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Research article

Fractional convex type contraction with solution of fractional differential equation

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Abstract: The focus of this paper is to present a new idea of fractional convex type contraction and establish some new results for such contraction under the improved approach of fractional convex type contractive condition in the context of $\mathcal{F}$-complete $\mathcal{F}$-metric space. The authors derive some results for Suzuki type contractions, orbitally T-complete and orbitally continuous mappings in $\mathcal{F}$-metric spaces and obtain some consequences by using graphic contraction. The motivation of this paper is to observe the solution of fractional order differential equation with one of the boundary condition using fixed point technique in $\mathcal{F}$-metric space.

Keywords: fractional convex α-η-contraction; $\mathcal{F}$-metric space; $\mathcal{F}$-Cauchy; $\mathcal{F}$ -complete
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1. Preliminaries and scope

Fixed Point Theory has been generalized in different ways by Reich [11, 16]. Erdal et al. [12], initiate an idea of interpolative type contractions and established some new fixed point results in partial metric space. Recently, Jleli and Samet [10] introduced a new generalization of metric space name it as $\mathcal{F}$-metric space.

Definition 1.1. [10] Let $\mathcal{F}$ be the set of functions $f : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:
($\mathcal{F}_1$) f is non decreasing, i.e., for all $0 < s < t, \Longrightarrow f(s) \leq f(t)$;
($\mathcal{F}_2$) For every sequence $\{t_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to+\infty} t_n = 0 \iff \lim_{n\to+\infty} f(t_n) = -\infty.$$

The generalized notion of metric space is as follows:

Definition 1.2. [10] Let X be a nonempty set and let $D : X \times X \to [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \mu) \in \mathcal{F} \times [0, +\infty)$ such that (D_1) $(x, y) \in X \times X$, $D(x, y) = 0 \iff x = y$.

(D_2) $D(x, y) = D(y, x)$ for all $(x, y) \in X \times X$.

(D_3) For every $(x, y) \in X \times X$, for every $N \in \mathbb{N}$, $N \geq 2$, and for every $(u_i)_{i=1}^{N} \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$D(x, y) > 0 \implies f(D(x, y)) \leq f(\sum_{i=1}^{N-1} d(u_i, u_{i+1})) + \mu.$$

Then D is said to be an $\mathcal{F}$-metric on X, and the pair (X, D) is said to be an $\mathcal{F}$-metric space.

Example 1.3. The set of natural numbers $\mathbb{N} = X$ is an $\mathcal{F}$-metric space if we define D by

$$D(x, y) = \begin{cases} (x-y)^2, & \text{if } (x, y) \in [0, 3] \times [0, 3] \\ |x-y|, & \text{if } (x, y) \notin [0, 3] \times [0, 3], \end{cases}$$

for all $(x, y) \in X \times X$ with $f(t) = \ln(t)$ and $\mu = \ln(3)$. Moreover $D(.,.)$ is not a metric but it is a $\mathcal{F}$-metric.

Definition 1.4. [10] Let (X, D) be an $\mathcal{F}$-metric space. Let $\{x_n\}$ be a sequence in X.

(i) We say that $\{x_n\}$ is $\mathcal{F}$-Cauchy, if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.

(ii) We say that (X, D) is $\mathcal{F}$-complete, if every $\mathcal{F}$ -Cauchy sequence in X is $\mathcal{F}$-convergent to a certain element in X.

Jleli and Samet presented Banach Contraction Principle:

Theorem 1.5. *[10] Let (X, D) be an $\mathcal{F}$-metric space, and let $g : X \to X$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) (X, D) is $\mathcal{F}$-complete.

(ii) There exists $k \in (0, 1)$ such that

$$D(g(x), g(y)) \leq kD(x, y), \ (x, y) \in X \times X.$$

Then g has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by $x_{n+1} = g(x_n)$, $n \in \mathbb{N}$, is $\mathcal{F}$-convergent to x^*.

In 2012, Samet et al. introduced a class of α-admissible mapping and established Banach contraction Principle by using α-admissible mappings. They define α-admissible mapping as follows:

Definition 1.6. [18] Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that T is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.

Next, Salimi et al. [17] modified the concept of α-admissible mapping as follows:

Definition 1.7. [17] Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

If $\eta(x, y) = 1$, then above definition reduces to Definition 1.6. If $\alpha(x, y) = 1$, then T is called an η-subadmissible mapping.

Definition 1.8. [7] Let (X, d) be a metric space. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-η-continuous mapping on (X, d) if for given $x \in X$, and sequence $\{x_n\}$ with

$$x_n \to x \text{ as } n \to \infty,\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Rightarrow Tx_n \to Tx.$$

For more details see [13, 14].

A mapping $T : X \to X$ is called orbitally continuous at $v \in X$ if $\lim_{n\to\infty} T^n x = v$ implies that $\lim_{n\to\infty} TT^n x = Tv$. The mapping T is orbitally continuous on X if T is orbitally continuous for all $v \in X$.

2. Main results

In this section, we present a new fractional convex Reich-type α-η-contraction and establish some new fixed point theorems for fractional interpolative Reich-type α-η-contraction in the setting of $\mathcal{F}$-complete $\mathcal{F}$-metric spaces.

Definition 2.1. Let (X, D) be a $\mathcal{F}$-metric space. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is fractional convex Reich-type α-η-contraction if there are constants $\lambda \in [0, 1)$ and $\beta, \gamma \in (0, 1)$ such that whenever $\alpha(x, y) \geq \eta(x, y)$, we have

$$D(Tx, Ty)^p \leq \lambda \left[D(x, y)^{p\beta} \cdot D(y, Ty)^{p\gamma} \cdot D(x, Tx)^{p(1-\beta-\gamma)} \right], \tag{2.1}$$

for all $x, y \in X \backslash Fix(T)$, where $p \in [1, \infty)$.

Example 2.2. Let $X = \{0, 1, 2, 3\}$ endowed with $\mathcal{F}$-metric D given by

$$D(x, y) = \begin{cases} (x-y)^2, & \text{if } (x, y) \in X \times X \\ |x-y|, & \text{if } (x, y) \notin X \times X, \end{cases}$$

with $f(t) = \ln(t)$ and $\mu = \ln(3)$. Define $T : X \to X$ by

$$T0 = 0, T1 = 1, T2 = T3 = 0.$$

and $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in X \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \eta(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in X \\ 0, & \text{otherwise} \end{cases}.$$

If $x, y \in X$. Clearly $\alpha(x, y) \geq \eta(x, y)$ such that

$$\begin{aligned} D(T2, T3)^p &= 0 \leq \lambda \left[D(2, 3)^{p\beta} \cdot D(2, T2)^{p(1-\beta-\gamma)} \cdot D(3, T3)^{p\gamma} \right] \\ &= \lambda \left[(1)^\beta \cdot D(2, 0)^{1-\beta-\gamma} \cdot D(3, 0)^\gamma \right] \\ &= \lambda \left[(4)^{p(1-\beta)} \cdot (\frac{9}{4})^{p\gamma} \right] \end{aligned}$$

By taking any value of constants $\lambda \in [0, 1)$, $\beta, \gamma \in (0, 1)$ and $p \in [1, \infty)$. Clearly, (2.1) holds for all $x, y \in X \backslash Fix(T)$. Note that T has two fixed points, which are 0 and 1. For more detail and examples see [12].

Now we state and prove our main theorem.

Theorem 2.3. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a fractional convex Reich-type α-η-contraction satisfying the following assertions:*
(i) T is an α-admissible mapping with respect to η;
(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
(iii) T is α-η-continuous.
Then T has a fixed point in X.

Proof. Let x_0 in X such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$. Continuing this process, $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. Now since, T is an α-admissible mapping with respect to η then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing in this process we have,

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}. \tag{2.2}$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$ then $x_n = x^*$, is a fixed point of T. So, we assume that $x_n \neq x_{n+1}$ with

$$D(Tx_{n-1}, Tx_n) = D(x_n, Tx_n) > 0, \text{ for all } n \in \mathbb{N}.$$

Since T is fractional convex type α-η-contraction, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x_n, x_{n+1})^p &= D(Tx_{n-1}, Tx_n)^p \\ &\leq \lambda \left[D(x_{n-1}, x_n)^{p\beta} \cdot D(x_{n-1}, Tx_{n-1})^{p(1-\beta-\gamma)} \cdot D(x_n, Tx_n)^{p\gamma} \right], \\ &= \lambda \left[D(x_{n-1}, x_n)^{p\beta} \cdot D(x_{n-1}, x_n)^{p(1-\beta-\gamma)} \cdot D(x_n, x_{n+1})^{p\gamma} \right], \\ &= \lambda \left[D(x_{n-1}, x_n)^{p(1-\gamma)} \cdot D(x_n, x_{n+1})^{p\gamma} \right], \end{aligned}$$

we deduce that

$$D(x_n, x_{n+1})^{p(1-\gamma)} \leq \lambda D(x_{n-1}, x_n)^{p(1-\gamma)}. \tag{2.3}$$

We conclude that $\{D(x_{n-1}, x_n)\}$ is a non-increasing sequence with non-negative terms. Thus, there is a nonnegative constant ϱ such that $\lim_{n\to\infty} D(x_{n-1}, x_n) = \varrho$. Note that $\varrho \geq 0$. we deduce from (2.3), we have

$$D(x_n, x_{n+1}) \leq \lambda D(x_{n-1}, x_n) \leq \lambda^n D(x_0, x_1).$$

Which provide

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leq \frac{\lambda^n}{1-\lambda} D(x_0, x_1), \; m > n.$$

Since

$$\lim_{n\to+\infty} \frac{\lambda^n}{1-\lambda} D(x_0, x_1) = 0,$$

there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1-\lambda} D(x_0, x_1) < \delta, \; n \geq N.$$

Let $\epsilon > 0$ be fixed and $(f, \mu) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \text{ implies } f(t) < f(\epsilon) - a. \tag{2.4}$$

Hence by (2.4) and $(\mathcal{F}_1)$, we got

$$f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) \leq f\left(\frac{\lambda^n}{1-\lambda} D(x_0, x_1)\right) < f(\epsilon) - \mu, \tag{2.5}$$

where $m, n \in \mathbb{N}$ such that $m > n \geq N$ with $D(x_n, x_m) > 0$. Therefore by using (D_3) and (2.5), we have

$$f(D(x_m, x_n)) \leq f\left(\sum_{i=n}^{m-1} (D(x_i, x_{i+1}))\right) + \mu < f(\epsilon),$$

which implies by $(\mathcal{F}_1)$ we have

$$D(x_m, x_n) < \epsilon, m > n \geq N.$$

Hence $\{x_n\}$ is a $\mathcal{F}$-Cauchy sequence. Since (X, D) is a $\mathcal{F}$-complete metric space there exists $x^* \in X$ such that x_n is $\mathcal{F}$-convergent to x^*, that is,

$$\lim_{n\to\infty} D(x_n, x^*) = 0. \tag{2.6}$$

T is α-η-continuous and $\alpha(x_{n-1}, x_n) \geq \eta(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$ then $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. That is, $x^* = Tx^*$. Now we are going to prove that x^* is a fixed point of T. We argue by contradiction by supposing that $D(Tx^*, x^*) > 0$. By $(D3)$, we have

$$f(D(Tx^*, x^*)) \leq f(D(Tx^*, Tx_n)^p + D(Tx_n, x^*)) + \mu, \ n \in \mathbb{N}.$$

By using $(\mathcal{F}_1)$ and the contractive condition gives

$$f(D(Tx^*, x^*)) \leq f\left(\lambda D(x^*, x_n)^{p\beta} \cdot D(Tx^*, x^*)^{p(1-\beta-\gamma)} \cdot D(x_n, x_{n+1})^{p\gamma} + D(x_{n+1}, x^*)\right) + \mu, \ n \in \mathbb{N}.$$

In otherway, by using $(\mathcal{F}_2)$ and (2.6), we have

$$\lim_{n\to\infty} f\left(\lambda D(x^*, x_n)^{p\beta} + D(x_{n+1}, x^*)\right) + \mu = -\infty,$$

which give a contradiction. Therfore $D(Tx^*, x^*) = 0$, hence x^* is a fixed point of T. □

Theorem 2.4. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a fractional convex Reich-type α-η-contraction satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x^*$ as $n \to \infty$ then $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof. On similar lines of proof of Theorem 2.3, we obtain $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ for all $n \in \mathbb{N}$. By (D_3), we have

$$f\,(D(Tx^*, x^*)) \leq f\,(D(Tx^*, Tx_n) + D(x_n, x^*)) + \mu$$

From (2.1) and $(\mathcal{F}_1)$, we have

$$\begin{aligned} f\,(D(Tx^*, x^*)) &\leq f\,((D(Tx^*, Tx_n)^p) + D(Tx_n, x^*)) + \mu \\ &\leq f\left(\lambda\left[D(x^*, x_n)^{p\beta} \cdot D(x^*, Tx^*)^{p(1-\beta-\gamma)} \cdot D(x_n, x_{n+1})^{p\gamma}\right] + D(x_n, x^*)\right) + \mu \end{aligned}$$

Using (2.6) the fact that

$$\lim_{n\to\infty} D(x_n, x^*) = 0 = \lim_{n\to\infty} D(x_{n+1}, x^*),$$

we obtain

$$f\,(D(x^*, Tx^*)) \leq f\,(D(x^*, Tx^*)) + \mu$$

Using $(\mathcal{F}_2)$, we have

$$\lim_{n\to\infty} f\,(D(x^*, Tx^*)) + \mu = -\infty,$$

which is a contradiction. Therefore $D(x^*, Tx^*) = 0$ that is x^* is a fixed point of T. □

Example 2.5. Let $X = \mathbb{R}$ with an $\mathcal{F}$-metric $D : X \times X \to \mathbb{R}^+$ by

$$D\,(x, y) = \begin{cases} (x-y)^2\,, \text{ if } (x, y) \in \mathbb{N} \times \mathbb{N} \\ |x-y|\,, \text{ if } (x, y) \notin \mathbb{N} \times \mathbb{N}, \end{cases}$$

with $f(t) = \ln(t)$ and $\mu = \ln(100)$. Define $T : X \to X$ by

$$Tx = \begin{cases} 1 - \frac{x}{2}, & \text{if } x \in \mathbb{N} \\ 0, & \text{if } x \notin \mathbb{N} \end{cases}$$

and $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 2, \text{ if } x, y \in [0, \infty) \\ 0, \text{ otherwise} \end{cases} \quad \text{and } \eta(x, y) = \begin{cases} 1, \text{ if } x, y \in [0, \infty) \\ 0, \text{ otherwise} \end{cases}.$$

Also let $\psi(t) = \frac{t}{3}$, for all $t > 0$.

Case I. If $x = y$. Clearly $D(x, y) = 0$. Hence all the conditions of Theorem 2.3 are satisfied.

Case II. If x, y are in $\mathbb{N}$, but $Tx \notin \mathbb{N}, Ty \notin \mathbb{N}$, then

$$D\,(Tx, Ty)^p = D(1 - \frac{x}{2}, 1 - \frac{y}{2}) = \left[\frac{1}{2}\,|x-y|\right]^p.$$

Clearly T is α-admissible mapping with respect to η, whenever $\alpha(x, y) \geq \eta(x, y)$, such that

$$D\,(Tx, Ty)^p = \left[\frac{1}{2}\,|x-y|\right]^p \leq \lambda\left[(x-y)^{2p\beta} \cdot \left|\frac{3}{2}x - 1\right|^{p(1-\beta-\gamma)} \cdot \left|\frac{3}{2}y - 1\right|^{p\gamma}\right],$$

by taking constants $\lambda \in [0, 1)$, $p \in [1, \infty)$ and $\beta, \gamma \in (0, 1)$, for all $x, y \in \mathbb{N} \backslash Fix(T)$.
Other cases when(i). both x, y are not in $\mathbb{N}$, we obtain

$$D\,(Tx, Ty)^p = 0.$$

then clearly T is α-admissible mapping with respect to η, whenever $\alpha(x,y) \geq \eta(x,y)$, such that

$$D(Tx,Ty)^p = 0 \leq \lambda \left[|x-y|^{p\beta} \cdot |x|^{p(1-\beta-\gamma)} \cdot |y|^{p\gamma} \right],$$

where $\lambda \in [0,1)$, $p \in [1,\infty)$ and $\beta, \gamma \in (0,1)$, for all $x, y \in \mathbb{N} \backslash Fix(T)$.

(ii). If one of x, y are in $\mathbb{N}$ and other not in $\mathbb{N}$, we obtain

$$D(Tx,Ty) = D(1-\frac{x}{2}, 0) = \left|1-\frac{x}{2}\right|^p.$$

Clearly T is α-admissible mapping with respect to η, whenever $\alpha(x,y) \geq \eta(x,y)$, such that

$$D(Tx,Ty)^p = \left|1-\frac{x}{2}\right|^p \leq \lambda \left[|x-y|^{p\beta} \cdot \left|\frac{3}{2}x - 1\right|^{p(1-\beta-\gamma)} \cdot |y|^{p\gamma} \right].$$

Therefore, all the conditions of our Theorem 2.3 are satisfied. Hence T is fractional convex Reich-type α-η-contraction.

Definition 2.6. Let (X, D) be a $\mathcal{F}$-metric space and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. The $\mathcal{F}$ -metric space X is said to be α-η-complete if and only if every $\mathcal{F}$-Cauchy sequence $\{x_n\}$ with

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N}.$$

$\mathcal{F}$ -converges in X.

Remark 2.7. Theorems 2.3 and 2.4 also hold for α-η-complete $\mathcal{F}$ -metric space instead of $\mathcal{F}$-complete $\mathcal{F}$-metric space (see for details [5]).

3. Fractional convex Kannan type α-η -contraction

In this section, we introduce new fractional convex Kannan type α-η-contraction and establish some fixed point theorems in the setting of $\mathcal{F}$-complete $\mathcal{F}$-metric space. We define fractional interpolative Kannan type α-η-contraction as follows:

Definition 3.1. Let (X, D) be a $\mathcal{F}$-metric space. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is fractional convex Kannan type α-η-contraction if there exist a constants $\lambda \in [0,1)$, and $\beta \in (0,1)$ with $\alpha(x,y) \geq \eta(x,y)$ such that

$$D(Tx,Ty)^p \leq \lambda \left[D(x,Tx)\right]^{p(1-\beta)} \cdot \left[D(y,Ty)\right]^{p\beta}, \tag{3.1}$$

where $p \in [1,\infty)$ for all $x, y \in X$ with $x \neq Tx$.

Now we state and prove our next theorem.

Theorem 3.2. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a fractional convex Kannan type α-η-contraction satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iii) T is α-η-continuous.

Then T has a fixed point in X.

Proof. Let x_0 in X such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$. Continuing this process, $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. Now since, T is an α-admissible mapping with respect to η then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing in this process we have,

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, there is nothing to prove. So, we assume that $x_n \neq x_{n+1}$ with

$$d(Tx_{n-1}, Tx_n) = d(x_n, Tx_n) > 0, \forall n \in \mathbb{N}.$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$ then $x_n = x^*$, is a fixed point of T. So, we assume that $x_n \neq x_{n+1}$ with

$$D(Tx_{n-1}, Tx_n) = D(x_n, Tx_n) > 0, \text{ for all } n \in \mathbb{N}.$$

Since T is fractional convex Kannan type α-η-contraction, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} D(x_n, x_{n+1})^p &= D(Tx_{n-1}, Tx_n)^p \leq \lambda\, [D(x_{n-1}, Tx_{n-1})]^{p(1-\beta)} \cdot [D(x_n, Tx_n)]^{p\beta}, \\ &= \lambda\, [D(x_{n-1}, x_n)]^{p(1-\beta)} \cdot [D(x_n, x_{n+1})]^{p\beta} \end{aligned} \tag{3.3}$$

By Eq (3.3) implies that

$$[D(x_n, x_{n+1})]^{p(1-\beta)} \leq \lambda\, [D(x_{n-1}, x_n)]^{p(1-\beta)}. \tag{3.4}$$

Thus, we conclude that $\{D(x_{n-1}, x_n)\}$ is non-increasing and non-negative. As a result, there is a nonnegative constant ρ such that $\lim_{n\to\infty} D(x_{n-1}, x_n) = \rho$. We shall indicate that $\rho > 0$. Indeed, from (3.4), we derive that

$$D(x_n, x_{n+1}) \leq \lambda D(x_{n-1}, x_n) \leq \lambda^n D(x_0, x_1). \tag{3.5}$$

Rest of the proof follows similar lines of Theorem 2.3. □

Theorem 3.3. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a fractional convex Kannan type α-η-contraction satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x^*$ as $n \to \infty$ then $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof. As similar lines of the Theorem 2.4, Since, by (iii), $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ for all $n \in \mathbb{N}$. By (D_3), we have

$$f\,(D(Tx^*, x^*)) \leq f\,(D(Tx^*, Tx_n) + D(Tx_n, x^*)) + \mu$$

From (3.1) and $(\mathcal{F}_1)$, we have

$$f\,(D(Tx^*, x^*)^p) \leq f\left(\lambda\left(D(x^*, Tx^*)^{p(1-\beta)} \cdot D(x^*, x_n)^{p\beta}\right) + D(x_{n+1}, x^*)\right) + \mu$$

Using (2.6) and fact that

$$\lim_{n\to\infty} D(x_n, x^*) = 0 = \lim_{n\to\infty} D(x_{n+1}, x^*),$$

we obtain

$$f(D(x^*, Tx^*)) \leq f(D(x^*, Tx^*)) + \mu,$$

which is a contradiction. Therefore $D(x^*, Tx^*) = 0$ that is x^* is a fixed point of T. □

If $\eta(x, y) = 1$, in the Theorem 2.3, Theorem 2.4, Theorem 3.2 and Theorem 3.3, we have the following corollaries.

Corollary 3.4. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a fractional interpolative Reich-type α-η-contraction satisfying the following assertions:*
(i) T is an α-admissible mapping;
(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point in X.

Corollary 3.5. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a fractional interpolative Reich-type α-η-contraction satisfying the following assertions:*
(i) T is an α-admissible mapping;
(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x^*$ as $n \to \infty$ then $\alpha(x_n, x^*) \geq 1$ holds for all $n \in \mathbb{N}$.
Then T has a fixed point in X.

Corollary 3.6. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a fractional interpolative Kannan type α-contraction satisfying the following assertions:*
(i) T is an α-admissible mapping;
(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point in X.

Corollary 3.7. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a fractional interpolative Kannan type α-contraction satisfying the following assertions:*
(i) T is an α-admissible mapping;
(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ with $x_n \to x^*$ as $n \to \infty$ then $\alpha(x_n, x^*) \geq 1$ holds for all $n \in \mathbb{N}$.
Then T has a fixed point in X.

4. Consequences

As consequence of our results we derive some results for Suzuki type contractions, orbitally T-complete and orbitally continuous mappings in $\mathcal{F}$-metric spaces.

Theorem 4.1. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a continuous self mapping on X. Assume that there exists $r \in [0, 1)$ and $\beta, \gamma \in (0, 1)$ such that*

$$D(x, Tx) \leq D(x, y) \implies D(Tx, Ty)^p \leq r\,[D(x, y)]^{p\beta} \cdot [D(x, Tx)]^{p(1-\beta-\gamma)} \cdot [D(y, Ty)]^{p\gamma},$$

where $p \in [1, \infty)$ *, for all* $x, y \in X$.

Then T *has a fixed point in* X.

Proof. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = D(x, y) \text{ and } \eta(x, y) = D(x, y), \text{ for all } x, y \in X,$$

and $\beta, \gamma \in (0, 1)$, and $r \in [0, 1)$. It is clear that

$$\eta(x, y) \leq \alpha(x, y), \text{ for all } x, y \in X,$$

that is, conditions (i)–(iii) of our Theorem 2.3 hold true. Let

$$\eta(x, Tx) \leq \alpha(x, y) \text{ then } D(x, Tx) \leq D(x, y),$$

which implies contractive condition

$$D(Tx, Ty)^p \leq r\,[D(x, y)]^{p\beta} \cdot [D(x, Tx)]^{p(1-\beta-\gamma)} \cdot [D(y, Ty)]^{p\gamma}.$$

Thus all conditions of Theorem 2.3 holds true. Hence T has a fixed point in X. □

Theorem 4.2. *Let* (X, D) *be an* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* T *be a continuous mapping. Assume that there exists* $r \in [0, 1)$ *and and* $\beta \in (0, 1)$ *such that*

$$D(x, Tx) \leq D(x, y) \implies D(Tx, Ty)^p \leq r\,[D(x, Tx)]^{p(1-\beta)} \cdot [D(y, Ty)]^{p\beta}$$

where $p \in [1, \infty)$ *, for all* $x, y \in X$*. Then* T *has a fixed point in* X.

Corollary 4.3. *Let* (X, D) *be an* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* T *be a continuous mapping. Assume that there exists* $r \in [0, 1)$ *such that*

$$D(x, Tx) \leq D(x, y) \implies D(Tx, Ty) \leq rD(x, y),$$

for all $x, y \in X$*. Then* T *has a fixed point in* X.

Theorem 4.4. *Let* (X, D) *be a* $\mathcal{F}$*-metric space and* T *be a self mapping of* X*. Suppose the following assertions hold:*

(i) (X, D) *is an orbitally* T*-complete* $\mathcal{F}$*-metric space;*

(ii) there exists $r \in [0, 1)$ *and* $\beta, \gamma \in (0, 1)$ *such that*

$$D(Tx, Ty)^p \leq r\,[D(x, y)]^{p\beta} \cdot [D(x, Tx)]^{p(1-\beta-\gamma)} \cdot [D(y, Ty)]^{p\gamma}$$

where $p \in [1, \infty)$ *for all* $x, y \in O(\omega)$ *for some* $\omega \in X$*, where* $O(\omega)$ *is an orbit of* ω*;*

(iii) if $\{x_n\}$ *is a sequence such that* $\{x_n\} \subseteq O(\omega)$ *with* $x_n \to x^*$ *as* $n \to \infty$ *then* $x^* \in O(\omega)$.

Then T *has a fixed point.*

Proof. Define $\alpha, \eta : X \times X \to [0, +\infty)$, by $\alpha(x, y) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(x, y) = 0$ otherwise and $\eta(x, y) = 1$ for all $x, y \in X$(see Remark 6 [5]). Then (X, D) is an α-η-complete $\mathcal{F}$-metric and T is α-admissible mapping with respect to η. Let $\alpha(x, y) \geq \eta(x, y)$, then $x, y \in O(\omega)$, then from (ii) we have

$$D(Tx, Ty) \leq r\,[D(x, y)]^{\beta} \cdot [D(x, Tx)]^{1-\beta-\gamma} \cdot [D(y, Ty)]^{\gamma}.$$

That is T is fractional convex Reich-type α-η-contraction. Let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x^*$ as $n \to \infty$. So $\{x_n\} \subseteq O(\omega)$. From (iii) $x^* \in O(\omega)$, that is $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$. Hence all conditions of Theorem 2.4 hold true. Thus T has a fixed point. □

Theorem 4.5. *Let (X, D) be a $\mathcal{F}$-metric space and T be a self mapping. Suppose the following Assertions hold:*

(i) (X, D) is an orbitally T-complete $\mathcal{F}$-metric space;

(ii) there exists $r \in [0, 1)$ and $\beta \in (0, 1)$ such that

$$D(Tx, Ty)^p \leq r\,[D(x, Tx)]^{p(1-\beta)} \cdot [D(y, Ty)]^{p\beta}$$

where $p \in [1, \infty)$for all $x, y \in O(\omega)$ for some $\omega \in X$;

(iii) if $\{x_n\}$ is a sequence such that $\{x_n\} \subseteq O(\omega)$ with $x_n \to x^*$ as $n \to \infty$ then $x^* \in O(\omega)$.

Then T has a fixed point.

Theorem 4.6. *Let (X, D) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a self mapping of X. Suppose the following assertions hold:*

(i) for all $x, y \in O(\omega)$, there exists $r \in [0, 1)$ and $\beta, \gamma \in (0, 1)$, $p \in [1, \infty)$ such that

$$D(Tx, Ty)^p \leq r\,[D(x, y)]^{p\beta} . [D(x, Tx)]^{p(1-\beta-\gamma)} \cdot [D(y, Ty)]^{p\gamma}$$

for some $\omega \in X$;

(ii) T is an orbitally continuous function.

Then T has a fixed point.

Proof. Define $\alpha, \eta : X \times X \to [0, +\infty)$, by $\alpha(x, y) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(x, y) = 0$ otherwise with $\eta(x, y) = 1$(see Remark 1.1 [6]), we know that T is an α-η-continuous mapping. Let $\alpha(x, y) \geq \eta(x, y)$, then $x, y \in O(\omega)$. So $Tx, Ty \in O(\omega)$ that is $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. Therefore T is α-admissible mapping with respect to η. From (i) we have

$$D(Tx, Ty)^p \leq r\,[D(x, y)]^{p\beta} . [D(x, Tx)]^{p(1-\beta-\gamma)} \cdot [D(y, Ty)]^{p\gamma} .$$

That is T is fractional convex Reich-type α-η-contraction. Hence all condition of Theorem 2.3 holds true. Thus T has a fixed point. □

Theorem 4.7. *Let (X, D) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a self mapping. Suppose the following assertions hold:*

(i) for all $x, y \in O(\omega)$, there exists $r \in [0, 1)$ and $\beta \in (0, 1)$, $p \in [1, \infty)$ such that

$$D(Tx, Ty)^p \leq r\,[D(x, Tx)]^{p(1-\beta)} \cdot [D(y, Ty)]^{p\beta}$$

for some $\omega \in X$;

(ii) T is an orbitally continuous function.

Then T has a fixed point.

Corollary 4.8. *Let (X, D) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a self mapping. Suppose the following assertions hold:*

(i) for all $x, y \in O(\omega)$, there exists $r \in [0, 1)$ such that

$$D(Tx, Ty) \leq r\,(D(x, y)$$

for some $\omega \in X$;

(ii) T is an orbitally continuous function.

Then T has a fixed point.

4.1. Results for graphic contraction in $\mathcal{F}$-metric space

Consistent with Jachymski [9], let (X, D) be a $\mathcal{F}$ -metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [9]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length m ($m \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{m}$ of $m + 1$ vertices such that $x_0 = x$, $x_m = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., m$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected where $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges(see for details [1, 3, 8, 9]).

Definition 4.9. [9] We say that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G, i.e.,

$$\forall x, y \in X,\ (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$$

and T decreases weights of edges of G in the following way:

$$\exists\, k \in (0, 1),\ \forall x, y \in X,\ (x, y) \in E(G) \Rightarrow d(Tx, Ty) \leq kd(x, y).$$

Definition 4.10. [9] A self-map T is called G-continuous, if given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x,\ \text{as } n \to \infty,\ (x_n, x_{n+1}) \in E(G),\ \forall n \in \mathbb{N} \implies Tx_n \to Tx.$$

Now we extend concept of G-contraction in $\mathcal{F}$ -metric space as follows.

Definition 4.11. Let (X, D) be a $\mathcal{F}$ -metric space endowed with a graph G and $T : X \to X$ be a mapping with $r \in [0, 1)$, $p \in [1, \infty)$ and $\beta, \gamma \in (0, 1)$, the following conditions hold:

$$\forall x, y \in X, ,\ (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$$

$$\forall x, y \in X,\ (x, y) \in E(G) \Rightarrow D(Tx, Ty)^p \leq r\,[D(x, y)]^{p\beta} \cdot [D(x, Tx)]^{p(1-\beta-\gamma)} \cdot [D(y, Ty)]^{p\gamma},$$

Then the mapping T is called a fractional interpolative Reich-type graphic contractive mapping.

Theorem 4.12. *Let (X, D) be a $\mathcal{F}$ -complete metric space endowed with a graph G and $T : X \to X$ be a fractional interpolative Reich-type graphic contractive mapping. Suppose that the following assertions hold:*

(i) there exists an $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;

(ii) T is G-continuous;

(iii) if $\{x_n\}$ is a $\mathcal{F}$ -Cauchy sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Define, $\alpha : X^2 \to (-\infty, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in E(G) \\ 0, & \text{otherwise} \end{cases}.$$

We prove that the mapping T is α-admissible. Let $x, y \in X$ with $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$. As T is fractional interpolative Reich-type graphic contractive mapping, we have, $(Tx, Ty) \in E(G)$. That is, $\alpha(Tx, Ty) \geq 1$. From (i) there exists x_0 such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \geq 1$. Thus T is α-admissible mapping. If $x, y \in X$ with $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$. Now, since, T is fractional interpolative Reich-type graphic contractive mapping, so $D(Tx, Ty) \leq r\,[D(x, y)]^{\beta} \cdot [D(x, Tx)]^{1-\beta-\gamma} \cdot [D(y, Ty)]^{\gamma}$. That is,

$$\alpha(x, y) \geq 1 \Longrightarrow D(Tx, Ty)^p \leq r\,[D(x, y)]^{p\beta} \cdot [D(x, Tx)]^{p(1-\beta-\gamma)} \cdot [D(y, Ty)]^{p\gamma}.$$

T be G-continuous on (X, D) then

$$x_n \to x, \text{ as } n \to \infty, \ (x_n, x_{n+1}) \in E(G), \ \forall n \in \mathbb{N} \implies Tx_n \to Tx.$$

That is

$$x_n \to x \text{ as } n \to \infty \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N},$$

which implies that T is α-continuous on (X, D). Further, if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then by (iii) we have, $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n, x) \geq 1$. Hence, all conditions of Corollary 3.4 and 3.5 are satisfied and T has a fixed point.

□

Corollary 4.13. *Let (X, D) be a $\mathcal{F}$-complete metric space endowed with a graph G and and $T : X \to X$ be a mapping. Suppose that the following assertions hold:*

(i) T is Banach G-contraction on X;

(ii) there exists an $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

5. Application to fractional order differential equation

The fractional order differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. The fractional order differential equations have numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. For more detail see [2, 4, 15]. Our aim is to give the existence of bounded solution of conformable fractional order differential equation given in (5.1). Suppose that a function $f : (0, \infty) \to \mathbb{R}$ the (conformable) fractional derivative of order α of f at $t > 0$ was defined by

$$D_t^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}$$

D_t^{α} is conformable fractional α order derivative. We consider the following boundary value problem of a fractional order differential equation:

$$\begin{aligned} D_t^{\alpha} x(t) &= f(t, x(t), D_t^{\alpha-1} x(t)), \ t \in [0, 1] \text{ where } \alpha \in (1, 2] \quad (5.1) \\ \text{with } x(0) &= A \text{ and } D_t^{\alpha-1} x(1) = B \text{ or } x(1) = B \text{ and } D_t^{\alpha-1} x(0) = A \end{aligned}$$

Suppose $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function. Now we define the Green function under the assumption of (5.1) is given by

$$G(t,s) = \begin{cases} s.s^{\alpha-2}, & 0 \le s \le t \le 1 \\ t.s^{\alpha-2}, & 0 \le t \le s \le 1 \end{cases}$$

Let $C^\alpha(I)$ be the linear space of all continuous functions defined on I, where $I = [0,1]$, $\alpha > 0$ and let $D(x,y)^p = \|x-y\|_\infty^{2p} = \max_{t\in I} |x(t) - y(t)|^{2p}$ for all $x, y \in C^\alpha(I)$. Then $(C^\alpha(I), d)$ is a $\mathcal{F}$ -complete metric space.

We consider the following conditions:

(a) there exists $r \in [0,1)$ and $\beta, \gamma \in (0,1)$, $p \in [1,\infty)$ with a function $\zeta : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in I$, for all $a, b \in \mathbb{R}$ with $\zeta(a,b) \ge \xi(a,b)$, such that

$$\left| f(s,x(s),D_t^{\alpha-1}x(s))ds - f(s,y(s),D_t^{\alpha-1}y(s))ds \right| \le r\left[|x(s)-y(s)|^{2p\beta} \cdot |x(s)-Tx(s)|^{2p(1-\beta-\gamma)} \cdot |x(s)-Ty(s)|^{2p\gamma} \right];$$

(b) there exists $x_1 \in C^\alpha(I)$ such that for all $t \in I$,

$$\zeta\left(x_1(t), \int_0^1 G(t,s)f(s,x_1(s),D_t^{\alpha-1}x_1(s))ds\right) \ge \xi\left(x_1(t), \int_0^1 G(t,s)f(s,x_1(s),D_t^{\alpha-1}x_1(s))ds\right);$$

(c) for all $t \in I$ and for all $x, y \in C^\alpha(I)$,

$$\begin{aligned} \zeta\left(x(t),y(t)\right) \ \ge \ \xi\left(x(t),y(t)\right) \text{ implies } & \zeta\left(\int_0^1 G(t,s)f(s,x(s),D_t^{\alpha-1}x_1(s))ds, \int_0^1 G(t,s)f(s,y(s),D_t^{\alpha-1}y_1(s))ds\right) \\ \ge \ & \xi\left(\int_0^1 G(t,s)f(s,x(s),D_t^{\alpha-1}x_1(s))ds, \int_0^1 G(t,s)f(s,y(s),D_t^{\alpha-1}y_1(s))ds\right); \end{aligned}$$

(d) for any cluster point x of a sequence $\{x_n\}$ of points in $C^\alpha(I)$ with

$$\zeta\left(x_n, x_{n+1}\right) \ge \xi\left(x_n, x_{n+1}\right), \ \lim_{n\to\infty} \inf \zeta\left(x_n, x\right) \ge \lim_{n\to\infty} \inf \xi\left(x_n, x\right).$$

Theorem 5.1. *Suppose that conditions (a)–(d) are satisfied. Then* (5.1) *has at least one solution* $x^* \in C^\alpha(I)$.

Proof. We known that $x \in C^\alpha(I)$ is a solution of (5.1) if and only if $x \in C^\alpha(I)$ is a solution of the fractional order integral equation

$$x(t) = \lambda \int_0^1 G(t,s)f(s,x(s),D_t^{\alpha-1}x(s))ds \text{ for all } \lambda, t \in I.$$

We define a map $T : C^\alpha(I) \to C^\alpha(I)$ by

$$Tx(t) = \lambda \int_0^1 G(t,s)f(s,x(s),D_t^{\alpha-1}x(s))ds + g(t) \text{ for all } t \in I.$$

Then problem (5.1) is equivalent to finding $x^* \in C^\alpha(I)$ that is a fixed point of T. Let $x, y \in C^\alpha(I)$, such that $\zeta\left(x(t), y(t)\right) \ge 0$, for all $t \in I$. For using (a), we get

$$|Tx(t) - Ty(t)| \ = \ \left| \int_0^1 G(t,s)\left[f(s,x(s),D_t^{\alpha-1}x(s))ds - f(s,y(s),D_t^{\alpha-1}y(s)) \right] ds \right|$$

$$
\begin{aligned}
&\leq \int_0^1 G(t,s)\left|f(s,x(s),D_t^{\alpha-1}x(s))ds - f(s,y(s),D_t^{\alpha-1}y(s))ds\right| \\
&\leq |\lambda|\int_0^1 G(t,s)rds\left[|x(s)-y(s)|^{2p\beta}\cdot|x(s)-Tx(s)|^{2p(1-\beta-\gamma)}\cdot|x(s)-Ty(s)|^{2p\gamma}\right] \\
&\leq \max_{t\in I}\int_0^1 G(t,s)dsr\left[\|x(s)-y(s)\|_\infty^{2p\beta}\cdot\|x(s)-Tx(s)\|_\infty^{2p(1-\beta-\gamma)}\cdot\|x(s)-Ty(s)\|_\infty^{2p\gamma}\right] \\
&\leq r\left(\|x(s)-y(s)\|_\infty^{2p\beta}\cdot\|x(s)-Tx(s)\|_\infty^{2p(1-\beta-\gamma)}\cdot\|x(s)-Ty(s)\|_\infty^{2p\gamma}\right).
\end{aligned}
$$

Thus we have $D(Tx,Ty)^p < |x(s)-y(s)|^{2p\beta}\cdot|x(s)-Tx(s)|^{2p(1-\beta-\gamma)}\cdot|x(s)-Ty(s)|^{2p\gamma}$ for all $x,y\in C^\alpha(I)$ such that $\zeta(x(t),y(t))\geq\xi(x(t),y(t))$ for all $t\in I$. We define $\alpha : C^\alpha(I)\times C^\alpha(I)\to[0,\infty)$ by

$$
\alpha(x,y)=\left\{\begin{array}{ll} 1 & \text{if } \zeta(x(t),y(t))\geq 0,\ t\in I, \\ 0 & \text{otherwise}\end{array}\right\} \text{ and } \eta(x,y)=\left\{\begin{array}{ll} \frac{1}{2} & \text{if } \xi(x(t),y(t))\geq 0,\ t\in I, \\ 0 & \text{otherwise}\end{array}\right\}
$$

Then, for all $x,y\in C^\alpha(I)$, $\alpha(x,y)\geq\eta(x,y)$, we have

$$
D(Tx,Ty)^p\leq r(|x(s)-y(s)|^{2p\beta}\cdot|x(s)-Tx(s)|^{2p(1-\beta-\gamma)}\cdot|x(s)-Ty(s)|^{2p\gamma}).
$$

Obviously, $\alpha(x,y)\geq\eta(x,y)$ for all $x,y\in C^\alpha(I)$.If $\alpha(x,y)\geq\eta(x,y)$ for all $x,y\in C(I)$, then $\zeta(x(t),y(t))\geq\xi(x(t),y(t))$. From (c), we have $\zeta(Tx(t),Ty(t))\geq\xi(Tx(t),Ty(t))$,and so $\alpha(Tx,Ty)\geq\eta(Tx,Ty)$. Thus, T is α-admissible mapping with respect to η. From (b) there exists $x_1\in C^\alpha(I)$ such that $\alpha(x_1,Tx_1)=\eta(x_1,Tx_1)$. By (d), we have that for any cluster point x of a sequence $\{x_n\}$ of points in $C^\alpha(I)$ with $\alpha(x_n,x_{n+1})=\eta(x_n,x_{n+1})$, $\lim_{n\to\infty}\inf\alpha(x_n,x)=\lim_{n\to\infty}\inf\eta(x_n,x)$. By applying Theorem 2.3, T has a fixed point in $C^\alpha(I)$, i.e., there exists $x^*\in C^\alpha(I)$ such that $Tx^*=x^*$, and x^* is a solution of (5.1). □

6. Applications

The fractional order differential equations emerge in various areas of engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, control theory, biology, economics, blood flow phenomena, signal and image processing, biophysics, aerodynamics, fitting of experimental data.

7. Conclusion

This research focuses on new idea of fractional convex Reich-type and Kannan type α-η-contraction in the setting of $\mathcal{F}$-metric space which is different and more general from ordinary metric. This paper will open a new domain of fixed point theory. We developed here Suzuki type fixed point results in orbitally complete $\mathcal{F}$-metric space. These new investigations and applications would enhance the impact of new setup.

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Conflict of interest

The author declare that he has no competing interest.

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