 2k^* + 2 \le j \le r, k^* - j + 1 \le i \le n - k^* - j + 1\}$: The set of all edges which have endpoints tagged with 0 and $2\lceil \frac{k}{4} \rceil$,
- $A_3 = \{x_i x_{i+j} : 1 \le j \le n - 2k^*, k^* + 1 \le i \le n - 2k^* - j + 1\}$: The set of all edges which have endpoints tagged with $2\lceil \frac{k}{4} \rceil$,
- $A_4 = \{x_i x_{i+j} : 1 \le j \le n - 2k^* + 1, n - k^* - j + 2 \le i \le n - k^* + 1\} \bigcup \{x_i x_{i+j} : n - 2k^* + 2 \le j \le r, k^* + 1 \le i \le n - k^* + 1\}$: The set of all edges which have endpoints tagged with $2\lceil \frac{k}{4} \rceil$ and k,
- $A_5 = \{x_i x_{i+j} : 1 \le j \le k^* - 2, n - k^* + 2 \le i \le n - j\}$: The set of all edges which have endpoints tagged with k,
- $A_6 = \{x_i, x_{i+j}, n - 2k^* + 2 \le j \le r, n - k^* - j + 2 \le i \le k^*\}$: The set of all edges which have endpoints tagged with 0 and k.

In Subcase 2.1.1.3, $A_1 = \{x_i x_{i+j} : 1 \le j \le r - 1, 1 \le i \le k^* - j\}$ and other subsets A_s, $2 \le s \le 6$ as in the Subcase 2.1.1.1, and in Subcase 2.1.1.2, $A_1 = \{x_i x_{i+j} : 1 \le j \le k^* - 1, 1 \le i \le k^* - j\}$, $A_2 = \{x_i x_{i+j} : 1 \le j \le n - 2k^* + 1, k^* - j + 1 \le i \le k^*\} \bigcup \{x_i x_{i+j} : n - 2k^* + 2 \le j \le k^*, k^* - j + 1 \le i \le n - k^* - j + 1\} \bigcup \{x_1 x_{n-k^*+1}\}$ and other subsets A_s, $3 \le s \le 6$ as in the Subcase 2.1.1.1. Therefore, we obtain the edge weights for the Subcases 2.1.1.1, 2.1.1.2, and 2.1.1.3 as follows:

1. The edges of the set A_1, obtain weights from the set of sequential integers $\{1, 2, ..., \frac{r(2k^*-r-1)}{2}\}$,

2. The edges of the set A_2, obtain weights from the set of sequential integers $\{\frac{r(2k^*-r-1)}{2}+1,...,\frac{r(2k^*-r-1)}{2}+\frac{(n-2k^*+1)(2r-n+2k^*)}{2}\}$,
3. The edges of the set A_3, obtain weights from the set of sequential integers $\{rk^*+1,...,rk^*+\frac{(n-2k^*+1)(n-2k^*)}{2}\}$,
4. The edges of the set A_4, obtain weights from the set of sequential integers $\{rk^*+\frac{(n-2k^*+1)(n-2k^*)}{2}+1,...,\frac{r(2n-r-1)}{2}-\frac{(k^*-1)(k^*-4)}{2}+k^*+1)\}$,
5. The edges of the set A_5, get weights from the set of sequential integers $\{\frac{r(2n-r-1)}{2}-\frac{(k^*-1)(k^*-4)}{2}+k^*+2,...,\frac{r(2n-r-1)}{2}\}$,
6. Finally, the edges of the set A_6, get weights from the set of sequential integers $\{\frac{r(2k^*-r-1)}{2}+\frac{(n-2k^*+1)(2r-n+2k^*)}{2}+1,...,rk^*\}$.

An explanation of above corresponding weights is depicted in Figure 6.

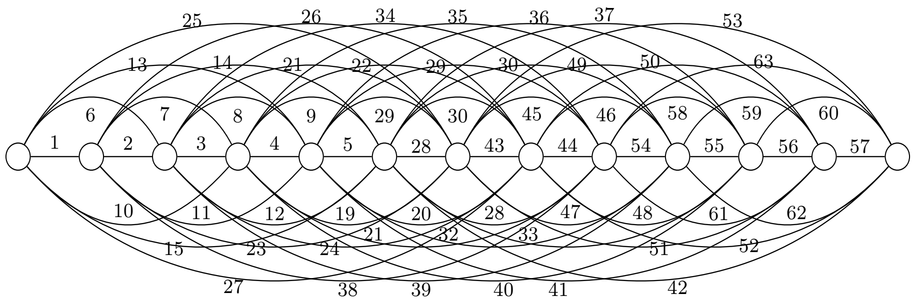

Figure 6. The edge weights of P_{13}^{7}.

In the Subcase 2.1.2, the edge weights are obtained as follows :

1. The edges of the set A_1, get weights from the set of sequential integers $\{1,2,...,\frac{k^*(k^*-1)}{2}\}$,
2. The edges of the set A_2, get weights from the set of sequential integers $\{\frac{k^*(k^*-1)}{2}+1,...,\frac{k^*(6n-7k^*-1)-n(n-1)+2}{2}-1)\}$,
3. The edges of the set A_3, admit weights from the set of sequential integers $\{k^{*2}+k^*+1,...,\frac{n(n+1)+2k^*(3k^*-2n)+2}{2}-1\}$,
4. The edges of the set A_4, admit weights from the set of sequential integers $\{\frac{n(n+1)+2k^*(3k^*-2n)+2}{2},...,k^*(n-k^*)+n-2\}$,
5. The edges of the set A_5, receive weights from the set of sequential integers $\{k^*(n-k^*)+n-1,...,\frac{r(2n-r-1)}{2}\}$,
6. Finally, The edges of the set A_6, receive weights from the set of sequential integers $\{\frac{k^*(6n-7k^*-1)-n(n-1)+2}{2},...,k^{*2}+k^*\}$.

It is not hard to see that the weights of edges are distinct numbers from the set $\{1,2,3,...,\frac{(r(2n-r-1)}{2}\}$.

Subcase 2.2. If $k^*=\frac{n}{2}$. Define the total k-labeling φ of P_n^r as follows:
The corresponding labeling for P_8^4 is illustrated in Figure 10. Otherwise we have following labeling:

$$\varphi(x_i)=\begin{cases}0, & 1\le i\le k^*-1\\ 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor, & k^*\le i\le k^*+1\\ k, & k^*+2\le i\le n\end{cases}$$

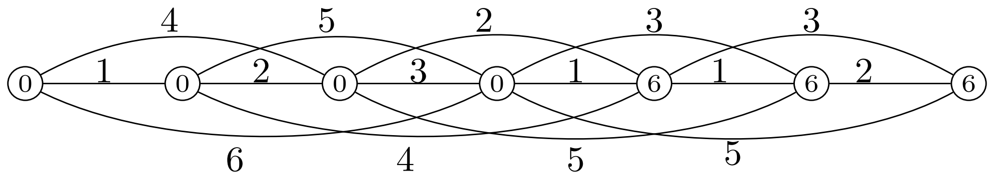

Figure 7. A reflexive irregular 6-labeling of P_7^3.

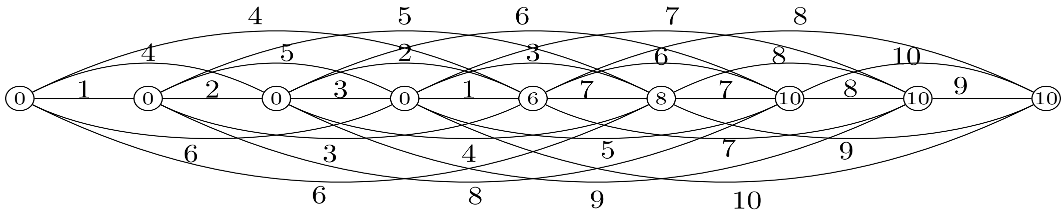

Figure 8. A reflexive irregular 10-labeling of P_9^5.

Now we define the edge labels as follows:
For $1 \leq j \leq 3$ we have two subcases.
Subcase 2.2.1 If $\frac{(k^*-1)(k^*+2)}{2} \geq k$,

$$\varphi(x_i x_{i+1}) = \begin{cases} i, & 1 \leq i \leq k^*-2 \\ \frac{(k^*-1)(k^*-2)}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + 1, & i = k^*-1 \\ \frac{k^*(4r-k^*-1)-r(r+1)}{2} - 4\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + 1, & i = k^* \\ \frac{k^*(4r-k^*-1)-r(r+1)}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor - k + 2, & i = k^*+1 \\ \frac{r(2n-r-1)}{2} - \frac{k^{*2}-k^*+4}{2} - 2k + i, & k^*+2 \leq i \leq n-1 \end{cases}$$

$$\varphi(x_i x_{i+2}) = \begin{cases} k^* + i - 2, & 1 \leq i \leq k^*-3 \\ \frac{k^{*2}-5k^*+2}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + 4 + i, & k^*-2 \leq i \leq k^*-1 \\ \frac{k^*(4r-k^*-3)-r(r+1)}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + i + 3, & k^* \leq i \leq k^*+1 \\ (1+r)n - \frac{r(r+1)}{2} - \frac{k^{*2}+k^*+8}{2} - 2k + i, & k^*+2 \leq i \leq n-2 \end{cases}$$

$$\varphi(x_i x_{i+3}) = \begin{cases} 2k^* + i - 5, & 1 \leq i \leq k^*-4 \\ \frac{k^{*2}-5k^*+2}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + 7 + i, & k^*-3 \leq i \leq k^*-2 \\ \frac{k^{*2}+k^*-2}{2} - k + 1, & i = k^*-1 \\ \frac{k^*(4r-k^*-3)-r(r+1)}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor - & \\ -k + i + 5, & k^* \leq i \leq k^*+1 \\ (2+r)n - \frac{r(r+1)}{2} - \frac{k^{*2}+3k^*+14}{2} - 2k + i, & k^*+2 \leq i \leq n-3 \end{cases}$$

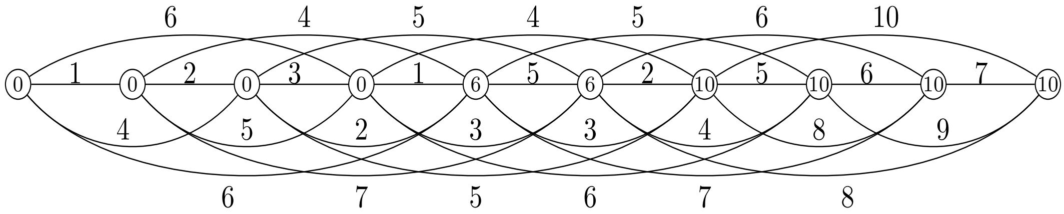

Figure 9. A reflexive irregular 10-labeling of P^4_{10}.

Subcase 2.2.2 If $\frac{(k^*-1)(k^*+2)}{2} < k$,

$$\varphi(x_i x_{i+1}) = \begin{cases} i, & 1 \le i \le k^*-2 \\ \frac{(k^*-1)(k^*-2)}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + 1, & i = k^*-1 \\ (r-k^*)(k^*-1) + \frac{(2k^*-r-1)(r-2)}{2} - \\ -4\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + k + 1, & i = k^* \\ (r-k^*)(k^*-1) + \frac{(2k^*-r-1)(r-2)}{2} - \\ -2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + 2, & i = k^*+1 \\ (r-k^*+2)(k^*-1) + \frac{(2k^*-r-1)(r-2)}{2} - \\ -k^* - k + i, & k^*+2 \le i \le n-1 \end{cases}$$

$$\varphi(x_i x_{i+2}) = \begin{cases} k^* + i - 2, & 1 \le i \le k^*-3 \\ \frac{(k^*-1)(k^*-2)}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor - \\ -k^* + i + 4, & k^*-2 \le i \le k^*-1 \\ (r-k^*)(k^*-1) + \frac{(2k^*-r-1)(r-2)}{2} - \\ -4\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor + k - k^* + i + 3, & k^* \le i \le k^*+1 \\ (r-k^*+2)(k^*-1) + \frac{(2k^*-r-1)(r-2)}{2} + \\ +n - 2k^* - k + i - 2, & k^*+2 \le i \le n-2 \end{cases}$$

$$\varphi(x_i x_{i+3}) = \begin{cases} 2k^* + i - 5, & 1 \le i \le k^*-4 \\ \frac{(k^*-1)(k^*-2)}{2} - 2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor - \\ -k^* + i + 7, & k^*-3 \le i \le k^*-2 \\ 1, & i = k^*-1 \\ (r-k^*)(k^*-1) + \frac{(2k^*-r-1)(r-2)}{2} - \\ -2\lfloor\frac{(k^*-1)(k^*-2)}{4}\rfloor - k^* + i + 5, & k^* \le i \le k^*+1 \\ (r-k^*+2)(k^*-1) + \frac{(2k^*-r-1)(r-2)}{2} + \\ +2n - 3k^* - k + i - 5, & k^*+2 \le i \le n-3 \end{cases}$$

For $4 \le j \le k^*-2$ (if $k^* \ge 6$),

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{k^*-j} x_{k^*-1}) + i, & 1 \le i \le k^*-j-1 \\ \varphi(x_{k^*-j-2} x_{k^*+1}) - k^* + j + i + 1, & k^*-j \le i \le k^*-j+1 \\ \varphi(x_{k^*-1} x_{k^*+j-2}) - k^* + j + i, & k^*-j+1 \le i \le k^*-1 \\ \varphi(x_{k^*+1} x_{k^*+j}) - k^* + i + 1, & k^* \le i \le k^*+1 \\ \varphi(x_{n-j-1} x_n) - k^* + i - 1, & k^*+2 \le i \le n-j\,. \end{cases}$$

$$\varphi(x_ix_{i+k^*-1}) = \begin{cases} \varphi(x_3x_{k^*+1}) + i, & 1 \le i \le 2 \\ \varphi(x_{k^*-1}x_{2k^*-3}) + i - 1, & 3 \le i \le k^* - 1 \\ \varphi(x_{k^*+1}x_{2k^*-1}) - k^* + i + 1, & k^* \le i \le k^* + 1 . \end{cases}$$

$$\varphi(x_ix_{i+k^*}) = \begin{cases} \varphi(x_2x_{k^*+1}) + 1, & i = 1 \\ \varphi(x_{k^*-1}x_{2k^*-2}) + i - 1, & 2 \le i \le k^* - 1 \\ \varphi(x_{k^*+1}x_{2k^*}) - k^* + i + 1, & k^* \le i \le k^* + 1 . \end{cases}$$

$$\varphi(x_ix_{i+k^*+1}) = \varphi(x_{k^*-1}x_{2k^*-1}) + i, \qquad k^* \le i \le n - k^* - 1 .$$

For $k^* + 2 \le j \le 2r - k^* + 1$,

$$\varphi(x_ix_{i+j}) = \varphi(x_{n-j+1}x_n) + i, \qquad 1 \le i \le n - j .$$

Thus the vertices are labeled with even numbers. Now we will calculate the weights of edges under

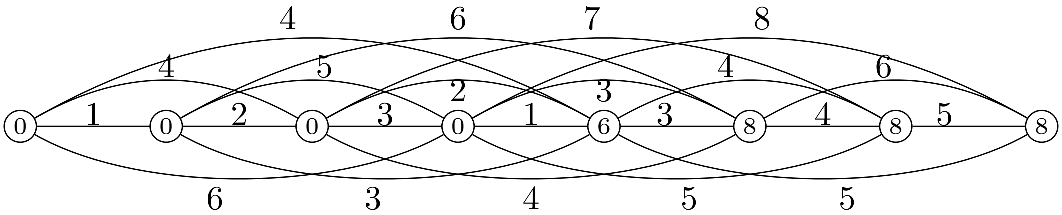

Figure 10. A reflexive irregular 8-labeling of P_8^4.

the labeling φ:
Like in the previous case the edge set of P_n^r can be partied into six mutually separated subsets A_s, $1 \le i \le 6$ as follows:

- $A_1 = \{x_ix_{i+j} : 1 \le j \le k^* - 2, 1 \le i \le k^* - j - 1\}$: The set of all edges which have endpoints tagged with 0,
- $A_2 = \{x_ix_{i+j} : 1 \le j \le 2, k^* - j \le i \le k^* - 1\} \bigcup \{x_ix_{i+j} : 3 \le j \le k^* - 1, k^* - j \le i \le k^* - j + 1\} \bigcup \{x_1x_{k^*+1}\}$: The set of all edges which have endpoints tagged with 0 and $2\lfloor \frac{(k^*-1)(k^*-2)}{4} \rfloor$,
- $A_3 = \{x_{k^*}x_{k^*+1}\}$: The set of only one edge which has endpoints $2\lfloor \frac{(k^*-1)(k^*-2)}{4} \rfloor$,
- $A_4 = \{x_{k^*}x_{k^*+j}, x_{k^*+1}x_{k^*+j+1} : 3 \le j \le n - 1, \} \bigcup \{x_{k^*+1}x_{k^*+2}, x_{k^*}x_n\}$: The set of all edges which have endpoints tagged with $2\lceil \frac{k}{4} \rceil$ and k,
- $A_5 = \{x_ix_{i+j} : 1 \le j \le k^* - 2, k^* + 2 \le i \le n - j\}$: The set of all edges which have endpoints tagged with k,
- $A_6 = \{x_ix_{i+j}, 3 \le j \le r, k^* - j + 2 \le i \le k^* - 1\}$: The set of all edges which have endpoints tagged with 0 and k.

Accordingly we obtain the edge weights as follows:
In the Subcase 2.2.1,

1. The edge weights of the set A_1, admit the successive numbers from the set $\{1, 2, ..., \frac{(k^*-1)(k^*-2)}{2}\}$,

2. The edge weights of the set A_2, receive the successive numbers from the set $\{\frac{(k^*-1)(k^*-2)}{2}+1,...,\frac{k^*(k^*+1)-2}{2}\}$,
3. The edge weight of the set A_3, admits the number $\{\frac{k^*(4r-k^*-1)-r(r+1)}{2}+1\}$;
4. The edge weights of the set A_4, admit the successive numbers from the set $\{\frac{k^*(4r-k^*-1)-r(r+1)}{2}+2,...,\frac{r(2n-r-1)}{2}-\frac{k^{*2}-3k^*}{2}\}$,
5. The edge weights of the set A_5, receive the successive numbers from the set $\frac{r(2n-r-1)}{2}-\frac{k^{*2}-3k^*}{2}+1,...,\frac{r(2n-r-1)}{2}\}$,
6. Finally, the edge weights of the set A_6, receive the successive numbers from the set $\{\frac{k^{*2}+k^*-2}{2}+1,...,\frac{k^*(4r-k^*-1)-r(r+1)}{2}\}$.

In the Subcase 2.2.2,

1. The edge weights of the set A_1, admit the successive numbers from the set $\{1,2,...,\frac{(k^*-1)(k^*-2)}{2}\}$,
2. The edge weights of the set A_2, receive the successive numbers from the set $\{\frac{(k^*-1)(k^*-2)}{2}+1,...,\frac{(k^*-1)(k^*+2)}{2}\}$,
3. The edge weight of the set A_3, admits the number $\{(r-k^*)(k^*-1)+\frac{(2k^*-r-1)(r-2)}{2}+k+1\}$,
4. The edge weights of the set A_4, admit the successive numbers from the set $\{(r-k^*)(k^*-1)+\frac{(2k^*-r-1)(r-2)}{2}+k,...,(r-k^*+2)(k^*-1)+\frac{(2k^*-r-1)(r-2)}{2}+k+1\}$,
5. The edge weights of the set A_5, receive the successive numbers from the set $\{(r-k^*+2)(k^*-1)+\frac{(2k^*-r-1)(r-2)}{2}+k+2,...,\frac{(2r-k^*+2)(k^*-1)}{2}+\frac{(2k^*-r-1)(r-2)}{2}+k+1\}$,
6. The edge weights of the set A_6, receive the successive numbers from the set $\{k+1,...,(r-k^*)(k^*-1)+\frac{(2k^*-r-1)(r-2)}{2}+k\}$.

Hence the edge weights in the Subcases 2.2.1 and 2.2.2 are distinct numbers from the sets $\{1,2,3,...,\frac{r(2n-r-1)}{2}\}$ and $\{1,2,3,...,\frac{(2r-k^*+2)(k^*-1)}{2}+\frac{(2k^*-r-1)(r-2)}{2}+k+1\}$ respectively.
Subcase 2.3 If $k^*>\frac{n}{2}$, then we have the following subcases:
Subcase 2.3.1 If n is odd, we construct the total k-labeling φ of P_n^r as follows:

$$\varphi(x_i)=\begin{cases}0, & 1\le i\le n-k^*-1\\ 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor, & n-k^*\le i\le\frac{n+1}{2}\\ 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor+2\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor, & \frac{n+3}{2}\le i\le k^*+1\\ k, & k^*+2\le i\le n\,.\end{cases}$$

Now, to define the edge labeling we have to consider the following two subcases:
Subcase 2.3.1.1 If $2k^*-n=1$,

$$\varphi(x_ix_{i+1})=\begin{cases}i, & 1\le i\le k^*-3\\ \frac{(k^*-2)(k^*-3)}{2}-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor+1, & i=k^*-2\end{cases}$$

$$\varphi(x_{k^*-1}x_{k^*})=\begin{cases}\frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+3)}{2}-\\ -4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-2, & \text{if } 4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor\ge k\\ \frac{(k^*-2)(k^*+3)}{2}-4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor+1, & \text{if } 4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor<k\,.\end{cases}$$

$$\varphi(x_ix_{i+1})=\begin{cases}\frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+7)}{2}-\\-4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-1, & i=k^*\\ \frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+3)}{2}-\\-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-k+1, & i=k^*+1\\ \frac{r(2n-r-1)}{2}-\frac{k^{*2}-3k^*+8}{2}-2k+i, & k^*+2\le i\le n-1\,.\end{cases}$$

$$\varphi(x_ix_{i+2})=\begin{cases}k^*-3+i, & 1\le i\le k^*-4\\ \frac{k^{*2}-7k^*+16}{2}-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor+i, & k^*-3\le i\le k^*-2\\ \frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+7)}{2}-\\-4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor, & i=k^*-1\\ \frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+3)}{2}-\\-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-k+1, & i=k^*\\ \frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+3)}{2}-\\-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-k+2, & i=k^*+1\\ \frac{(2n-r)(r+1)}{2}-\frac{k^{*2}-k^*-6}{2}-2k+i, & k^*+2\le i\le n-2\,.\end{cases}$$

$$\varphi(x_ix_{i+3})=\begin{cases}k^*+i-7, & 1\le i\le k^*-5\\ \frac{k^{*2}-7k^*+22}{2}-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor+i, & k^*-4\le i\le k^*-3\\ \frac{(k^*-2)(k^*-3)}{2}-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor+1, & i=k^*-2\\ \frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+5)}{2}-\\-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-k+i+3, & k^*-1\le i\le k^*\\ \frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+3)}{2}-\\-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-k+3, & i=k^*+1\\ \frac{(2n-r)(r+1)}{2}-\frac{k^{*2}+k^*+18}{2}+n-2k+i, & k^*+2\le i\le n-3\,.\end{cases}$$

$$\varphi(x_ix_{i+4})=\begin{cases}k^*+i-12, & 1\le i\le k^*-6\\ \frac{k^{*2}-7k^*+28}{2}-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor+i, & k^*-5\le i\le k^*-4\\ \frac{(k^*-2)(k^*-3)}{2}-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor+2, & i=k^*-3\,.\end{cases}$$

$$\varphi(x_{k^*-2}x_{k^*+2})=\begin{cases}\frac{(k^*-2)(k^*+3)}{2}-k+1, & \text{if } 4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor\ge k\\ \frac{(k^*-2)(k^*+3)}{2}-k+2, & \text{if } 4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor<k\,.\end{cases}$$

$$\varphi(x_ix_{i+4})=\begin{cases}\frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+5)}{2}-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-\\-k+i+5, & k^*-1\le i\le k^*\\ \frac{r(2n-r-1)}{2}-\frac{(k^*-2)(k^*+3)}{2}-2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor-\\-k+4, & i=k^*+1\\ \frac{(2n-r)(r+1)}{2}-\frac{k^{*2}+3k^*+26}{2}+2n-2k+i, & k^*+2\le i\le n-4\,.\end{cases}$$

For $5 \le j \le k^* - 3$ (if $k^* \ge 8$),

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{k^*-j-1} x_{k^*-2}) + i, & 1 \le i \le k^* - j - 2 \\ \varphi(x_{k^*-j+1} x_{k^*}) - k^* + j + i + 2, & k^* - j - 1 \le i \le k^* - j \\ \varphi(x_{k^*-j+2} x_{k^*+1}) + 1, & i = k^* - j + 1 \\ \varphi(x_{k^*-2} x_{k^*+j-3}) - k^* + j + i - 1, & k^* - j + 2 \le i \le k^* - 2 \\ \varphi(x_{k^*} x_{k^*+j-1}) - k^* + i + 2, & k^* - 1 \le i \le k^* \\ \varphi(x_{k^*+1} x_{k^*+j}) + 1, & i = k^* + 1 \\ \varphi(x_{n-j+1} x_n) - k^* + i - 1, & k^* + 2 \le i \le n - j\,. \end{cases}$$

$$\varphi(x_i x_{i+k^*-2}) = \begin{cases} \varphi(x_3 x_{k^*}) + i, & 1 \le i \le 2 \\ \varphi(x_4 x_{k^*+1}) + 1, & i = 3 \\ \varphi(x_{k^*-2} x_{2k^*-5}) - 3, & 4 \le i \le k^* - 2 \\ \varphi(x_{k^*} x_{2k^*-3}) - k^* + i + 2, & k^* - 1 \le i \le k^* \\ \varphi(x_{k^*+1} x_{2k^*-2}) + 1, & i = k^* + 1\,. \end{cases}$$

$$\varphi(x_i x_{i+k^*-1}) = \begin{cases} \varphi(x_2 x_{k^*}) + 1, & i = 1 \\ \varphi(x_3 x_{k^*+1}) + 1, & i = 2 \\ \varphi(x_{k^*-2} x_{2k^*-4}) + i - 2, & 3 \le i \le k^* - 2 \\ \varphi(x_{k^*} x_{2k^*-2}) - k^* + i + 2, & k^* - 1 \le i \le k^*\,. \end{cases}$$

$$\varphi(x_i x_{i+k^*}) = \begin{cases} \varphi(x_2 x_{k^*+1}) + 1, & i = 1 \\ \varphi(x_{k^*-2} x_{2k^*-3}) + 1, & 2 \le i \le k^* - 2 \\ \varphi(x_{k^*} x_{2k^*-1}) + 1, & i = k^* - 1\,. \end{cases}$$

$$\varphi(x_i x_{i+k^*+1}) = \varphi(x_{k^*-2} x_{2k^*-2}) + i, \qquad 1 \le i \le n - k^* - 1.$$

For $k^* + 2 \le j \le r$,

$$\varphi(x_i x_{i+j}) = \varphi(x_{n-j+1} x_n) + i, \qquad 1 \le i \le n - j.$$

An explanation of above labeling is depicted in Figure 11. In this case, we split edge set of the P_n^r into nine mutually separated subsets as follows:

- $A_1 = \{x_i x_{i+j} : 1 \le j \le k^* - 3, 1 \le i \le k^* - j - 2\}$: The set of all edges which have endpoints tagged with 0,
- $A_2 = \{x_i x_{i+j} : 2 \le j \le k^* - 2, k^* - j \le i \le k^* - j + 1\} \bigcup \{x_{k^*-2} x_{k^*-1}, x_1 x_{k^*}\}$: The set of all edges which have endpoints tagged with 0 and $2\lfloor \frac{(k^*-2)(k^*-3)}{4} \rfloor$,
- $A_3 = \{x_{k^*-1} x_{k^*}\}$: The set of only one edge which has endpoints labeled with $2\lfloor \frac{(k^*-2)(k^*-3)}{4} \rfloor$,
- $A_4 = \{x_{k^*-1} x_{k^*+1}, x_{k^*-1} x_{k^*+1}\}$: The set of two edges which has endpoints labeled with $2\lfloor \frac{(k^*-2)(k^*-3)}{4} \rfloor$ and $2k^* + 2\lfloor \frac{(k^*-2)(k^*-3)}{4} \rfloor - 4$,
- $A_5 = \{x_i x_{i+j} : 4 \le j \le k^* + 1, k^* - j + 2 \le i \le k^* - 2\} \bigcup \{x_i x_{i+j} : k^* + 1 \le j \le r, 1 \le i \le n - j\}$: The set of all edges which have endpoints tagged with 0 and k,

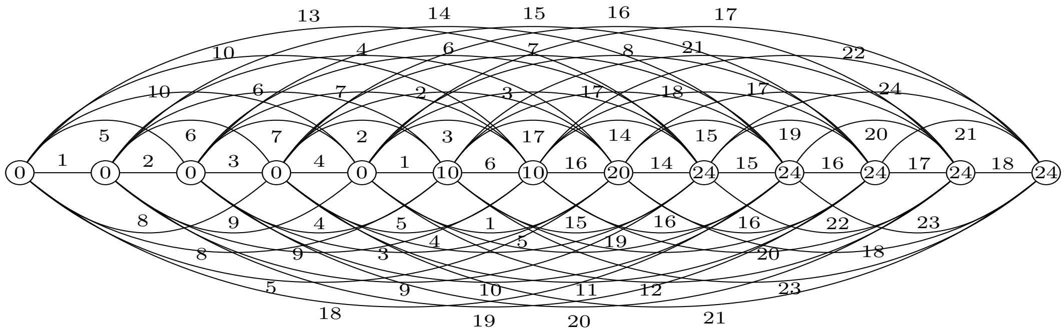

Figure 11. A reflexive irregular 24-labeling of P_{13}^{9}.

- $A_6 = \{x_{k^*-1}x_{k^*+j-1}, x_{k^*}x_{k^*+j} : 3 \le j \le k^* - 1\} \bigcup \{x_2x_{k^*+2}, x_{k^*-1}x_n\}$: The set of all edges which have endpoints tagged with $2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor$ and k,
- $A_7 = \{x_{k^*+1}x_{k^*+j} : 1 \le j \le k^* - 1\}$: The set of all edges which have endpoints tagged with $2k^* + 2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor - 4$ and k,
- $A_8 = \{x_{k^*-j+1}x_{k^*+1} : 3 \le j \le k^*\}$: The set of all edges which have endpoints tagged with 0 and $2k^* + 2\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor - 4$,
- $A_9 = \{x_ix_{i+j} : 1 \le j \le k^* - 3, k^* + 2 \le i \le n - j\}$: The set of all edges which have endpoints tagged with k.

Observe that under the total k-labeling φ the edge (edges):

1. from the set A_1, receive the weights from the successive numbers $\{1, 2, ..., \frac{(k^*-2)(k^*-3)}{2}\}$,
2. from the set A_2, receive the weights from the successive numbers $\{\frac{(k^*-2)(k^*-3)}{2} + 1, ..., \frac{(k^*-1)(k^*+2)}{2}\}$,
3. from the set A_3, receives the weight$\{\frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*+3)}{2} - 2\}$ if $4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor \ge k$ or receives the weight $\{\frac{(k^*-2)(k^*+3)}{2} + 1\}$ if $4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor < k$,
4. from the set A_4, admit the two weights $\{\frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*+3)}{2} - 1, \frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*+3)}{2}\}$,
5. from the set A_5, receive the weights from the successive numbers $\{\frac{(k^*-1)(k^*+2)}{2} + 1, ..., \frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*+3)}{2} - 3\}$, if $4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor \ge k$ or receive the weights from the set $\{\frac{(k^*-2)(k^*+3)}{2} + 2, ..., \frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*+3)}{2}\}$, if $4\lfloor\frac{(k^*-2)(k^*-3)}{4}\rfloor < k$,
6. from the set A_6, admit the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*+3)}{2} + 1, ..., \frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*+1)}{2}\}$,
7. from the set A_7, receive the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*-1)}{2} + 1, ..., \frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*-3)}{2}\}$,
8. from the set A_8, admit the weights from the successive numbers $\{\frac{(k^*-2)(k^*+1)}{2} + 1, ..., \frac{(k^*-2)(k^*+3)}{2}\}$,
9. from the set A_9, admit the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*-3)}{2}, ..., \frac{r(2n-r-1)}{2}\}$.

An explanation of above corresponding weights is depicted in Figure 12.

Subcase 2.3.1.2 If $2k^* - n \ne 1$, hence we define the edge labeling as follows:

$$\varphi(x_ix_{i+1}) = \begin{cases} i, & 1 \le i \le n - k^* - 2 \\ \frac{(n-k^*-1)(n-k^*-2)}{2} - 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + 1, & i = n - k^* - 1. \end{cases}$$

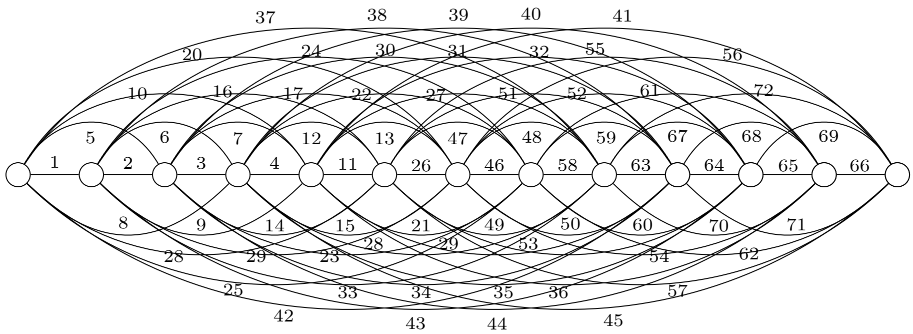

Figure 12. The edge weights of P_{13}^{9}.

For $n-k^* \le i \le \frac{n-1}{2}$,

$$\varphi(x_ix_{i+1}) = \begin{cases} \frac{r(2n-r-1)}{2} - \frac{(k^*-2)(k^*+1)}{2} - \\ -4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - n + i, & \text{if } 4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor \ge k \\ \frac{(n-k^*-1)(3k^*-n)}{2} - \\ -4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + i, & \text{if } 4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor < k \end{cases}$$

$$\varphi(x_ix_{i+1}) = \begin{cases} \frac{r(2n-r-1)}{2} - \frac{(4k^*(n-1)-n(n-4)-3)}{8} - \\ -4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - 2\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor + 1, & i = \frac{n+1}{2} \\ \frac{r(2n-r-1)}{2} - \frac{(n^2-4n+3)}{8} - 4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - \\ -4\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor - \frac{n+3}{2} + i + 1, & \frac{n+3}{2} \le i \le k^* \\ \frac{r(2n-r-1)}{2} - \frac{(n-k^*-1)(k^*-1)}{2} - 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - \\ -2\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor - k + 1, & i = k^*+1 \\ \frac{r(2n-r-1)}{2} - \frac{(n-k^*-1)(n-k^*-2)}{2} - 2k - k^* + i - 1, & k^*+2 \le i \le n-1\,. \end{cases}$$

For $1 < j \le \frac{2k^*-n-1}{2}$,

$$\varphi(x_ix_{i+j}) = \begin{cases} \varphi(x_{n-k^*-j}x_{n-k^*-1}) + i, & 1 \le i \le n-k^*-j-1 \\ \varphi(x_{n-k^*-1}x_{n-k^*+j-2}) - n + k^* + \\ +j + i + 1, & n-k^*-j \le i \le n-k^*-1 \\ \varphi(x_{\frac{n+1}{2}-j+1}x_{\frac{n+1}{2}}) - n + k^* + i + 1, & n-k^* \le i \le \frac{n+1}{2} - j \\ \varphi(x_{\frac{n+1}{2}-j+1}x_{\frac{n+1}{2}}) - \frac{n+1}{2} + j + i, & \frac{n+1}{2} - j + 1 \le i \le \frac{n+1}{2} \\ \varphi(x_{k^*-j+2}x_{k^*+1}) - \frac{n+3}{2} + i + 1, & \frac{n+3}{2} \le i \le k^*-j+1 \\ \varphi(x_{k^*+1}x_{k^*+j}) - k^* + j + i - 1, & k^*-j+2 \le i \le k^*+1 \\ \varphi(x_{n-j+1}x_n) - k^* + i - 1, & k^*+2 \le i \le n-j\,. \end{cases}$$

$$\varphi(x_ix_{i+\frac{2k^*-n+1}{2}}) = \begin{cases} \varphi(x_{\frac{n-3}{2}}x_{n-k^*-1}) + i, & 1 \le i \le \frac{3n-4k^*-3}{2} \\ \varphi(x_{n-k^*-1}x_{\frac{n+1}{2}}) - \frac{3n-4k^*-3}{2} + i, & \frac{3n-4k^*-1}{2} \le i \le n-k^*-1 \\ \varphi(x_{\frac{2n-2k^*+2}{2}}x_{\frac{n+1}{2}}) + 1, & i = n-k^* \\ \varphi(x_{n-k^*+1}) - n + k^* + i, & n-k^*+1 \le i \le \frac{n+1}{2} \\ \varphi(x_{k^*+1}x_{\frac{4k^*-n+1}{2}}) - \frac{n+1}{2} + i, & \frac{n+3}{2} \le i \le k^*+1 \\ \varphi(x_{\frac{3n-2k^*+1}{2}}x_n) - k^* + i - 1, & k^*+2 \le i \le n-j\,. \end{cases}$$

$$\varphi(x_i x_{i+\frac{2k^*-n+3}{2}}) = \begin{cases} \varphi(x_{\frac{3n-4k^*-3}{2}} x_{n-k^*-1}) + i, & 1 \le i \le \frac{3n-4k^*-5}{2} \\ \varphi(x_{n-k^*-1} x_{\frac{n-1}{2}}) - \frac{3n-4k^*-5}{2} + i, & \frac{3n-4k^*-3}{2} \le i \le n-k^*-1 \\ \varphi(x_{\frac{n+1}{2}} x_{k^*+1}) - n + k^* + i + 1, & n-k^* \le i \le \frac{n-1}{2} \\ \frac{4k^*(2k^*+5)-12k^*n+3n(n-4)+17}{8} + \\ +\frac{r(2n-r-1)}{2} - 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - k, & i = \frac{n+1}{2} \\ \varphi(x_{k^*+1} x_{\frac{4k^*-n+3}{2}}) - \frac{n+1}{2} + i, & \frac{n+3}{2} \le i \le k^*+1 \\ \varphi(x_{\frac{3n-2k^*-1}{2}} x_n) - k^* + i - 1, & k^*+2 \le i \le \frac{3n-2k^*-3}{2}\ . \end{cases}$$

$$\varphi(x_i x_{i+\frac{2k^*-n+5}{2}}) = \begin{cases} \varphi(x_{\frac{3n-4k^*-5}{2}} x_{n-k^*-1}) + i, & 1 \le i \le \frac{3n-4k^*-7}{2} \\ \varphi(x_{n-k^*-1} x_{\frac{n+1}{2}}) - \frac{3n-4k^*-5}{2} + i + 1, & \frac{3n-4k^*-5}{2} \le i \le n-k^*-2 \\ \frac{(n-k^*-1)(k^*+1)}{2} - 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - \\ -2\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor + 1 & i = n-k^*-1 \\ \varphi(x_{\frac{n-1}{2}} x_{k^*+1}) - n + k^* + i + 1, & n-k^* \le i \le \frac{n-3}{2} \\ \varphi(x_{\frac{n+1}{2}} x_{k^*+2}) - \frac{n-3}{2} + i, & \frac{n-1}{2} \le i \le \frac{n+1}{2} \\ \varphi(x_{k^*+1} x_{\frac{4k^*-n+5}{2}}) - \frac{n+1}{2} + i, & \frac{n+3}{2} \le i \le k^*+1 \\ \varphi(x_{\frac{3n-2k^*-3}{2}} x_n) - k^* + i - 1, & k^*+2 \le i \le \frac{3n-2k^*-5}{2}\ . \end{cases}$$

For $\frac{2k^*-n+7}{2} \le j \le 2k^*-n+2$,

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{n-k^*-j} x_{n-k^*-1}) + i, & 1 \le i \le n-k^*-j-1 \\ \varphi(x_{\frac{n+1}{2}-j+1} x_{\frac{n+1}{2}}) - n + k^* + j + i + 1, & n-k^*-j \le i \le \frac{n+1}{2} - j \\ \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) - \frac{n+1}{2} + j + i, & \frac{n+3}{2} - j \le i \le n-k^*-1 \\ \varphi(x_{k^*-j+2} x_{k^*+1}) - n + k^* + i + 1, & n-k^* \le i \le k^*-j+1 \\ \varphi(x_{\frac{n+1}{2}} x_{\frac{n+1}{2}+j-1}) - k^* + j + i - 1, & k^*-j+2 \le i \le \frac{n+1}{2} \\ \varphi(x_{k^*+1} x_{k^*+j}) - \frac{n+1}{2} + i, & \frac{n+3}{2} \le i \le k^*+1 \\ \varphi(x_{n-j+1} x_n) - k^* + i - 1, & k^*+2 \le i \le n-j\ . \end{cases}$$

$$\varphi(x_i x_{i+2k^*-n+3}) = \begin{cases} \varphi(x_{2n-3k^*-3} x_{n-k^*-1}) + i, & 1 \le i \le 2n-3k^*-4 \\ \varphi(x_{\frac{3n-4k^*-3}{2}-j+1} x_{\frac{n+1}{2}}) + \\ +3k^* - 2n + i + 4, & 2n-3k^*-3 \le i \le \frac{3n-4k^*-5}{2} \\ \varphi(x_{n-k^*-1} x_{k^*+1}) + \\ +i - \frac{3n-4k^*-5}{2}, & \frac{3n-4k^*-3}{2} \le i \le n-k^*-2 \end{cases}$$

$$\varphi(x_{n-k^*-1} x_{k^*+2}) = \begin{cases} \frac{(n-k^*-1)(3k^*-n+2)}{2} - k + 1, & \text{if} \quad 4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor \ge k \\ \frac{(12nk^*-8k^{*2}-12k^*+8n-3n^2-5)}{8} - \\ -k + 1, & \text{if} \quad 4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor < k \end{cases}$$

$$\varphi(x_i x_{i+2k^*-n+3}) = \begin{cases} \varphi(x_{\frac{n+1}{2}} x_{\frac{4k^*-n+5}{2}}) - \\ -n + k^* + i + 1, & n-k^* \le i \le \frac{n+1}{2} \\ \varphi(x_{k^*+1} x_{3k^*-n+3}) - \\ -\frac{n+1}{2} + i, & \frac{n+3}{2} \le i \le k^*+1 \\ \varphi(x_{2n-2k^*-2} x_n) - \\ -k^* + i - 1, & k^*+2 \le i \le 2n-2k^*-3\ . \end{cases}$$

For $2k^* - n + 4 \leq j \leq n - k^* - 2$ (if $2n - 3k^* - 5 > 0$),

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{n-k^*-j} x_{n-k^*-1}) + i, & 1 \leq i \leq n - k^* - j - 1 \\ \varphi(x_{\frac{n+1}{2}-j+1} x_{\frac{n+1}{2}}) - n + \\ +k^* + j + i + 1, & 1 \leq i \leq \frac{n+1}{2} - j \\ \varphi(x_{k^*-j+2} x_{k^*+1}) - \frac{n+1}{2} + \\ +j + i, & \frac{n+3}{2} - j \leq i \leq k^* - j + 1 \\ \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) - \\ -k^* + j + i - 1, & k^* - j + 2 \leq i \leq n - k^* - 1 \\ \varphi(x_{\frac{n+1}{2}} x_{\frac{n+1}{2}+j-1}) + k^* - \\ -n + i + 1, & n - k^* \leq i \leq \frac{n+1}{2} \\ \varphi(x_{\frac{k^*+1}{2}} x_{k^*+j}) - \frac{n+1}{2} + i, & \frac{n+3}{2} \leq i \leq k^* + 1 \\ \varphi(x_{n-j+1} x_n) - k^* + i - 1, & k^* + 2 \leq i \leq n - j \,. \end{cases}$$

$$\varphi(x_i x_{i+n-k^*-1}) = \begin{cases} \varphi(x_{\frac{2k^*-n+5}{2}} x_{\frac{n+1}{2}}) + i, & 1 \leq i \leq \frac{3k^*-n+3}{2} \\ \varphi(x_{2k^*-n+3} x_{k^*+1}) - \\ -\frac{2k^*-n+5}{2} + i + 1, & \frac{2k^*-n+5}{2} \leq i \leq \frac{4k^*-2n+4}{2} \\ \varphi(x_{n-k^*-1} x_{2n-2k^*-3}) - \\ -2k^* + n + i - 2, & 2k^* - n + 3 \leq i \leq n - k^* - 1 \\ \varphi(x_{\frac{n+1}{2}} x_{\frac{3n-2k^*-3}{2}}) - n + \\ +k^* + i + 1, & n - k^* \leq i \leq \frac{n+1}{2} \\ \varphi(x_{k^*+1} x_{n-1}) - \frac{n+1}{2} + \\ +k^* + i, & \frac{n+3}{2} \leq i \leq n - j \,. \end{cases}$$

For $n - k^* \leq j \leq \frac{n-3}{2}$,

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{\frac{n+3}{2}-j} x_{\frac{n+1}{2}}) + i, & 1 \leq i \leq \frac{n+1}{2} - j \\ \varphi(x_{k^*-j+2} x_{k^*+1}) - \frac{n+1}{2} + j + i, & \frac{n+3}{2} - j \leq i \leq k^* - j + 1 \\ \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) - k^* + j + i - 1, & k^* - j + 2 \leq i \leq n - k^* - 1 \\ \varphi(x_{\frac{n+1}{2}} x_{\frac{n-1}{2}+j}) - n + k^* + i + 1, & n - k^* \leq i \leq \frac{n+1}{2} \\ \varphi(x_{k^*+1} x_{k^*-j}) - \frac{n-1}{2} + i, & \frac{n+3}{2} \leq i \leq n - j \,. \end{cases}$$

$$\varphi(x_i x_{i+\frac{n-1}{2}}) = \begin{cases} \varphi(x_2 x_{\frac{n+1}{2}}) + i, & i = 1 \\ \varphi(x_{\frac{2k^*-n+5}{2}} x_{k^*+1}) + i - 1, & 2 \leq i \leq \frac{2k^*-n+3}{2} \\ \varphi(x_{n-k^*-1} x_{\frac{3n-2k^*-5}{2}}) - \frac{2k^*-n+3}{2} + i, & \frac{2k^*-n+5}{2} \leq i \leq n - k^* - 1 \\ \varphi(x_{\frac{n+1}{2}} x_{n-1}) - n + k^* + i + 1, & n - k^* \leq i \leq \frac{n+1}{2}. \end{cases}$$

For $\frac{n+1}{2} \leq j \leq k^*$,

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{k^*-j+2} x_{k^*+1}) + i, & 1 \leq i \leq k^* - j + 1 \\ \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) - k^* + j + i - 1, & k^* - j + 2 \leq i \leq n - k^* - 1 \\ \varphi(x_{n-j+1} x_n) - n + k^* + i, & n - k^* \leq i \leq n - j. \end{cases}$$

For $k^*+1 \leq j \leq r$,

$$\varphi(x_i x_{i+j}) = \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) + i, \qquad 1 \leq i \leq n-j.$$

Note that in this subcase the edge set of P_n^r can be partied into ten mutually separated subsets as follows:

- $A_1 = \{x_i x_{i+j} : 1 \leq j \leq n-k^*-2, 1 \leq i \leq n-k^*-j-1\}$: The set of all edges which have endpoints tagged with 0,
- $A_2 = \{x_i x_{i+j} : 1 \leq j \leq \frac{2k^*-n+3}{2}, n-k^*-j \leq i \leq n-k^*-1\} \bigcup \{x_i x_{i+j} : \frac{2k^*-n+5}{2} \leq j \leq n-k^*-2, n-k^*-j \leq i \leq \frac{n+1}{2}-j\} \bigcup \{x_i x_{i+j} : n-k^*-1 \leq j \leq \frac{n-1}{2}, 1 \leq i \leq \frac{n+1}{2}-j\}$: The set of all edges which have endpoints tagged with 0 and $2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor$,
- $A_3 = \{x_i x_{i+j} : 1 \leq j \leq \frac{2k^*-n+1}{2}, n-k^* \leq i \leq \frac{n+1}{2}-j\}$: The set of all edges which have endpoints tagged with $2\lfloor\frac{(n-k^*-1)(k^*-2)}{4}\rfloor$,
- $A_4 = \{x_i x_{i+j} : 1 \leq j \leq \frac{2k^*-n+1}{2}, \frac{n+3}{2}-j \leq i \leq \frac{n+1}{2}\} \bigcup \{x_i x_{i+\frac{2k^*-n+3}{2}} : n-k^* \leq i \leq \frac{n-1}{2}\} \bigcup \{x_i x_{i+j} : \frac{2k^*-n+5}{2} \leq j \leq 2k^*-n+1, n-k \leq i \leq k^*-j+1\}$: The set of all edges which have endpoints tagged with $2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor$ and $2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + 2\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor$,
- $A_5 = \{x_i x_{i+j} : 2k^*-n+3 \leq j \leq k^*+1, k^*-j+2 \leq i \leq n-k^*-1\} \bigcup \{x_i x_{i+j} : k^*+2 \leq j \leq r, 1 \leq i \leq n-j\}$: The set of all edges which have endpoints tagged with 0 and k,
- $A_6 = \{x_i x_{i+j} : \frac{2k^*-n+3}{2} \leq j \leq 2k^*-n+2, k^*-j+2 \leq i \leq \frac{n+1}{2}\} \bigcup \{x_i x_{i+j} : 2k^*-n+3 \leq j \leq \frac{n-1}{2}, n-k^* \leq i \leq \frac{n+1}{2}\} \bigcup \{x_i x_{i+j} : \frac{n+1}{2} \leq j \leq k^*, n-k^* \leq i \leq n-j\}$: The set of all edges which have endpoints tagged with $2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor$ and k,
- $A_7 = \{x_i x_{i+j} : 1 \leq j \leq \frac{2k^*-n+1}{2}, k^*+2 \leq i \leq k^*+j+1\} \bigcup \{x_i x_{i+j} : \frac{2k^*-n+3}{2} \leq j \leq n-k^*-1, \frac{n+3}{2} \leq i \leq k^*+1\} \bigcup \{x_i x_{i+j} : n-k^* \leq j \leq \frac{n-3}{2}, \frac{n+3}{2} \leq i \leq n-j\}$: The set of all edges which have endpoints tagged with $2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + 2\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor$ and k,
- $A_8 = \{x_i x_{i+j} : \frac{2k^*-n+5}{2} \leq j \leq 2k^*-n+2, \frac{n+3}{2}-j \leq i \leq n-k^*-1\} \bigcup \{x_i x_{i+j} : 2k^*-n+3 \leq j \leq \frac{n+1}{2}, \frac{n+3}{2}-j \leq i \leq k^*-j+1\} \bigcup \{x_i x_{i+j} : \frac{n+3}{2} \leq j \leq k^*, 1 \leq i \leq k^*-j+1\}$: The set of all edges which have endpoints tagged with 0 and $2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + 2\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor$,
- $A_9 = \{x_i x_{i+j} : 1 \leq j \leq \frac{(2k^*-n-1)}{2}, \frac{n+3}{2} \leq i \leq k^*+j-1\}$: The set of all edges which have endpoints tagged with $2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + 2\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor$,
- $A_{10} = \{x_i x_{i+j} : 1 \leq j \leq n-k^*-2, k^*+2 \leq i \leq n-j\}$: The set of all edges which have endpoints tagged with k.

It is obvious that under the total k-labeling φ the edge (edges):

1. from the set A_1, receive the weights from the successive numbers $\{1, 2, ..., \frac{(n-k^*-1)(n-k^*-2)}{2}\}$,
2. from the set A_2, receive the weights from the successive numbers $\{\frac{(n-k^*-1)(n-k^*-2)}{2}+1, ..., \frac{(n-k^*-1)(k^*+1)}{2}\}$,
3. from the set A_3, obtain the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{k^*(k^*+1)}{2} + 1, ..., \frac{r(2n-r-1)}{2} - \frac{(4k^*(n-1)-n(n-4)-3)}{8}\}$ if $4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor \geq k$ or obtain the weights from the set $\{\frac{(n-k^*-1)(3k^*-n+2)}{2}+1, ..., \frac{(12nk^*-8k^{*2}-12k^*+8n-3n^2-5)}{8}\}$ if $4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor < k$,
4. from the set A_4, admit the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{(4k^*(n-1)-n(n-4)-3)}{8} + 1, ..., \frac{r(2n-r-1)}{2} - \frac{12k^*n-4k^*(2k^*+5)-3n(n-4)-9}{8}\}$,

5. from the set A_5, admit the weights from the successive numbers $\{\frac{(n-k^*-1)(3k^*-n+2)}{2}+1, ..., \frac{r(2n-r-1)}{2}-\frac{(4k^*(n-1)-n(n-4)-3)}{8}\}$, if $4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor \geq k$ or admit the weights from the set $\{\frac{(12nk^*-8k^{*2}-12k^*+8n-3n^2-5)}{8}+1, ..., \frac{r(2n-r-1)}{2}-\frac{(4k^*(n-1)-n(n-4)-3)}{8}\}$, if $4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor < k$,
6. from the set A_6, admit the weights from the successive numbers $\{\frac{r(2n-r-1)}{2}-\frac{12k^*n-4k^*(2k^*+5)-3n(n-4)-9}{8}+1, ..., \frac{r(2n-r-1)}{2}-\frac{(n^2+4n+3)}{8}\}$,
7. from the set A_7, obtain the weights from the successive numbers $\{\frac{r(2n-r-1)}{2}-\frac{(n-k^*-1)(k^*-1)}{2}+1, ..., \frac{r(2n-r-1)}{2}-\frac{(n-k^*-1)(n-k^*-2)}{2}\}$,
8. from the set A_8, obtain the weights from the successive numbers $\{\frac{(n-k^*-1)(k^*+1)}{2}+1, ..., \frac{(n-k^*-1)(3k^*-n+2)}{2}\}$,
9. from the set A_9, admit the weights from the successive numbers $\{\frac{r(2n-r-1)}{2}-\frac{(n^2+4n+3)}{8}+1, ..., \frac{r(2n-r-1)}{2}-\frac{(n-k^*-1)(k^*-1)}{2}\}$,
10. from the set A_{10}, admit the weights from the successive numbers $\{\frac{r(2n-r-1)}{2}-\frac{(n-k^*-1)(n-k^*-2)}{2}+1, ..., \frac{r(2n-r-1)}{2}\}$.

Subcase 2.3.2 If n is even, therefore we construct the total k-labeling φ of P_n^r as follows:

$$\varphi(x_i) = \begin{cases} 0, & 1 \leq i \leq n-k^*-1 \\ 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor, & n-k^* \leq i \leq \frac{n}{2} \\ 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + 2\lfloor\frac{(n-k^*-1)(2k^*-n+2)}{4}\rfloor, & \frac{n+2}{2} \leq i \leq k^*+1 \\ k, & k^*+2 \leq i \leq n\,. \end{cases}$$

$$\varphi(x_ix_{i+1}) = \begin{cases} i, & 1 \leq i \leq n-k^*-2 \\ \frac{(n-k^*-1)(n-k^*-2)}{2} - 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + 1, & i = n-k^*-1 \\ \frac{(n-k^*-1)(3k^*-n)}{2} - 4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor + i, & n-k^* \leq i \leq \frac{n}{2}-1 \\ \frac{r(2n-r-1)}{2} - \frac{n(4k^*-n+2)}{8} - 4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - & \\ -2\lfloor\frac{(n-k^*-1)(2k^*-n+2)}{4}\rfloor + 1, & i = \frac{n}{2} \\ \frac{r(2n-r-1)}{2} - \frac{(2k^*-n+2)(3n-2k^*-4)}{8} - \frac{k^*(n-k^*-1)}{2} - & \\ -4\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - 4\lfloor\frac{(n-k^*-1)(2k^*-n+3)}{4}\rfloor - & \\ -\frac{n}{2} + i, & \frac{n}{2}+1 \leq i \leq k^* \\ \frac{r(2n-r-1)}{2} - \frac{k^*(n-k^*-1)}{2} - 2\lfloor\frac{(n-k^*-1)(n-k^*-2)}{4}\rfloor - & \\ -2\lfloor\frac{(n-k^*-1)(2k^*-n+2)}{4}\rfloor - k + 1, & i = k^*+1 \\ \frac{r(2n-r-1)}{2} - \frac{(n-k^*-1)(n-k^*-2)}{2} - 2k - k^* + i - 1, & k^*+2 \leq i \leq n-1\,. \end{cases}$$

For $2 \leq j \leq k^* - \frac{n}{2}$ (if $k^* - \frac{n}{2} \neq 1$),

$$\varphi(x_ix_{i+j}) = \begin{cases} \varphi(x_{n-k^*-j}x_{n-k^*-1}) + i, & 1 \leq i \leq n-k^*-j-1 \\ \varphi(x_{n-k^*-1}x_{n-k^*+j-2}) - n + k^* + & \\ +j+i+1, & n-k^*-j \leq i \leq n-k^*-1 \\ \varphi(x_{\frac{n}{2}-j+1}x_{\frac{n}{2}}) - n + k^* + i + 1, & n-k^* \leq i \leq \frac{n}{2}-j \\ \varphi(x_{\frac{n}{2}}x_{\frac{n}{2}+j-1}) - \frac{n}{2} + j + i, & \frac{n}{2}-j+1 \leq i \leq \frac{n}{2} \\ \varphi(x_{k^*-j+2}x_{k^*+1}) - \frac{n}{2} + i, & \frac{n}{2}+1 \leq i \leq k^*-j+1 \\ \varphi(x_{k^*+1}x_{k^*+j}) - k^* + j + i - 1, & k^*-j+2 \leq i \leq k^*+1 \\ \varphi(x_{n-j+1}x_n) - k^* + i - 1, & k^*+2 \leq i \leq n-j\,. \end{cases}$$

$$\varphi(x_i x_{i+k^*-\frac{n}{2}+1}) = \begin{cases} \varphi(x_{\frac{3n}{2}-2k^*-1} x_{n-k^*-1}) + i, & 1 \le i \le \frac{3n}{2} - 2k^* - 2 \\ \varphi(x_{n-k^*-1} x_{n-k^*}) - \\ -\frac{3n}{2} + 2k^* + i + 2, & \frac{3n}{2} - 2k^* - 1 \le i \le n - k^* - 1 \\ \varphi(x_{\frac{n}{2}} x_{k^*}) - n + k^* + i + 1, & n - k^* \le i \le \frac{n}{2} \\ \varphi(x_{k^*+1} x_{2k^*-\frac{n}{2}+1}) - \frac{n}{2} + i, & \frac{n}{2} + 1 \le i \le k^* + 1 \\ \varphi(x_{\frac{3n}{2}-k^*} x_n) - k^* + i - 1, & k^* + 2 \le i \le \frac{3n}{2} - k^* - 1. \end{cases}$$

$$\varphi(x_i x_{i+k^*-\frac{n}{2}+2}) = \begin{cases} \varphi(x_{\frac{3n}{2}-2k^*-2} x_{n-k^*-1}) + i, & 1 \le i \le \frac{3n}{2} - 2k^* - 3 \\ \varphi(x_{\frac{n}{2}-n-k^*-2} x_{\frac{n}{2}}) - \frac{3n}{2} + \\ +2k^* + i + 3, & \frac{3n}{2} - 2k^* - 2 \le i \le n - k^* - 2 \\ \frac{k^*(n-k^*-1)}{2} - 2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor - \\ -2\lfloor \frac{(n-k^*-1)(2k^*-n+2)}{4} \rfloor + 1, & i = n - k^* - 1 \\ \varphi(x_{\frac{n}{2}} x_{k^*+1}) - n + k^* + i + 1, & n - k^* \le i \le \frac{n}{2} - 1 \\ \frac{r(2n-r-1)}{2} - \frac{(n-k^*-1)(3k^*-n+2)}{2} - \\ -2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor - k + 1, & i = \frac{n}{2} \\ \varphi(x_{k^*+1} x_{2k^*-\frac{n}{2}+2}) - \frac{n}{2} + i, & \frac{n}{2} + 1 \le i \le k^* + 1 \\ \varphi(x_{\frac{3n}{2}-k^*-1} x_n) - k^* + i - 1, & k^* + 2 \le i \le \frac{3n}{2} - k^* - 2. \end{cases}$$

For $k^* - \frac{n}{2} + 3 \le j \le 2k^* - n + 2$,

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{n-k^*-j} x_{n-k^*-1}) + i, & 1 \le i \le n - k^* - j - 1 \\ \varphi(x_{\frac{n}{2}-j+1} x_{\frac{n}{2}}) - n + k^* + j + i + 1, & n - k^* - j \le i \le \frac{n}{2} - j \\ \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) - \frac{n}{2} + j + i, & \frac{n}{2} - j + 1 \le i \le n - k^* - 1 \\ \varphi(x_{\frac{n}{2}} x_{\frac{n}{2}+j-1}) - n + k^* + i + 1, & n - k^* \le i \le \frac{n}{2} \\ \varphi(x_{k^*+1} x_{k^*+j}) - \frac{n}{2} + i, & \frac{n}{2} + 1 \le i \le k^* + 1 \\ \varphi(x_{n-j+1} x_n) - k^* + i - 1, & k^* + 2 \le i \le n - j. \end{cases}$$

$$\varphi(x_i x_{i+2k^*-n+3}) = \begin{cases} \varphi(x_{2n-3k^*-3} x_{n-k^*-1}) + i, & 1 \le i \le 2n - 3k^* - 4 \\ \varphi(x_{\frac{3n}{2}-2k^*-2} x_{\frac{n}{2}}) - 2n + 3k^* + \\ +i + 4, & 2n - 3k^* - 3 \le i \le \frac{3n}{2} - 2k^* - 3 \\ \varphi(x_{n-k^*-1} x_{k^*+1}) - \frac{3n}{2} + 2k^* + \\ +i + 3, & \frac{3n}{2} - 2k^* - 2 \le i \le n - k^* - 2 \\ \frac{4k^*(n-k^*-1)+(2k^*-n+2)(3n-2k^*-4)}{8} + \\ -k + 1, & i = n - k^* - 1 \\ \varphi(x_{\frac{n}{2}} x_{2k^*-\frac{n}{2}+2}) - n + k^* + i + 1, & n - k^* \le i \le \frac{n}{2} \\ \varphi(x_{k^*+1} x_{3k^*-n+3}) - \frac{n}{2} + i, & \frac{n}{2} + 1 \le i \le k^* + 1 \\ \varphi(x_{2n-2k^*-2} x_n) - k^* + i - 1, & k^* + 2 \le i \le 2n - 2k^* - 3. \end{cases}$$

For $2k^* - n + 4 \leq j \leq n - k^* - 2$ (if $2n - 3k^* - 5 > 0$),

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{n-k^*-j} x_{n-k^*-1}) + i, & 1 \leq i \leq n - k^* - j - 1 \\ \varphi(x_{\frac{n}{2}-j+1} x_{\frac{n}{2}}) - n + k^* + j+ \\ +i + 1, & n - k^* - j \leq i \leq \frac{n}{2} - j \\ \varphi(x_{k^*-j+2} x_{k^*+1}) - \frac{n}{2} + j + i, & \frac{n}{2} - j + 1 \leq i \leq k^* - j + 1 \\ \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) - k^* + \\ +j + i - 1, & k^* - j + 2 \leq i \leq n - k^* - 1 \\ \varphi(x_{\frac{n}{2}} x_{\frac{n}{2}+j-1}) - n + k^* + i + 1, & n - k^* \leq i \leq \frac{n}{2} \\ \varphi(x_{k^*+1} x_{k^*+j}) - \frac{n}{2} + i, & \frac{n}{2} + 1 \leq i \leq k^* + 1 \\ \varphi(x_{n-j+1} x_n) - k^* + i - 1, & k^* + 2 \leq i \leq n - j\ . \end{cases}$$

$$\varphi(x_i x_{i+n-k^*-1}) = \begin{cases} \varphi(x_{k^*-\frac{n}{2}+2} x_{\frac{n}{2}}) - n + k^* + \\ +i + 1, & 1 \leq i \leq k^* - \frac{n}{2} + 1 \\ \varphi(x_{2k^*-n+3} x_{k^*+1}) - \frac{n}{2} + i, & k^* - \frac{n}{2} + 2 \leq i \leq 2k^* - n + 2 \\ \varphi(x_{n-k^*-1} x_{n-k^*-3}) - 2k^* + \\ +n + i - 2, & 2k^* - n + 3 \leq i \leq n - k^* - 1 \\ \varphi(x_{\frac{n}{2}} x_{\frac{3n}{2}-k^*-2}) - n + k^* + \\ +i + 1, & n - k^* \leq i \leq \frac{n}{2} \\ \varphi(x_{k^*+1} x_{n-1}) - \frac{n}{2} + i, & \frac{n}{2} + 1 \leq i \leq k^* + 1\ . \end{cases}$$

For $n - k^* \leq j \leq \frac{n}{2} - 1$,

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{\frac{n}{2}-j+1} x_{\frac{n}{2}}) + i, & 1 \leq i \leq \frac{n}{2} - j \\ \varphi(x_{k^*-j+2} x_{k^*+1}) - \frac{n}{2} + j + i, & \frac{n}{2} - j + 1 \leq i \leq k^* - j + 1 \\ \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) - k^* + \\ +j + i - 1, & k^* - j + 2 \leq i \leq n - k^* - 1 \\ \varphi(x_{\frac{n}{2}} x_{\frac{n}{2}+j-1}) - n + k^* + i + 1, & n - k^* \leq i \leq \frac{n}{2} \\ \varphi(x_{k^*+1} x_{k^*+j}) - \frac{n}{2} + i, & \frac{n}{2} + 1 \leq i \leq n - j. \end{cases}$$

For $\frac{n}{2} \leq j \leq k^*$,

$$\varphi(x_i x_{i+j}) = \begin{cases} \varphi(x_{k^*-j+2} x_{k^*+1}) + i, & 1 \leq i \leq k^* - j + 1 \\ \varphi(x_{n-k^*-1} x_{n-k^*+j-2}) - k^* + j + i - 1, & k^* - j + 2 \leq i \leq n - k^* - 1 \\ \varphi(x_{\frac{n}{2}} x_{\frac{n}{2}+j-1}) - n + k^* + i + 1, & n - k^* \leq i \leq n - j. \end{cases}$$

For $k^* + 1 \leq j \leq r$,

$$\varphi(x_i x_{i+j}) = \varphi(x_{n-j+1} x_n) + i, \qquad 1 \leq i \leq n - j.$$

As the pervious Subcase 2.3.1.2, the edge set can be divided into ten mutually separated subsets as follows:

- $A_1 = \{x_i x_{i+j} : 1 \leq j \leq n - k^* - 2, 1 \leq j \leq n - k^* - j - 1\}$: The set of all edges which have endpoints tagged with 0,

- $A_2 = \{x_ix_{i+j} : 1 \le j \le \frac{2k^*-n+2}{2}, n-k^*-j \le i \le n-k^*-1\} \bigcup \{x_ix_{i+j} : \frac{2k^*-n+4}{2} \le j \le n-k^*-1, n-k^*-j \le i \le \frac{n}{2}-j\} \bigcup \{x_ix_{i+j} : n-k^* \le j \le \frac{n}{2}-1, 1 \le i \le \frac{n}{2}-j\}$: The set of all edges which have endpoints tagged with 0 and $2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor$,
- $A_3 = \{x_ix_{i+j} : 1 \le j \le \frac{2k^*-n}{2}, n-k^* \le i \le \frac{n}{2}-j\}$: The set of all edges which have endpoints tagged with $2\lfloor \frac{(k^*-2)(k^*-3)}{4} \rfloor$,
- $A_4 = \{x_ix_{i+j} : 1 \le j \le \frac{2k^*-n+2}{2}, \frac{n+2}{2}-j \le i \le \frac{n}{2}\} \bigcup \{x_ix_{i+j} : \frac{2k^*-n+4}{2} \le j \le 2k^*-n+1, n-k^* \le i \le k^*-j+1\}$: The set of all edges which have endpoints tagged with $2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor$ and $2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor + 2\lfloor \frac{(n-k^*-1)(2k^*-n+2)}{4} \rfloor$,
- $A_5 = \{x_ix_{i+j} : 2k^*-n+3 \le j \le k^*+1, k^*-j+2 \le i \le n-k^*-1\} \bigcup \{x_ix_{i+j} : k^*+2 \le j \le r, 1 \le i \le n-j\}$: The set of all edges which have endpoints tagged with 0 and k,
- $A_6 = \{x_ix_{i+j} : \frac{2k^*-n+4}{2} \le j \le 2k^*-n+2, k^*-j+2 \le i \le \frac{n}{2}\} \bigcup \{x_ix_{i+j} : 2k^*-n+3 \le j \le k^*, n-k^* \le i \le n-j\}$: The set of all edges which have endpoints tagged with $2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor$ and k,
- $A_7 = \{x_ix_{i+j} : 1 \le j \le \frac{2k^*-n+2}{2}, k^*-j+2 \le i \le k^*+1\} \bigcup \{x_ix_{i+j} : \frac{2k^*-n+4}{2} \le j \le n-k^*-1, \frac{n}{2}+1 \le i \le k^*+1\} \bigcup \{n-k^* \le j \le \frac{n}{2}-1, \frac{n}{2}+1 \le i \le n-j\}$: The set of all edges which have endpoints tagged with $2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor + 2\lfloor \frac{(n-k^*-1)(2k^*-n+2)}{4} \rfloor$ and k,
- $A_8 = \{x_ix_{i+j} : \frac{2k^*-n+4}{2} \le j \le 2k^*-n+2, \frac{n}{2}-j+1 \le i \le n-k^*-1\} \bigcup \{x_ix_{i+j} : 2k^*-n+3 \le j \le \frac{n}{2}, \frac{n}{2}+1-j \le i \le k^*-j+1\} \bigcup \{x_ix_{i+j} : \frac{n}{2}+1 \le j \le k^*, 1 \le i \le k^*-j+1\}$: The set of all edges which have endpoints tagged with 0 and $2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor + 2\lfloor \frac{(n-k^*-1)(2k^*-n+2)}{4} \rfloor$,
- $A_9 = \{x_ix_{i+j} : 1 \le j \le \frac{(2k^*-n)}{2}, \frac{n+3}{2} \le i \le k^*+j-1\}$: The set of all edges which have endpoints tagged with $2\lfloor \frac{(n-k^*-1)(n-k^*-2)}{4} \rfloor + 2\lfloor \frac{(n-k^*-1)(2k^*-n+2)}{4} \rfloor$,
- $A_{10} = \{x_ix_{i+j} : 1 \le j \le n-k^*-2, k^*+2 \le i \le n-j\}$: The set of all edges which have endpoints tagged with k.

One can easily check that under the total k-labeling φ the edge (edges):

1. from the set A_1, get the weights from the successive numbers $\{1, 2, ..., \frac{(n-k^*-1)(n-k^*-2)}{2}\}$,
2. from the set A_2, get the weights from the successive numbers $\{\frac{(n-k^*-1)(n-k^*-2)}{2}+1, ..., \frac{k^*(n-k^*-1)}{2}\}$,
3. from the set A_3, get the weights from the successive numbers $\{\frac{(n-k^*-1)(3k^*-n+2)}{2}+1, ..., \frac{4k^*(n-k^*-1)+(2k^*-n+2)(3n-2k^*-4)}{8}\}$,
4. from the set A_4, get the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{n(4k^*-n+2)}{8}+1, ..., \frac{r(2n-r-1)}{2} - \frac{(2k^*-n+2)(3n-2k^*-4)}{8}\}$,
5. from the set A_5, admit the weights from the successive numbers $\{\frac{4k^*(n-k^*-1)+(2k^*-n+2)(3n-2k^*-4)}{8}+1, ..., \frac{r(2n-r-1)}{2} - \frac{n(4k^*-n+2)}{8}\}$,
6. from the set A_6, admit the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{(n-k^*-1)(3k^*-n+2)}{2}+1, ..., \frac{r(2n-r-1)}{2} - \frac{n(4k^*-n+2)}{8}\}$,
7. from the set A_7, obtain the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{k^*(n-k^*-1)}{2}+1, ..., \frac{r(2n-r-1)}{2} - \frac{(n-k^*-1)(n-k^*-2)}{2}\}$,
8. from the set A_8, obtain the weights from the successive numbers $\{\frac{k^*(n-k^*-1)}{2}+1, ..., \frac{(n-k^*-1)(3k^*-n+2)}{2}\}$,
9. from the set A_9, admit the weights from the successive numbers $\{\frac{r(2n-r-1)}{2} - \frac{(2k^*-n+2)(3n-2k^*-4)}{8} + 1, ..., \frac{r(2n-r-1)}{2} - \frac{(n-k^*-1)(3k^*-n+2)}{2}\}$,
10. from the set A_{10}, admit the weights from the successive numbers

$\{\frac{r(2n-r-1)}{2} - \frac{(n-k^*-1)(n-k^*-2)}{2} + 1, ..., \frac{r(2n-r-1)}{2}\}$.

By direct computation we can verify that in the both subcases all vertices are labeled by even numbers and the edge weights of the edges are different numbers from the set $\{1, 2, 3, ..., \frac{r(2n-r-1)}{2}\}$. Hence we prove that in all cases vertices are even numbers and the labels of edges are less than or equal to $k = \lceil\frac{r(2n-r-1)}{6}\rceil + 1$ where $\frac{r(2n-r-1)}{2} \equiv 2, 3 \pmod 6$ and less than or equal to $k = \lceil\frac{r(2n-r-1)}{6}\rceil$ where $\frac{r(2n-r-1)}{2} \not\equiv 2, 3 \pmod 6$. Thus the edge irregular reflexive strength of the r-th power of the path P_n^r:

$$res(P_n^r) = \begin{cases} \lceil\frac{r(2n-r-1)}{6}\rceil, & \text{if } \frac{r(2n-r-1)}{2} \not\equiv 2, 3 \pmod 6 \\ \lceil\frac{r(2n-r-1)}{6}\rceil + 1, & \text{if } \frac{r(2n-r-1)}{2} \equiv 2, 3 \pmod 6 \end{cases}.$$

□

3. Conclusions

In this paper we discussed the edge irregular reflexive k-labeling of the r-th power of the path P_n, where $r \geq 2$, $n \geq r+4$. Also, we computed the precise value of the reflexive edge strength of P_n^r, $r \geq 2$, $n \geq r + 4$. In the future, we would like to calculate the reflexive edge strength, res, for r-th power of other graphs.

Conflict of interest

The authors declare that they have no competing interest.

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