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Research article

More on proper nonnegative splittings of rectangular matrices

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Abstract: In this paper, we further investigate the single proper nonnegative splittings and double proper nonnegative splittings of rectangular matrices. Two convergence theorems for the single proper nonnegative splitting of a semimonotone matrix are derived, and more comparison results for the spectral radii of matrices arising from the single proper nonnegative splittings and double proper nonnegative splittings of different rectangular matrices are presented. The obtained results generalize the previous ones, and it can be regarded as the useful supplement of the results in [Comput. Math. Appl., 67: 136–144, 2014] and [Results. Math., 71: 93–109, 2017].

Keywords: rectangular matrix; proper nonnegative splitting; convergence; comparison theorems; Moore-Penrose inverse
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1. Introduction

Let $\mathbb{R}^{m\times n}$ denote the set of all real $m \times n$ matrices. $O \in \mathbb{R}^{m\times n}$ represents a matrix with all zero elements. For $A \in \mathbb{R}^{m\times n}$, the notation $A \geq O$ ($A > O$) denotes that all elements of matrix A are nonnegative (positive), and in this case matrix A is called nonnegative (positive) matrix. For two matrices $A,\ B \in \mathbb{R}^{m\times n}$, $A \geq B$ ($A > B$) means that $A - B \geq O$ ($A - B > O$). The nonnegative (positive) vectors, by identifying them with $n\times 1$ matrices, are denoted by $x \geq 0$ ($x > 0$). A real rectangular matrix A is said to be semimonotone if $A^\dagger \geq O$ [14], here $A^\dagger$ is the Moore-Penrose inverse of A, see [2, 20] or Section 2.

Real rectangular linear system of the form

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times 1}$, appear in many areas of applications, for example, finite difference discretization of partial differential equations with suitable boundary conditions. There are two forms of splitting iteration methods for solving the rectangular linear system (1.1):

(1). Assume A has the single splitting [4]

$$A = U - V, \tag{1.2}$$

then the approximate solution of (1.1) is generated by

$$x_{k+1} = U^{\dagger} V x_k + U^{\dagger} b, \tag{1.3}$$

where $U^{\dagger}$ is the Moore-Penrose inverse of U, the matrix $U^{\dagger}V$ is called the iteration matrix of (1.3). The splitting $A = U - V$ is called a proper splitting if $R(A) = R(U)$ and $N(A) = N(U)$ [4], where $R(\cdot)$ and $N(\cdot)$ is the range and kernel of a given matrix, respectively. It should be remarked that the uniqueness of proper splittings was provided in [13]. Let $\rho(C)$ be the spectral radius of the real square matrix C, then for the proper splitting $A = U - V$, the iteration scheme (1.3) converges to the minimal norm least squares solution $x = A^{\dagger}b$ of (1.1) for any initial vector x_0 if and only if $\rho(U^{\dagger}V) < 1$ [4, Corollary 1]. Note that if $A = U - V$ is not a proper splitting, the iteration scheme (1.3) may not converge to the minimal norm least squares solution $x = A^{\dagger}b$ of (1.1) for any initial vector x_0 even for $\rho(U^{\dagger}V) < 1$, see [4, 11]. If the iteration scheme (1.3) is convergent, then we say that the proper single splitting $A = U - V$ is a convergent splitting. The convergence of the iteration scheme (1.3) for proper splittings of A has been studied extensively in [4, 6, 7, 9, 11, 12, 14].

(2). Assume A has the double splitting

$$A = P - R - S, \tag{1.4}$$

then the approximate solution of (1.1) is generated by [9]

$$x_{k+1} = P^{\dagger} R x_k + P^{\dagger} S x_{k-1} + P^{\dagger} b \tag{1.5}$$

with the aid of the Moore-Penrose inverse of P. It should be remarked that the double splitting was first introduced by Woźnicki in [19] for nonsingular matrix, and was extended to rectangular matrices in [9, 11]. The iteration scheme (1.5) can be rewritten in the following equivalent form

$$\begin{pmatrix} x_{k+1} \\ x_k \end{pmatrix} = \begin{pmatrix} P^{\dagger}R & P^{\dagger}S \\ I & O \end{pmatrix} \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix} + \begin{pmatrix} P^{\dagger}b \\ 0 \end{pmatrix}, \ i = 1, 2, \cdots, \tag{1.6}$$

here I is the identity matrix with appropriate size, and $W = \begin{pmatrix} P^{\dagger}R & P^{\dagger}S \\ I & O \end{pmatrix}$ is the iteration matrix of (1.6). The splitting $A = P - R - S$ is called a double proper splitting if $R(A) = R(P)$ and $N(A) = N(P)$ [9]. For double proper splitting (1.4), the iterative method (1.5) or (1.6) converges to the unique least squares solution of minimum norm of (1.1) if and only if $\rho(W) < 1$. The convergence of the iteration scheme (1.6) for double proper splittings of A has been studied in [9, 11, 16].

Comparison theorems between the spectral radii of iteration matrices are useful tools to analyze the convergence rate of iteration methods or to judge the effectiveness of preconditioners [8–10, 12, 15]. Comparison theorems between the spectral radii of iteration matrices arising from different splittings of one matrix are actually the comparison of convergence rate between the different iteration methods, while comparison theorems between the spectral radii of matrices arising from the splittings of different matrices are in fact the comparison of effectiveness of different preconditioners [12, 15]. Some comparison theorems of single proper splittings of a semimonotone matrix are established recently in [3, 9, 11], and comparison theorems of single proper splittings of different semimonotone matrices are proposed in [3, 12]. Comparison theorems for double proper splittings of a rectangular matrix can be found in [1,3], and which for double proper splittings of different rectangular matrices can be found in [3, 9, 11].

In this paper, we further investigate the proper nonnegative splitting (see Section 2) of a rectangular matrix. New convergence theorems for the single proper nonnegative splitting of a semimonotone matrix are given, and the comparison theorems of proper nonegative splittings of different rectangular matrices are presented. The remainder of the paper is organized as follows. In Section 2, we give some relevant definitions, notations and earlier results, which are used in the paper. In Section 3, we present the new convergence theorems for the single proper nonnegative splitting of a semimonotone matrix and the comparison theorems of single proper nonnegative splittings of different semimonotone matrices. In Section 4, some comparison theorems of double proper nonegative splittings of different rectangular matrices are presented. We end this paper with some conclusions in Section 5.

2. Preliminaries

For a matrix $A \in \mathbb{R}^{m\times n}$, the matrix $X \in \mathbb{R}^{n\times m}$, satisfying the four Penrose equations: $AXA = A$, $XAX = X$, $(AX)^T = AX$ and $(XA)^T = XA$, is called the Moore-Penrose inverse of A (B^T denotes the transpose of B). It always exists and is unique, and is denoted by $A^\dagger$, i.e., $X = A^\dagger$, see [2, 20].

For nonnegative matrix, there are well known results which are shown next.

Lemma 2.1. [21, Theorem 2.20] *Let A be the nonnegaitve $n\times n$ matrix, then A has a nonnegative real eigenvalue equal to its spectral radius.*

Lemma 2.2. [21, Theorem 2.21] *Let A, B be $n \times n$ matrices, if $A \geq B \geq O$, then $\rho(A) \geq \rho(B)$.*

Lemma 2.3. [5, Theorem 2-1.11] *Let $B \geq O$ and $x \geq 0$ be such that $Bx - \alpha x \geq 0$, then $\alpha \leq \rho(B)$.*

Lemma 2.4. [21, Theorem 3.16] *Let $X \in \mathbb{R}^{n\times n}$ and $X \geq O$. Then $\rho(X) < 1$ if and only if $(I - X)^{-1}$ exists and $(I - X)^{-1} = \sum_{k=0}^{\infty} X^k \geq O$.*

Using the notation of the nonnegative matrix, single proper regular, single proper weak regular and single proper nonnegative splittings of a real rectangular matrix, which are the natural extensions of the regular, weak regular and nonnegative splittings of a real square matrix [5, 21], are defined as

Definition 2.5. *For $A \in \mathbb{R}^{m\times n}$, the splitting $A = U - V$ is called*

(1). *a single proper regular splitting if it is a proper splitting such that $U^\dagger \geq O$ and $V \geq O$* [7, Definition 1], [9, Definition 1.2]*;*
(2). *a single proper weak regular splitting of the first type if it is a proper splitting such that $U^\dagger \geq O$ and $U^\dagger V \geq O$; a proper weak regular splitting of the second type if it is a proper splitting such that $U^\dagger \geq O$ and $VU^\dagger \geq O$* [7, Definition 1], [9, Definition 1.2]*;*
(3). *a single proper nonnegative splitting if it is a proper splitting such that $U^\dagger V \geq O$* [11, Definition 3.1].

It should be remarked that Jena et al. [9] only considered the proper weak regular splitting of the first type, they name it as proper weak regular splitting. The existence of the proper splitting is discussed in [4], there is an example in [4] to show how to construct such splitting.

For the proper splitting $A = U - V$ of a semimonotone matrix A, the fact that $U = A + V$ is a proper splitting implies that $\rho(A^\dagger V) < 1$ and $I + A^\dagger V$ is invertible, so we have $U^\dagger = (I + A^\dagger V)^{-1} A^\dagger$ [14, Theorem 2.2] and $U^\dagger V = (I + A^\dagger V)^{-1} A^\dagger V$. The next lemma shows the relation between the eigenvalues of $U^\dagger V$ and $A^\dagger V$.

Lemma 2.6. [14, Lemma 2.6] *Let $A = U - V$ be a proper splitting of real $m \times n$ matrix A. Let μ_i, $1 \le i \le s$ and λ_j, $1 \le j \le s$ be the eigenvalues of $U^\dagger V$ and $A^\dagger V$, respectively. Then for every j, we have $1 + \lambda_j \neq 0$. Also, for every i, there exists j such that $\mu_i = \frac{\lambda_j}{1+\lambda_j}$, and for every j, there exists i such that that $\lambda_j = \frac{\mu_i}{1-\mu_i}$.*

The definitions of double proper regular, double proper weak regular and double proper nonnegative splittings for a real rectangular matrix can be given in a similar way.

Definition 2.7. *For $A \in \mathbb{R}^{m \times n}$, the splitting $A = P - R - S$ is called*

(1). *a double proper regular splitting if it is a proper splitting such that $P^\dagger \ge O$, $R \ge O$ and $S \ge O$* [9, Definition 3.4], [1, Definition 2.7]*;*
(2). *a double proper weak regular splitting if it is a proper splitting such that $P^\dagger \ge O$, $P^\dagger R \ge O$ and $P^\dagger S \ge O$* [9, Definition 3.5], [1, Definition 2.7]*;*
(3). *a double proper nonnegative splitting if it is a proper splitting such that $P^\dagger R \ge O$ and $P^\dagger S \ge O$* [11, Definition 4.1].

The double proper splittings of a rectangular matrix are generalizations of the double splittings of a square matrix. Double splittings of a square nonsingular matrix are given in [17–19].

3. Single proper nonnegative splittings

In this section, two convergence theorems of the single proper nonnegative splitting of a semimonotone matrix are given, and the comparison theorems of the single proper nonegative splittings of different semimonotone matrices are presented.

Recall that for the convergent proper nonnegative splitting $A = U - V$ of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$, $A^\dagger \ge U^\dagger$ holds, see [11, Theorem 3.9 (a)]. In fact, for the proper nonnegative splitting $A = U - V$ of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$, we have the same result, which is shown in the following lemma.

Lemma 3.1. *Let $A = U - V$ be a proper nonnegative splitting of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$, then $A^\dagger \ge U^\dagger$.*

Proof. Given that $A = U - V$ is a proper nonnegative splitting of a semimonotone matrix A, so we have $A^\dagger \ge O$ and $U^\dagger V \ge O$. The fact $A = U - V$ is a proper splitting yields $A^\dagger = (I - U^\dagger V)^{-1} U^\dagger$, so $U^\dagger = (I - U^\dagger V) A^\dagger = A^\dagger - U^\dagger V A^\dagger$ [4, Theorem 1]. Therefor $A^\dagger - U^\dagger = U^\dagger V A^\dagger \ge O$, i.e., $A^\dagger \ge U^\dagger$. □

Now we are going to the new convergence results.

Theorem 3.2. *Let $A = U - V$ be a proper nonnegative splitting of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$, and $U \geq O$, then $\rho(U^\dagger V) = \frac{\rho(A^\dagger U)-1}{\rho(A^\dagger U)} < 1$.*

Proof. Note that for semimonotone matrix A and $U \geq O$, we have $A^\dagger U \geq O$. The following proof is the same as that in [11, Lemma 3.4], we omit it here. □

Theorem 3.3. *Let $A = U - V$ be a proper nonnegative splitting of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$, and $V \geq O$, then $\rho(U^\dagger V) = \frac{\rho(A^\dagger V)}{1+\rho(A^\dagger V)} < 1$.*

Proof. Note that A is a semimonotone matrix and $V \geq O$, therefore $A^\dagger V \geq O$, the following proof is omitted because it is the same as that in [11, Lemma 3.5]. □

Remark 3.4. *For a general rectangular matrix A, $A^\dagger U \geq O$ or $A^\dagger V \geq O$ can guarantee the convergence of the single proper nonnegative splitting [11], while for a semimonotone matrix A, $U \geq O$ or $V \geq O$ is sufficient to ensure the convergence of the single proper nonnegative splitting. For the single proper regular or single proper weak regular splitting of a semimonotone matrix A, $\rho(U^\dagger V) = \frac{\rho(A^\dagger V)}{1+\rho(A^\dagger V)} < 1$ holds without additional conditions [4,9].*

The following example shows that even $U \geq O$, $\rho(U^\dagger V) < 1$ does not hold for the single proper nonnegative splitting of a general rectangular matrix.

Example 3.5. *Let $A = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{20} & -\frac{1}{8} & 0 \end{pmatrix}$ be splitted as $A = U - V$ with $U = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{15} & \frac{1}{4} & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{60} & \frac{3}{8} & 0 \end{pmatrix}$. Then we have $A^\dagger = \begin{pmatrix} 5 & 0 \\ 2 & -8 \\ 0 & 0 \end{pmatrix}$, $U^\dagger = \begin{pmatrix} 5 & 0 \\ -\frac{4}{3} & 4 \\ 0 & 0 \end{pmatrix}$ and $U^\dagger V = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{15} & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $A = U - V$ is a single proper nonnegative splitting of general rectangular matrix A. Although $U \geq O$, $\rho(U^\dagger V) = 1.5000 > 1$.* □

Another example given below demonstrates that the condition $U \geq O$ or $V \geq O$ can not be dropped for the single proper nonnegative splitting of a semimonotone matrix.

Example 3.6. *Let $A = \begin{pmatrix} 5 & -1 & 0 \\ -5 & 2 & 0 \end{pmatrix}$ be splitted as $A = U - V$ with $U = \begin{pmatrix} 0 & -1 & 0 \\ -8 & 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} -5 & 0 & 0 \\ -3 & -2 & 0 \end{pmatrix}$. Then $A^\dagger = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \geq O$, $U^\dagger V = \begin{pmatrix} \frac{3}{8} & \frac{1}{4} & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq O$, so $A = U - V$ is a single proper nonnegative splitting of semimonotone matrix A, but $U \ngeq O$, hence $\rho(U^\dagger V) < 1$ does not hold, in fact, $\rho(U^\dagger V) = 1.3211 > 1$.* □

In what follows, we consider the comparison results between the spectral radii of matrices arising from the single proper nonnegative splittings of different semimonotone matrices. Let $A_1, A_2 \in \mathbb{R}^{m\times n}$ be two semimonotone matrices, $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be the proper nonnegative splittings of A_1 and A_2, respectively. Comparing $\rho(U_1^\dagger V_1)$ with $\rho(U_2^\dagger V_2)$, we have the following comparison theorem.

Theorem 3.7. *Let $A_1, A_2 \in \mathbb{R}^{m\times n}$ be two semimonotone matrices, $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be the proper nonnegative splittings of A_1 and A_2 respectively. If $A_2^\dagger \geq A_1^\dagger$ and $U_2 \geq U_1 \geq O$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

Proof. As A_1 and A_2 are semimonotone matrices, $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ are the proper nonnegative splittings and $U_2 \geq U_1 \geq O$, it follows from Theorem 3.2 that $\rho(U_i^\dagger V_i) < 1$ for $i = 1,\ 2$. Thus all we need to show is $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$.

For $i = 1, 2$, we know that

$$\rho(U_i^\dagger V_i) = \frac{\rho(A_i^\dagger U_i) - 1}{\rho(A_i^\dagger U_i)}.$$

Note that $U_1 \geq O$, then $A_2^\dagger \geq A_1^\dagger$ and $U_2 \geq U_1 \geq O$ leads to $A_2^\dagger U_2 \geq A_1^\dagger U_1 \geq O$, and Lemma 2.2 yields $\rho(A_1^\dagger U_1) \leq \rho(A_2^\dagger U_2)$. Let $f(\lambda) = \frac{\lambda - 1}{\lambda}$, then $f(\lambda)$ is a strictly increasing function for $\lambda > 0$. Hence the inequality $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$ holds. □

From Theorem 3.7, the following corollaries can be obtained.

Corollary 3.8. *Let $A \in \mathbb{R}^{m\times n}$ be a semimonotone matrix, $A = U_1 - V_1 = U_2 - V_2$ be two proper nonnegative splittings of A. If $U_2 \geq U_1 \geq O$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

From Corollary 3.8, it is easy to see that for a semimonotone matrix A, the assumption $U_2 \geq U_1$ is equivalent to $V_2 \geq V_1$. Hence, based on Corollary 3.8 and Theorem 3.3, we can give out the similar result for different semimonotone matrices A_1 and A_2.

Theorem 3.9. *Let $A_1,\ A_2 \in \mathbb{R}^{m\times n}$ be two semimonotone matrices, $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be the proper nonnegative splittings of A_1 and A_2 respectively. If $A_2^\dagger \geq A_1^\dagger$ and $V_2 \geq V_1 \geq O$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

Proof. As A_1 and A_2 are semimonotone matrices, $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ are the proper nonnegative splittings and $V_2 \geq V_1 \geq O$, it follows from Theorem 3.3 that $\rho(U_i^\dagger V_i) < 1$ for $i = 1,\ 2$. Thus all we need to show is $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$.

For $i = 1, 2$, it follows from Theorem 3.3 that

$$\rho(U_i^\dagger V_i) = \frac{\rho(A_i^\dagger V_i)}{1 + \rho(A_i^\dagger V_i)}.$$

Note that $V_1 \geq O$, then $A_2^\dagger \geq A_1^\dagger$ and $V_2 \geq V_1 \geq O$ leads to $A_2^\dagger V_2 \geq A_1^\dagger V_1 \geq O$, and Lemma 2.2 yields $\rho(A_1^\dagger V_1) \leq \rho(A_2^\dagger V_2)$. Let $f(\lambda) = \frac{\lambda}{1+\lambda}$, then $f(\lambda)$ is a strictly increasing function for $\lambda > 0$. Hence the inequality $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2)$ holds. □

If we consider the proper nonnegative splittings $A = U_1 - V_1 = U_2 - V_2$ of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$, we have the next corollary.

Corollary 3.10. *Let $A \in \mathbb{R}^{m\times n}$ be a semimonotone matrix, $A = U_1 - V_1 = U_2 - V_2$ be two proper nonnegative splittings of A. If $V_2 \geq V_1 \geq O$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

Theorem 3.9 extends Theorem 6 in [12] from single proper regular splittings to single proper nonnegative splittings of different semimonotone matrices. Corollary 3.10 extends Theorem 3.2 in [9] from single proper regular splittings to single proper nonnegative splittings of a semimonotone matrix A.

An example given below to shows that $\rho(U_1^\dagger V_1) \le \rho(U_2^\dagger V_2) < 1$ holds under the conditions $A_2^\dagger \ge A_1^\dagger$ and $V_2 \ge V_1 \ge O$ for single proper nonnegative splittings instead of single proper regular splittings of semimonotone matrices A_1 and A_2.

Example 3.11. *Let* $A_1 = \begin{pmatrix} 4 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. *Set* $U_1 = \begin{pmatrix} 5 & -1 & 0 \\ 0 & 4 & 0 \end{pmatrix}$, $V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ *and* $U_2 = \begin{pmatrix} 5 & -1 & 0 \\ 0 & 4 & 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. *It is easy to see that* $A_1^\dagger = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$, $A_2^\dagger = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$ *and* $U_1^\dagger V_1 = \begin{pmatrix} \frac{1}{5} & \frac{1}{10} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $U_2^\dagger V_2 = \begin{pmatrix} \frac{3}{5} & \frac{1}{10} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Moreover,* $V_2 \ge V_1 \ge O$ *but* $U_1 = U_2 \not\ge O$. *So,* $A_1 = U_1 - V_1$ *and* $A_2 = U_2 - V_2$ *are two single proper nonnegative splittings, instead of single proper regular splittings, of semimonotone matrices* A_1 *and* A_2. *But we still have* $\rho(U_1^\dagger V_1) = 0.5 \le \rho(U_2^\dagger V_2) = 0.6$. □

In what follows, we are moving to present a comparison result when both proper nonnegative splittings $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ are convergent splittings.

Theorem 3.12. *Let* A_1 *and* A_2 *be two semimonotone matrices,* $A_1 = U_1 - V_1$ *and* $A_2 = U_2 - V_2$ *be the convergent proper nonnegative splittings of* A_1 *and* A_2 *respectively. Let* $x \ge 0$ *and* $y \ge 0$ *be two nonzero vectors such that* $U_1^\dagger V_1 x = \rho(U_1^\dagger V_1)x$ *and* $U_2^\dagger V_2 y = \rho(U_2^\dagger V_2)y$. *Suppose that either* $V_1 x \ge 0$ *with* $\rho(U_1^\dagger V_1)x > 0$ *or* $V_2 y \ge 0$ *with* $y > 0$ *and* $\rho(U_2^\dagger V_2)y > 0$. *Further, assume that* $A_1^\dagger \le A_2^\dagger$ *and* $O \le U_2^\dagger \le U_1^\dagger$. *Then*

$$\rho(U_1^\dagger V_1) \le \rho(U_2^\dagger V_2) < 1.$$

Proof. Let us consider the case of $V_1 x \ge 0$ with $\rho(U_1^\dagger V_1)x > 0$. It follows from the convergence of the proper nonnegative splitting $A_2 = U_2 - V_2$ and Lemma 2.4, we get $(I - U_2^\dagger V_2)^{-1} \ge O$, so that

$$\begin{aligned} A_1^\dagger \le A_2^\dagger &= (U_2 - V_2)^\dagger \\ &= [U_2(I - U_2^\dagger V_2)]^\dagger \\ &= (I - U_2^\dagger V_2)^{-1} U_2^\dagger \\ &\le (I - U_2^\dagger V_2)^{-1} U_1^\dagger. \end{aligned}$$

Multiplying it on the right of both sides by $V_1 x$ gets

$$A_1^\dagger V_1 x \le (I - U_2^\dagger V_2)^{-1} U_1^\dagger V_1 x.$$

Note that $A_1^\dagger V_1 x = (I - U_1^\dagger V_1)^{-1} U_1^\dagger V_1 x = \frac{\rho(U_1^\dagger V_1)}{1-\rho(U_1^\dagger V_1)} x$ and $U_1^\dagger V_1 x = \rho(U_1^\dagger V_1)x$, we have

$$\frac{\rho(U_1^\dagger V_1)}{1 - \rho(U_1^\dagger V_1)} x \le \rho(U_1^\dagger V_1)(I - U_2^\dagger V_2)^{-1} x,$$

i.e.,

$$\frac{1}{1-\rho(U_1^{\dagger}V_1)}x \le (I - U_2^{\dagger}V_2)^{-1}x,$$

which, by Lemma 2.3, implies

$$\frac{1}{1-\rho(U_1^{\dagger}V_1)} \le \frac{1}{1-\rho(U_2^{\dagger}V_2)}.$$

Therefore, the required inequality $\rho(U_1^{\dagger}V_1) \le \rho(U_2^{\dagger}V_2)$ holds.

The case of $V_2y \ge 0$ with $y > 0$ and $\rho(U_2^{\dagger}V_2)y > 0$ can be proved in a similar way. □

When we pay our attention to different convergent proper nonnegative splittings of a semimonotone matrix A, from Theorem 3.12, the next corollary is obtained.

Corollary 3.13. *Let A be a semimonotone matrix, $A = U_1 - V_1 = U_2 - V_2$ be convergent proper nonnegative splittings of A. Let $x \ge 0$ and $y \ge 0$ be two nonzero vectors such that $U_1^{\dagger}V_1x = \rho(U_1^{\dagger}V_1)x$ and $U_2^{\dagger}V_2y = \rho(U_2^{\dagger}V_2)y$. Suppose that either $V_1x \ge 0$ with $\rho(U_1^{\dagger}V_1)x > 0$ or $V_2y \ge 0$ with $y > 0$ and $\rho(U_2^{\dagger}V_2)y > 0$. Further, assume that $O \le U_2^{\dagger} \le U_1^{\dagger}$. Then*

$$\rho(U_1^{\dagger}V_1) \le \rho(U_2^{\dagger}V_2) < 1.$$

In addition to the requirement of A be semimonotone, Corollary 3.13 is the same as Theorem 3.11 in [11]. For single proper regular splittings $A = U_1 - V_1 = U_2 - V_2$, [9] has a more concise result, see Theorem 3.3 of [9].

4. Double proper nonnegative splittings

In this part, we will provide the comparison theorem of double proper nonnegative splittings of different rectangular matrices.

Let $A_1,\ A_2 \in \mathbb{R}^{m\times n}$, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be double proper nonnegative splittings of A_1 and A_2, respectively. Then, we define

$$W_1 = \begin{pmatrix} P_1^{\dagger}R_1 & P_1^{\dagger}S_1 \\ I & 0 \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} P_2^{\dagger}R_2 & P_2^{\dagger}S_2 \\ I & 0 \end{pmatrix}.$$

First result comparing $\rho(W_1)$ with $\rho(W_2)$ is stated as the following theorem, which concerns the semimonotone matrices A_1 and A_2.

Theorem 4.1. *Let $A_1,\ A_2 \in \mathbb{R}^{m\times n}$ be two semimonotone matrices having the same null space, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be their double proper nonnegative splittings such that $P_1 \ge O$ and $P_2 \ge O$. If $P_1^{\dagger}A_1 \ge P_2^{\dagger}A_2$ and $P_1^{\dagger}S_1 \le P_2^{\dagger}S_2$, then*

$$\rho(W_1) \le \rho(W_2) < 1.$$

Proof. Note that A_1 and A_2 are semimonotone matrices and $P_1 \ge O$ and $P_2 \ge O$, then it follows from [11, Theorem 4.5] that both double proper nonnegative splittings are convergent, i.e., $\rho(W_1) < 1$ and $\rho(W_2) < 1$. Assume that $\rho(W_1) = 0$, then the conclusion holds clearly. Assume that $\rho(W_1) \ne 0$, from Definition 2.7 we have $W_1,\ W_2 \geq O$, then by Lemma 2.1 (Perron-Frobenius theorem), there exists a vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0,\ x \neq 0,$$

in conformity with W_1 such that $W_1 x = \rho(W_1)x$, i.e.,

$$\begin{aligned} P_1^{\dagger} R_1 x_1 + P_1^{\dagger} S_1 x_2 &= \rho(W_1) x_1, \\ x_1 &= \rho(W_1) x_2. \end{aligned}$$

Hence, we have

$$\begin{aligned} W_2 x - \rho(W_1)x &= \begin{pmatrix} P_2^{\dagger} R_2 x_1 + P_2^{\dagger} S_2 x_2 - \rho(W_1) x_1 \\ x_1 - \rho(W_1) x_2 \end{pmatrix} \\ &= \begin{pmatrix} (P_2^{\dagger} R_2 - P_1^{\dagger} R_1) x_1 - \frac{1}{\rho(W_1)} (P_1^{\dagger} S_1 - P_2^{\dagger} S_2) x_1 \\ 0 \end{pmatrix} \\ &:= \begin{pmatrix} \Delta \\ 0 \end{pmatrix}. \end{aligned}$$

Since A_1 and A_2 have the same null space, then $P_1^{\dagger} P_1 = P_2^{\dagger} P_2$ [9,11]. As $P_1^{\dagger} S_1 \leq P_2^{\dagger} S_2$ and $0 < \rho(W_1) < 1$ then

$$\begin{aligned} \Delta &= (P_2^{\dagger} R_2 - P_1^{\dagger} R_1) x_1 - \frac{1}{\rho(W_1)} (P_1^{\dagger} S_1 - P_2^{\dagger} S_2) x_1 \\ &\geq (P_2^{\dagger} (P_2 - A_2) x_1 - P_1^{\dagger} (P_1 - A_1) x_1). \end{aligned}$$

Therefore, in terms of $P_1^{\dagger} A_1 \geq P_2^{\dagger} A_2$, we have

$$\begin{aligned} W_2 x - \rho(W_1)x &\geq \begin{pmatrix} P_2^{\dagger} (P_2 - A_2) x_1 - P_1^{\dagger} (P_1 - A_1) x_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (P_1^{\dagger} A_1 - P_2^{\dagger} A_2) x_1 \\ 0 \end{pmatrix} \\ &\geq 0. \end{aligned}$$

Thus, by Lemma 2.3, we have $\rho(W_1) \leq \rho(W_2) < 1$. □

When we consider the double proper nonnegative splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ of a semimonotone matrix A, the following Corollary is a direct result of Theorem 4.1.

Corollary 4.2. *Let $A \in \mathbb{R}^{m \times n}$ be a semimonotone matrix, $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be double proper nonnegative splittings such that $P_1 \geq O$ and $P_2 \geq O$. If $P_1^{\dagger} \geq P_2^{\dagger}$ and $P_1^{\dagger} S_1 \leq P_2^{\dagger} S_2$, then*

$$\rho(W_1) \leq \rho(W_2) < 1.$$

As for general rectangular matrices A_1 and A_2, comparing $\rho(W_1)$ with $\rho(W_2)$, we have the following comparison result, which is a slight modification of Theorem 4.1.

Theorem 4.3. *Let $A_1,\ A_2 \in \mathbb{R}^{m\times n}$ be two matrices having the same null space, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be their double proper nonnegative splittings such that $A_1^{\dagger}P_1 \geq O$ and $A_2^{\dagger}P_2 \geq O$. If $P_1^{\dagger}A_1 \geq P_2^{\dagger}A_2$ and $P_1^{\dagger}S_1 \leq P_2^{\dagger}S_2$, then*

$$\rho(W_1) \leq \rho(W_2) < 1.$$

The next example shows that the converse of Theorem 4.3 is not true.

Example 4.4. *Let* $A_1 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -4 & 0 \end{pmatrix}$, *$A_1$ and A_2 have the same null space. If A_1 and A_2 be splitted as $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$, respectively, here* $P_1 = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -4 & 0 \end{pmatrix}$, $R_1 = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, $S_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ *and* $P_2 = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -5 & 0 \end{pmatrix}$, $R_2 = \begin{pmatrix} -3 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, $S_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *then we have* $P_1^{\dagger}A_1 = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2^{\dagger}A_2 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & \frac{4}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_1^{\dagger}R_1 = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2^{\dagger}R_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_1^{\dagger}S_1 = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2^{\dagger}S_2 = \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_1^{\dagger}P_1 = \begin{pmatrix} \frac{5}{2} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $A_2^{\dagger}P_2 = \begin{pmatrix} 3 & \frac{5}{8} & 0 \\ 0 & \frac{5}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *So $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ are two double proper nonnegative splittings which satisfy the conditions $A_1^{\dagger}P_1 \geq O$ and $A_2^{\dagger}P_2 \geq O$. We then have $\rho(W_1) = 0.6899 < 0.7287 = \rho(W_2)$, but $P_1^{\dagger}S_1 \nleq P_2^{\dagger}S_2$, $P_1^{\dagger}A_1 \ngeq P_2^{\dagger}A_2$.* □

For general rectangular matrices A_1 and A_2, comparing $\rho(W_1)$ with $\rho(W_2)$, we also have comparison result:

Theorem 4.5. *Let $A_1,\ A_2 \in \mathbb{R}^{m\times n}$ be two matrices, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be their double proper nonnegative splittings. If $P_1^{\dagger}S_1 \leq P_2^{\dagger}S_2$ and $P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \leq P_2^{\dagger}R_2 - P_1^{\dagger}R_1$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

Theorem 4.5 is a generalization of [1, Theorem 4.9], the proof is similar to that of [1, Theorem 4.9], hence we omit it.

What we need to pay attention to here is that when A_1 and A_2 have the same null space, the assumption $P_1^{\dagger}S_1 - P_2^{\dagger}S_2 \leq P_2^{\dagger}R_2 - P_1^{\dagger}R_1$ in Theorem 4.5 becomes $P_1^{\dagger}A_1 \geq P_2^{\dagger}A_2$, so we have the following corollary.

Corollary 4.6. *Let $A_1,\ A_2 \in \mathbb{R}^{m\times n}$ be two matrices having the same null space, $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be their double proper nonnegative splittings. If $P_1^{\dagger}A_1 \geq P_2^{\dagger}A_2$ and $P_1^{\dagger}S_1 \leq P_2^{\dagger}S_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

5. Conclusion

In this paper, new convergence theorems for single proper nonnegative splitting of a semimonotone matrix, and some comparison theorems for single and double proper nonnegative splittings of different rectangular matrices are given. The obtained results generalize the corresponding results in [1,3,9,12] and supplement the comparison results of proper nonnegative spllitings of matrices in [9,11]. Applying the comparison results to judge the efficiency of the preconditioners for rectangular linear systems need further study.

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Conflict of interest

The authors declare no conflict of interest.

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