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Research article

New Cusa-Huygens type inequalities

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Abstract: Using the monotone form of the L'Hôspital rule, we discuss the (absolute) monotonicity of the functions $U(x) = \frac{1}{x^4} - \frac{1}{x^5}\frac{3\sin x}{\cos x+2}$, $G(x) = \frac{1}{x^2}\left[\frac{\ln\sin x-\ln x}{\ln(2+\cos x)-\ln 3} - 1\right]$ and $J(x) = \frac{1-(\sin x)/x}{1-(2+\cos x)/3}$ to improve the Cusa-Huygens inequality in several directions on wider ranges. Our results are much better than those existing ones.

Keywords: Cusa-Huygens type inequality; circular function; Bernoulli number
Mathematics Subject Classification: 26D05, 26D15, 33B10

1. Introduction

It is well known inequality plays an irreplaceable role in the development of mathematics. Very recently, many inequalities such as Hermite-Hadamard type inequality [1–6], Petrović type inequality [7], Pólya-Szegö and Ćebyšev type inequalities [8], Ostrowski type inequality [9], reverse Minkowski inequality [10], Jensen type inequality [11–13], Cauchy-Schwarz type inequality [14], Bessel function inequality [15], trigonometric and hyperbolic functions inequalities [16–19], Grötzsch ring function inequality [20], Ramanujan transformation inequality [21], fractional integral inequality [22–27], complete and generalized elliptic integrals inequalities [28–33], generalized convex function inequality [34–36] and mean values inequality [37–39] have attracted the attention of many researchers.

The classical and well-known Cusa-Huygens inequality states that

$$\frac{\sin x}{x} < \frac{2+\cos x}{3} \tag{1.1}$$

for $0 < x < \pi/2$.

Chen and Cheung [40] gave the bounds for $\sin x/x$ in term of $((2+\cos x)/3)^{\delta}$ as follows

$$\left(\frac{2+\cos x}{3}\right)^{\theta_0} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{\vartheta_0} \tag{1.2}$$

for $0 < x < \pi/2$, where $\vartheta_0 = 1$ and $\theta_0 = (\ln\pi - \ln 2)/(\ln 3 - \ln 2)$ are the best possible constants such that the double inequality (1.2) holds for all $0 < x < \pi/2$. Inequality (1.2) was proved by Sun and Zhu in [41]. Recently, the generalizations, improvements and variants for the Cusa-Huygens inequality (1.1) have been the subject of much research.

Inspired by inequalities (1.1) and (1.2), the first aim of this paper is to improve the Cusa-Huygens inequality by considering the monotonicity of the functions

$$U(x) = \frac{1}{x^4} - \frac{1}{x^5}\frac{3\sin x}{\cos x + 2} \tag{1.3}$$

and

$$G(x) = \frac{1}{x^2}\left[\frac{\ln\sin x - \ln x}{\ln(2+\cos x) - \ln 3} - 1\right] \tag{1.4}$$

on a wider range $(0,\pi)$ instead of $(0,\pi/2)$. Our first aim of the article is to prove the following Theorems 1.1 and 1.2.

Theorem 1.1. Let $U(x)$ be defined by (1.3). Then the following statemsnts are true:

(i) There exists $x_0 \in (\pi/2,\pi)$ such that $U(x)$ is increasing on $(0,x_0)$ and decreasing on (x_0,π), and the double inequality

$$\left(1-\alpha_1 x^4\right)\frac{\cos x + 2}{3} < \frac{\sin x}{x} < \left(1-\beta_1 x^4\right)\frac{\cos x + 2}{3} \tag{1.5}$$

holds for $x \in (0,\pi/2)$ with the best possible constants $\alpha_1 = 16(\pi - 3)/\pi^5 = 0.007403\cdots$ and $\beta_1 = 1/180 = 0.005555\cdots$. Moreover, the right hand side inequality of (1.5) also holds for $x \in (0,\pi)$.

(ii) The function

$$xU(x) = \frac{1}{x^3} - \frac{1}{x^4}\frac{3\sin x}{\cos x + 2} \tag{1.6}$$

is increasing on $(0,\pi)$, and the inequality

$$\left(1-\frac{x^3}{\pi^3}\right)\frac{2+\cos x}{3} < \frac{\sin x}{x} \tag{1.7}$$

holds for $x \in (0,\pi)$.

From Theorem 1.1, we get Corollary 1.1 immediately.

Corollary 1.1. The double inequality

$$\left(1-\frac{x^3}{\pi^3}\right)\frac{2+\cos x}{3} < \frac{\sin x}{x} < \left(1-\frac{x^4}{180}\right)\frac{\cos x + 2}{3} \tag{1.8}$$

holds for all $x \in (0,\pi)$ with the best possible constants π^3 and 180.

Theorem 1.2. The function $G(x)$ defined by (1.4) is strictly increasing on $(0,\pi)$.

Let

$$\vartheta_1 = G(0^+) = \frac{1}{30}, \tag{1.9}$$

$$\theta_1 = G\left(\left(\frac{\pi}{2}\right)^-\right) = \frac{4}{\pi^2}\left(\frac{\ln(2/\pi)}{\ln(2/3)} - 1\right) = 0.046097\cdots \tag{1.10}$$

and

$$G(\pi^-) = \infty.$$

Then Theorem 1.2 leads to Corollary 1.2 immediately.

Corollary 1.2. (i) The double inequality

$$\left(\frac{2+\cos x}{3}\right)^{1+\theta_1 x^2} < \frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{1+\vartheta_1 x^2} \tag{1.11}$$

holds for all $x \in (0, \pi/2)$ with the best possible constants ϑ_1 and θ_1 given in (1.9) and (1.10).

(ii) The inequality

$$\frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{1+\theta_1 x^2} \tag{1.12}$$

holds for all $x \in (\pi/2, \pi)$ with the best constant θ_1 given by (1.9).

A real-valued function f is said to be absolutely monotonic on the interval I if f has derivatives of all orders on I such that

$$f^{(n)}(x) > 0$$

for all $x \in I$ and $n \geq 0$.

The second aim of the article is to provide an absolute monotonicity result for a special function and derive a new Cusa-Huygens type inequality.

Theorem 1.3. The function

$$J(x) = \frac{1 - (\sin x)/x}{1 - (2 + \cos x)/3} \tag{1.13}$$

is absolutely monotonic on $(0, 2\pi)$, and

From Theorem 1.3, we can easily obtain the following Corollary 1.3.

Corollary 1.3. Let $J(x)$ be defined by (1.13) Then the function

$$H_n(x) = \frac{J(x) - \sum_{k=1}^{n} \frac{6|B_{2k}|}{(2k-1)!} x^{2k-2}}{x^{2n}}$$

is absolutely monotonic on $(0, 2\pi)$, and the double inequality

$$\left[\sum_{k=1}^{n} \frac{6|B_{2k}|}{(2k-1)!} x^{2k-2} + \mu_n x^{2n}\right] \frac{2+\cos x}{3} - \left[\sum_{k=2}^{n} \frac{6|B_{2k}|}{(2k-1)!} x^{2k-2} + \mu_n x^{2n}\right]$$

$$< \frac{\sin x}{x} < \left[\sum_{k=1}^{n} \frac{6|B_{2k}|}{(2k-1)!} x^{2k-2} + \lambda_n x^{2n}\right] \frac{2+\cos x}{3} - \left[\sum_{k=2}^{n} \frac{6|B_{2k}|}{(2k-1)!} x^{2k-2} + \lambda_n x^{2n}\right]$$

holds for all $x \in (0, \pi/2)$ with the best possible constants

$$\lambda_n = \frac{6|B_{2n+2}|}{(2n+1)!}$$

and

$$\mu_n = \left[3\left(1-\frac{2}{\pi}\right) - \sum_{k=1}^{n}\frac{6|B_{2k}|}{(2k-1)!}\left(\frac{\pi}{2}\right)^{2k-2}\right]\left(\frac{2}{\pi}\right)^{2n},$$

where B_k is the Bernoulli number.

Remark 1.1. Let $n = 1$ and $n = 2$. Then Corollary 1.3 leads to the conclusion that

$$\left[1+\frac{8(\pi-3)}{\pi^3}x^2\right]\frac{2+\cos x}{3} - \frac{8(\pi-3)}{\pi^3}x^2 < \frac{\sin x}{x} < \left(1+\frac{1}{30}x^2\right)\frac{2+\cos x}{3} - \frac{1}{30}x^2 \tag{1.14}$$

and

$$\left(1+\frac{1}{30}x^2+\mu_2 x^4\right)\frac{2+\cos x}{3} - \left(\frac{1}{30}x^2+\mu_2 x^4\right) < \frac{\sin x}{x} < \left(1+\frac{1}{30}x^2+\lambda_2 x^4\right)\frac{2+\cos x}{3} - \left(\frac{1}{30}x^2+\lambda_2 x^4\right) \tag{1.15}$$

for $0 < x < \pi/2$ with $\lambda_2 = 1/840 = 0.001190\cdots$ and $\mu_2 = \left(16/\pi^4\right)\left(2-6/\pi-\pi^2/120\right) = 0.001296\cdots$.

2. Lemmas

In order to prove our main results, we need the monotone form of the L'Hôspital rule [42–44].

Lemma 2.1. (See [42–44]) Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $g' \neq 0$ on (a, b) and f'/g' is (strictly) increasing (decreasing) on (a, b). Then both the functions $(f(x)-f(b))/(g(x)-g(b))$ and $(f(x)-f(a))/(g(x)-g(a))$ are (strictly) increasing (decreasing) on $[a, b]$.

Lemma 2.2. Let B_n be the Bernoulli number. Then we have the following power series formulas

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}, \tag{2.1}$$

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty}\frac{(2n-1)\,2^{2n}}{(2n)!}|B_{2n}|x^{2n-2}, \tag{2.2}$$

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-1} \tag{2.3}$$

and

$$\frac{\cos x}{\sin^2 x} = \frac{1}{x^2} - \sum_{n=1}^{\infty}\frac{(2n-1)\left(2^{2n}-2\right)}{(2n)!}|B_{2n}|x^{2n-2} \tag{2.4}$$

for all $x \in (0, \pi)$.

Proof. The power series formulas (2.1) and (2.3) can be found in the literature [45], and the power series formulas (2.2) and (2.4) can be obtained from (2.1) and (2.3) together with the facts that

$$\frac{1}{\sin^2 x} = \csc^2 x = -(\cot x)'$$

and

$$\frac{\cos x}{\sin^2 x} = -\left(\frac{1}{\sin x}\right)'.$$

□

3. Proofs of Theorems 1.1–1.3

3.1. Proof of Theorem 1.1

(i) We clearly see that the function $U(x)$ can be rewritten as

$$U(x) = \frac{x^{-5}(2x - 3\sin x + x\cos x)}{\cos x + 2} := \frac{p(x)}{q(x)}.$$

Differentiation yields

$$p'(x) = \frac{15}{x^6}\sin x - \frac{1}{x^4}\sin x - \frac{7}{x^5}\cos x - \frac{8}{x^5}, \quad q'(x) = -\sin x,$$

$$\frac{p'(x)}{q'(x)} = \frac{1}{x^6}\left(8\frac{x}{\sin x} + 7\frac{x\cos x}{\sin x} + x^2 - 15\right).$$

Expanding in power series leads to

$$\frac{p'(x)}{q'(x)} = \frac{1}{x^6}\left[8 + 8\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n} + 7 - 7\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n} + x^2 - 15\right]$$

$$= \sum_{n=3}^{\infty}\frac{2^{2n}-16}{(2n)!}|B_{2n}|x^{2n-6},$$

which gives

$$\left[\frac{p'(x)}{q'(x)}\right]' = \sum_{n=3}^{\infty}\frac{(2n-6)\left(2^{2n}-16\right)}{(2n)!}|B_{2n}|x^{2n-7} > 0.$$

It follows from the identities

$$\left(\frac{p}{q}\right)' = \frac{q'}{q^2}\left(\frac{p'}{q'}q - p\right) = \frac{q'}{q^2}H_{p,q} \tag{3.1}$$

and

$$H'_{p,q} = \left(\frac{p'}{q'}\right)' q \tag{3.2}$$

given in [44] that $H'_{p,q} > 0$ due to $(p'/q')' > 0$ and $q > 0$.

From the formula

$$H_{p,q}(x) = \frac{p'(x)}{q'(x)} q(x) - p(x)$$
$$= \frac{1}{x^6}\left(8\frac{x}{\sin x} + 7\frac{x\cos x}{\sin x} + x^2 - 15\right)(\cos x + 2) - \frac{1}{x^5}(2x - 3\sin x + x\cos x).$$

we get

$$H_{p,q}(0^+) = -\frac{1}{84}, \quad H_{p,q}\left(\frac{\pi}{2}\right) = -\frac{1920 - 608\pi}{\pi^6} = -0.01031\cdots, \quad H_{p,q}(\pi) = \infty,$$

which implies that $H_{p,q}(x) < 0$ for $x \in (0, \pi/2)$, and there exists $x_0 \in (\pi/2, \pi)$ such that $H_{p,q}(x) < 0$ for $x \in (0, x_0)$ and $H_{p,q}(x) > 0$ for $x \in (x_0, \pi)$. It follows from $q' = -\sin x < 0$ and (3.1) that $(p/q)' > 0$ on $(0, \pi/2)$, and $(p/q)' > 0$ on $(0, x_0)$ and $(p/q)' < 0$ on (x_0, π).

Therefore, the double inequality (1.3) follows from the monotonicity of $U(x)$ on $(0, \pi/2)$.

Using the piecewise monotonicity of $U(x)$ on $(0, \pi)$, we arrive at

$$U(x) > \min\{U(0), U(\pi)\} = \min\left\{\frac{1}{180}, \frac{1}{\pi^4}\right\} = \frac{1}{180},$$

which prove that the right hand side inequality of (1.3) also holds for $x \in (0, \pi)$.

(*ii*) Differentiation yields

$$(xU)' = \frac{3(\cos x + 5)(\cos x + 1)}{x^5(\cos x + 2)^2} V(x),$$

where

$$V(x) = 4\frac{(\cos x + 2)\sin x}{(\cos x + 5)(\cos x + 1)} - x.$$

It follows from

$$V'(x) = \frac{(3 - \cos x)(1 - \cos x)^2}{(1 + \cos x)(5 + \cos x)^2} > 0$$

for $x \in (0, \pi)$ and $V(0) = 0$ that $V(x) > 0$ for $x \in (0, \pi)$, and so is $(xU)'$. Therefore, the inequality

$$xU(x) < \pi U(\pi) = \frac{1}{\pi^3}$$

holds for $x \in (0, \pi)$.

3.2. Proof of Theorem 1.2

Let

$$G(x) = \frac{\ln x - \ln 3 + \ln(2 + \cos x) - \ln \sin x}{x^2[\ln 3 - \ln(2 + \cos x)]} := \frac{a(x)}{b(x)}, \quad 0 < x < \pi.$$

Then from Lemma 2.1 we clearly see that it suffices to prove that $b'(x)/a'(x)$ is strictly decreasing on $(0, \pi)$ due to $a(0^+) = b(0^+) = 0$.

Elaborated computations lead to

$$\frac{b'(x)}{a'(x)} = \frac{2x[\ln 3 - \ln(2 + \cos x)] + x^2(\sin x)/(2 + \cos x)}{1/x - \cot x - (\sin x)/(2 + \cos x)} := h_1(x) \cdot h_2(x) + h_3(x),$$

where

$$h_1(x) = \frac{\ln 3 - \ln(2 + \cos x)}{x^2},$$

$$h_2(x) = \frac{2x^3}{1/x - \cot x - (\sin x)/(2 + \cos x)}$$

and

$$h_3(x) = \frac{x^2 (\sin x)/(2 + \cos x)}{1/x - \cot x - (\sin x)/(2 + \cos x)}.$$

Next, we prove that $h_i(x)$ is decreasing on $(0, \pi)$ for $i = 1, 2, 3$ and $h_i(x)$ is positive for $i = 1, 2$.

(i) Let

$$h_1(x) = \frac{\ln 3 - \ln(2 + \cos x)}{x^2} =: \frac{u(x)}{v(x)} = \frac{u(x) - u(0^+)}{v(x) - v(0^+)}, \quad 0 < x < \pi.$$

Then

$$u'(x) = \frac{\sin x}{\cos x + 2}, \quad v'(x) = 2x$$

and

$$\frac{v'(x)}{u'(x)} = 2\frac{x(\cos x + 2)}{\sin x} = 4\frac{x}{\sin x} + 2x\frac{\cos x}{\sin x} = 6 + \sum_{n=1}^{\infty} \frac{2^{2n+1} - 8}{(2n)!} |B_{2n}| x^{2n}$$

is clearly increasing on $(0, \pi)$. It follows from Lemma 2.1 that $h_1(x)$ is decreasing on $(0, \pi)$.

(ii) To prove that $h_2(x)$ is positive and decreasing on $(0, \pi)$, it suffices to prove that $1/h_2(x)$ is positive and increasing on $(0, \pi)$. Note that

$$\frac{2}{h_2(x)} = \frac{(2\sin x + \cos x \sin x - x - 2x\cos x)}{x^4 (\sin x)(\cos x + 2)}$$

$$= \left(\frac{1}{x^4} - \frac{1}{x^3}\frac{\cos x}{\sin x} - \frac{1}{3x^2}\right) + \left(\frac{1}{3x^2} - \frac{1}{x^3}\frac{\sin x}{\cos x + 2}\right)$$

$$= K(x) + \frac{1}{3}x\left[xU(x)\right], \tag{3.3}$$

where

$$K(x) = \frac{1}{x^4} - \frac{1}{x^3}\frac{\cos x}{\sin x} - \frac{1}{3x^2}$$

and $xU(x)$ is defined as (1.6), which is strictly increasing on $(0, \pi)$ by Theorem 1.1. We clearly see that it suffices to prove that $K(x)$ is strictly increasing on $(0, \pi)$. Indeed, by Lemma 2.2 we have

$$K(x) = \frac{1}{x^4} - \frac{1}{x^3}\left[\frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}\right] - \frac{1}{3x^2} = \sum_{n=2}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-4},$$

which is obviously increasing on $(0, \pi)$.

(iii) To prove that $h_3(x)$ is decreasing on $(0, \pi)$, it suffices to prove that $1/h_3(x)$ is positive and increasing on $(0, \pi)$. Note that

$$\frac{1}{h_3(x)} = \frac{1}{x^3}\left(\frac{\cos x}{\sin x} - \frac{x}{\sin^2 x} + \frac{2}{\sin x} - 2x\frac{\cos x}{\sin^2 x}\right).$$

It follows from Lemma 2.2 that

$$\frac{x^3}{h_3(x)} = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1} - \frac{1}{x} - \sum_{n=1}^{\infty} \frac{(2n-1)\,2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}$$

$$+\frac{2}{x} + 2\sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-1} - \frac{2}{x} + 2\sum_{n=1}^{\infty} \frac{(2n-1)\left(2^{2n}-2\right)}{(2n)!}|B_{2n}|x^{2n-1}$$

$$= \sum_{n=2}^{\infty} \frac{\left(2^{2n}-4\right)|B_{2n}|}{(2n-1)!}x^{2n-1}$$

and

$$\frac{1}{h_3(x)} = \sum_{n=2}^{\infty} \frac{\left(2^{2n}-4\right)|B_{2n}|}{(2n-1)!}x^{2n-4},$$

which is evidently positive and increasing on $(0, \pi)$. The proof of Theorem 1.2 is completed.

3.3. Proof of Theorem 1.3

It is obviously that $J(x)$ can be rewritten as

$$J(x) = \frac{3}{2\sin^2(x/2)} - \frac{3}{x}\frac{\cos(x/2)}{\sin(x/2)}$$

$$= \frac{3}{2}\left[\frac{4}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1)\,2^{2n}}{(2n)!}|B_{2n}|\left(\frac{x}{2}\right)^{2n-2}\right]$$

$$-\frac{3}{x}\left[\frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|\left(\frac{x}{2}\right)^{2n-1}\right] = \sum_{n=1}^{\infty} \frac{6|B_{2n}|}{(2n-1)!}x^{2n-2},$$

which is clearly absolutely monotonic on $(0, 2\pi)$.

4. Remarks

Remark 4.1. One of the referees asserted that the Cusa-Huygens inequality (1.1) holds for all $x \neq 0$. In fact, inequality (1.1) is equivalent to

$$D(x) = \frac{3\sin x}{2+\cos x} - x < 0.$$

Differentiation yields

$$D'(x) = -\frac{(\cos x - 1)^2}{(\cos x + 2)^2} \leq 0$$

for all $x \in \mathbb{R}$. If $x > 0$, then $D(x) < D(0) = 0$ and inequality (1.1) holds for $x > 0$. If $x < 0$, then $D(x) > D(0) = 0$ and inequality (1.1) also holds for $x < 0$.

Remark 4.2. We clearly see that the right hand side inequality of (1.5) is stronger than the Cusa-Huygens inequality (1.1).

Remark 4.3. Our double inequality (1.11) is clearly better than the inequality (1.2). Moreover, by Theorem 1.2 we deduce that the function

$$x^2G(x) = \frac{\ln\sin x - \ln x}{\ln(2+\cos x) - \ln 3}$$

is also strictly increasing on $(0, \pi)$. This conclusion immediately leads to the inequality (1.2) and the following new result: the inequality

$$\frac{\sin x}{x} < \left(\frac{2+\cos x}{3}\right)^{\theta_0} \tag{4.1}$$

holds for all $x \in (\pi/2, \pi)$ with the best constant $\theta_0 = (\ln\pi - \ln 2)/(\ln 3 - \ln 2)$.

Remark 4.4. The right-hand side inequality of (1.14) is stronger than the Cusa-Huygens inequality (1.1) due to

$$\left[\left(1+\frac{1}{30}x^2\right)\frac{2+\cos x}{3} - \frac{1}{30}x^2\right] - \frac{2+\cos x}{3} = \frac{1}{90}x^2(\cos x - 1) < 0$$

for all $x \in (0, \pi/2)$.

Remark 4.5. Numerical calculations and computer simulation experiments show that the double inequality (1.15) is stronger than the inequalities (1.5) and (1.11) on $(0, \pi/2)$.

Final, the following power series formula

$$G^*(x) = \frac{\ln[(\sin x)/x]}{\ln[(\cos x + 2)/3]} = 1 + \frac{1}{30}x^2 + \frac{1}{252}x^4 + \frac{1}{2592}x^6 + \frac{5}{149688}x^8 + O\left(x^{10}\right)$$

inspires us to propose the Conjecture 4.1.

Conjecture 4.1. The function $G^*(x)$ above mentioned is absolutely monotonic on $(0, \pi/2)$.

5. Conclusions

In the article, we have discussed the monotonicity of the functions $U(x)$, $xU(x)$ and $G(x)$ defined by (1.3) and (1.4) on the interval $(0, \pi)$, and the absolute monotonicity of the function $J(x)$ given in (1.13) on the interval $(0, 2\pi)$. Consequences, we have discovered several new Cusa-Huygens type inequalities, which are the improvements and refinements of some earlier known results.

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Conflict of interest

The author declares that he has no competing interest.

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