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Research article

Dynamics of a thermoelastic-laminated beam problem

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Abstract: The main goal of this work is to study the dynamics of a nonlinear thermoelastic laminated beam system with infinite memory acting on the effective rotation angle. We establish the well-posedness and prove the existence of a finite-dimensional global attractor.

Keywords: well-posedness; global attractor; finite-dimensional; infinite memory; laminated beam; thermoelasticity
Mathematics Subject Classification: 35B35, 35B40, 35D35, 37L05, 37L30, 93D15

1. Introduction

In this paper, we consider the following thermoelastic laminated beam system with infinite memory acting on the effective rotation angle, namely

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x + \theta_x = 0, \quad x \in (0,1), t > 0, \\ I_\rho (3w - \psi)_{tt} - D(3w-\psi)_{xx} + \int_0^{+\infty} g(s)(3w-\psi)_{xx}(x,t-s)ds \\ \qquad\qquad - G(\psi - \varphi_x) - \theta = f_1(x), \quad x \in (0,1), t > 0, \\ I_\rho w_{tt} - D w_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma h(w) + \frac{4}{3}\beta w_t = f_2(x), \quad x \in (0,1), t > 0, \\ k\theta_t - \tau\theta_{xx} + \varphi_{xt} + (3w-\psi)_t = 0, \quad x \in (0,1), t > 0, \end{cases} \tag{1.1}$$

where the functions $f_1, f_2 \in L^2(0,1)$ are external forcing terms, h and g are the nonlinear source term and relaxation function respectively. $\varphi = \varphi(x,t)$ is the transverse displacement, $\psi = \psi(x,t)$ is the rotation angle, $w = w(x,t)$ is proportional to the amount of slip along the interface, $3w-\psi$ is the effective rotational angle and $\theta = \theta(x,t)$ is the difference temperature. The positive parameters $\rho, I_\rho, G, D, \gamma, \beta, k$ and τ are the density, mass moment of inertia, shear stiffness, flexural rigidity, adhesive stiffness, adhesive damping parameter, capacity and the diffusivity respectively. We supplement system (1.1) with initial data

$$\begin{cases}\varphi(x,0)=\varphi_0(x),\ \psi(x,0)=\psi_0(x),\ w(x,0)=w_0(x),\ \theta(x,0)=\theta_0(x), & x\in[0,1],\\ \varphi_t(x,0)=\varphi_1(x),\ \psi_t(x,0)=\psi_1(x),\ w_t(x,0)=w_1(x), & x\in[0,1],\end{cases}\tag{1.2}$$

and boundary conditions

$$\begin{cases}\varphi_x(0,t)=\psi(0,t)=w(0,t)=\theta(0,t)=0, & t\geq 0,\\ \varphi(1,t)=\psi_x(1,t)=w_x(1,t)=\theta_x(1,t)=0, & t\geq 0.\end{cases}\tag{1.3}$$

System (1.1) models a vibrating structure, where two beams of the same layer and uniform thickness are fastened together by an adhesive force in a way that permits the beams to slip over each other while remaining in contact at all times. These types of structures are of great importance in the field of science and engineering and are formally called laminated beams. The negligible mass and thickness of the adhesive layer of the beams produces a damping mechanism which is proportional to slips frequency of the two beams, thus producing a structural frictional force in the interfacial slip, see Hansen and Spies [1].

In simple terms, the global attractor is a compact set on an infinite dimensional function space (the phase space), which attracts at a uniform rate any bounded subset of the phase space. In some cases, the global attractor may have finite dimension (Hausdorff and fractal dimension). Whenever the global attractor possesses a finite fractal dimension, an infinite dimensional dynamical system generated by a given PDE can be reduced to a finite dimensional systems of ODEs, for instance by making use of Hölder-Ma$\tilde{n}$é theorem. Readers may see [11, 12] and references there in for more details. The main novelty of this work is to show that system (1.1)–(1.3) possesses a global attractor which has a finite fractal dimension. Considering the complicated nature of system (1.1) with the presence of the nonlinear source term, the obvious challenge would be to establish that the system (1.1) is dissipative. We will achieve this by defining and estimating several Lyapunov functionals.

Now, we give a quick review of some models and results in the literature that are related to problem (1.1). Liu and Zhao [17] considered problem (1.1) with $h(w) = w$, $f_1 = f_2 = 0$ and established an exponential stability result. It is important to mention that in the settings of global attractors, the attractors in this case reduces to the singleton set $\{0\}$, which of course is simple. However, the presence of external forcing terms f_1, f_2 and the nonlinear source term h in system (1.1) creates a more interesting and much more complicated attractors compared to the case where $h(w) = w$ and $f_1 = f_2 = 0$. Raposo [23] studied

$$\begin{cases}\rho\varphi_{tt}+G\left(\psi-\varphi_x\right)_x=0, & \text{in } (0,1)\times(0,+\infty),\\ I_\rho\left(3w-\psi\right)_{tt}-D\left(3w-\psi\right)_{xx}-G\left(\psi-\varphi_x\right)+k_2(3w-\psi)_t=0, & \text{in } (0,1)\times(0,+\infty),\\ I_\rho w_{tt}-Dw_{xx}+3G\left(\psi-\varphi_x\right)+4\gamma w+4\beta w_t=0, & \text{in } (0,1)\times(0,+\infty).\end{cases}\tag{1.4}$$

and proved an exponential decay result. Apalara [6] considered a thermoelastic laminated beam with structural damping where the heat is given by the Cattaneo law. Precisely, he considered

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & \text{in } (0,1)\times(0,+\infty), \\ I_\rho(3w-\psi)_{tt} - D(3w-\psi)_{xx} - G(\psi-\varphi_x) + \delta\theta_x = 0, & \text{in } (0,1)\times(0,+\infty), \\ I_\rho w_{tt} - Dw_{xx} + G(\psi-\varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_3\theta_t + q_x + \delta(3w-\psi)_{tx} = 0, & \text{in } (0,1)\times(0,+\infty), \\ \tau q_t + \alpha q + \theta_x = 0, & \text{in } (0,1)\times(0,+\infty) \end{cases} \tag{1.5}$$

and proved the well-posedness as well as a uniform stability result. We refer the reader to [8, 16, 18–20, 24, 26] and the references cited therein for more related results. Let us mention that, the laminated beam problem (1.1) is closely related to the well-known Timoshenko problem. For instance, in the case of finite memory, setting $h(w) = w,\ u = 3w,\ \rho_1 = \rho, \rho_2 = I_\rho,\ k = G, b = D,\ \frac{3}{4}\delta = \gamma,\ \frac{3}{4}\alpha = \beta$, we get

$$\begin{cases} \rho_1\varphi_{tt} + k(\psi-\varphi_x)_x + \theta_x = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_2(u-\psi)_{tt} - D(u-\psi)_{xx} + \int_0^t g(t-s)(u-\psi)_{xx}(x,s)ds & \\ \qquad\qquad - k(\psi-\varphi_x) - \theta = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_2 u_{tt} - bu_{xx} + 3k(\psi-\varphi_x) + \delta u + \alpha u_t = 0, & \text{in } (0,1)\times(0,+\infty), \\ k\theta_t - \tau\theta_{xx} + \varphi_{xt} + (u-\psi)_t = 0, & \text{in } (0,1)\times(0,+\infty). \end{cases} \tag{1.6}$$

Using $(1.6)_2$ and $(1.6)_3$, then setting $u = 0$, we get the thermoelastic Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + \theta_x = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(s)\psi_{xx}(x,t-s)ds + k(\varphi_x+\psi) - \theta = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_3\theta_t - k\theta_{xx} + \varphi_{xt} + \psi_t = 0, & \text{in } (0,1)\times(0,+\infty). \end{cases} \tag{1.7}$$

Several authors have studied (1.7), for instance, Feng [14] considered (1.7) and established that the system is uniformly stable in cases of equal-wave speed and non equal wave speed of propagation. Apalara [2] studied (1.7) with Neumann–Dirichlet–Dirichlet boundary conditions and proved a stability result without any condition on the speed of wave propagation. Messaoudi and Fareh [21, 22] studied the following system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(x,s)ds + k(\varphi_x+\psi) - \theta = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_3\theta_t - k\theta_{xx} + \delta\psi_{xt} = 0, & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{1.8}$$

and proved a general decay result for the case of equal wave speed of propagation, as well as for non equal wave speed of propagation. For more related results, we refer the reader to [3–5, 9, 13, 15, 25] and references therein. This paper is organized as follows: In Section 2, we recall some preliminaries and assumptions on the relaxation and nonlinear functions g and h respectively. In Section 3, we prove a well-posedness result for system (1.1)–(1.3). In Section 4, we establish the existence of the global attractor for system (1.1)–(1.3). In Section 5, we show that the global attractor has a finite fractal dimension. Throughout this work, we denote the inner product and norm in $L^2(0,1)$ by $(,)$ and $\|.\|$ respectively. Also, the variables C_0, $\bar{C}$ or $C_i, i = 1,2,3,...$ are positive generic constants that may change from one line to another or within the same line.

2. Problem setting and preliminaries

In this section, we recall some useful materials and conditions. For this, we assume that the relaxation function g and the nonlinear function h satisfy:

(G1) $g : [0,+\infty) \longrightarrow (0,+\infty)$ is an absolutely continuous function, with

$$g(0) > 0, \quad D - \int_0^\infty g(s)ds = l_0 > 0; \tag{2.1}$$

(G2) there exists a positive constant λ such that for almost every $y \in \mathbb{R}^+$

$$g'(y) + \lambda g(y) \leq 0, \quad t \geq 0; \tag{2.2}$$

(G3) we assume $h \in C^1(\mathbb{R})$ and for the function $H(s) = \displaystyle\int_0^s h(\tau)d\tau$, there exist constants $C_1, C_2 > 0$ such that

$$\liminf_{|s|\to+\infty} \frac{H(s)}{s^2} \geq 0, \quad \liminf_{|s|\to+\infty} \frac{sh(s) - C_1 H(s)}{s^2} \geq 0, \quad h'(s) \geq -C_2. \tag{2.3}$$

We deduce from (2.3) that for every $\eta > 0$, there exist $C_\eta, C'_\eta > 0$ such that

$$H(s) + \eta s^2 \geq -C_\eta, \quad \forall s \in \mathbb{R}, \quad sh(s) - C_1 H(s) + \eta s^2 \geq -C'_\eta, \quad \forall s \in \mathbb{R}. \tag{2.4}$$

For example, the function $h(s) = s|s|^\gamma$, $0 \leq \gamma < +\infty$, satisfies (2.3).

To deal with the memory term, we set

$$\xi^t(x,s) = (3w - \psi)(x,t) - (3w - \psi)(x,t-s), \quad t,s \geq 0.$$

Simple calculations give

$$\xi_t(x,s) + \xi_s(x,s) - (3w - \psi)_t(x,t) = 0, \quad t,s \geq 0.$$

So problem (1.1)–(1.3) becomes

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x + \theta_x = 0, \quad x \in (0,1), t > 0, \\ I_\rho (3w - \psi)_{tt} - l_o (3w - \psi)_{xx} + \displaystyle\int_0^{+\infty} g(s)\xi_{xx}(x,s)ds \\ \qquad\qquad\qquad - G(\psi - \varphi_x) - \theta = f_1(x), \quad x \in (0,1), t > 0, \\ I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \dfrac{4}{3}\gamma h(w) + \dfrac{4}{3}\beta w_t = f_2(x), \quad x \in (0,1), t > 0, \\ k\theta_t - \tau\theta_{xx} + \varphi_{xt} + (3w - \psi)_t = 0, \quad x \in (0,1), t > 0, \\ \xi_t + \xi_s - (3w - \psi)_t = 0, \quad x \in (0,1), t, s > 0, \end{cases} \tag{2.5}$$

with boundary and initial conditions:

$$\begin{cases}\varphi_x(0,t)=\psi(0,t)=w(0,t)=\theta(0,t)=\xi(0,s)=0, \quad t\geq 0,\\ \varphi(1,t)=\psi_x(1,t)=w_x(1,t)=\theta_x(1,t)=\xi_x(1,s)=0, \quad t\geq 0,\\ \varphi(x,0)=\varphi_0(x),\ \psi(x,0)=\psi_0(x),\ w(x,0)=w_0(x),\ \theta(x,0)=\theta_0(x),\ x\in[0,1],\\ \varphi_t(x,0)=\varphi_1(x),\ \psi_t(x,0)=\psi_1(x),\ w_t(x,0)=w_1(x), \ \ x\in[0,1],\\ \xi(x,0)=0,\ \xi^0(x,s)=(3w_0-\psi_0)-(3w-\psi)(x,-s), \ \ x\in[0,1].\end{cases} \tag{2.6}$$

Let

$$W=(\varphi,u,3w-\psi,q,w,v,\theta,\xi)^T.$$

Then, we can re-write system (2.5)-(2.6) as follows:

$$\begin{cases}W_t+AW=F(W),\\ \\ W(x,0)=W_0(x),\end{cases} \tag{2.7}$$

where

$$W_0=(\varphi_0,\varphi_1,3w_0-\psi_0,3w_1-\psi_1,w_0,w_1,\theta_0,\xi^0)^T,$$
$$F(W)=(0,0,0,\frac{1}{I_\rho}f_1(x),0,-\frac{4\gamma}{3I_\rho}h(w)+\frac{1}{I_\rho}f_2(x),0,0)^T$$

and the linear operator A is given by

$$AW=\begin{pmatrix}-u\\ \frac{G}{\rho}(\psi-\varphi_x)_x+\frac{1}{\rho}\theta_x\\ -q\\ -\frac{I_0}{I_\rho}(3w-\psi)_{xx}-\frac{1}{I_\rho}\displaystyle\int_0^{+\infty}g(s)\xi_{xx}(x,s)ds-\frac{G}{I_\rho}(\psi-\varphi_x)-\frac{1}{I_\rho}\theta\\ -v\\ -\frac{D}{I_\rho}w_{xx}+\frac{G}{I_\rho}(\psi-\varphi_x)+\frac{4\beta}{3I_\rho}v\\ -\frac{\tau}{k}\theta_{xx}+\frac{1}{k}u_x+\frac{1}{k}q\\ \xi_s-q\end{pmatrix}.$$

We consider the following spaces:

$$H^1_*(0,1)=\{z\in H^1(0,1)/z(0)=0\},\quad \bar{H}^1_*(0,1)=\{z\in H^1(0,1)/z(1)=0\},$$

$$H^2_*(0,1)=\{z\in H^2(0,1)/z_x\in H^1_*(0,1)\},\quad \bar{H}^2_*(0,1)=\{z\in H^2(0,1)/z_x\in \bar{H}^1_*(0,1)\}$$

and set

$$\mathcal{H}=\bar{H}^1_*(0,1)\times L^2(0,1)\times H^1_*(0,1)\times L^2(0,1)\times H^1_*(0,1)\times L^2(0,1)\times L^2(0,1)\times\mathcal{M}, \tag{2.8}$$

where

$$\mathcal{M}=L^2_g\left(\mathbb{R}^+,H^1_*(0,1)\right)$$

is defined by

$$L_g^2\left(\mathbb{R}^+, H_*^1(0,1)\right) = \left\{z : \mathbb{R}^+ \to H_*^1(0,1) / \int_0^1 \int_0^\infty g(s)|z_x(x,s)|^2 ds dx < +\infty\right\} \tag{2.9}$$

and endowed with the inner product

$$\langle u, v\rangle_{\mathcal{M}} = \int_0^1 \int_0^\infty g(s) u_x(x,s) v_x(x,s) ds dx.$$

In addition, we define the space

$$\mathcal{D}(\mathcal{M}) = \{\xi, \xi_s \in \mathcal{M}, \xi(x,0) = 0\}.$$

We have that, the inner product

$$\begin{aligned}
&\langle (u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8),(v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8)\rangle_{\mathcal{H}} \\
&= \rho \int_0^1 u_2 v_2 dx + I_\rho \int_0^1 u_4 v_4 dx + 3I_\rho \int_0^1 u_6 v_6 dx + k \int_0^1 u_7 v_7 dx \\
&+ G \int_0^1 (3u_5 - u_3 - u_{1x})(3v_5 - v_3 - v_{1x}) dx + l_0 \int_0^1 u_{3x} v_{3x} dx \\
&+ 3D \int_0^1 u_{5x} v_{5x} dx + \int_0^1 \int_0^{+\infty} g(s) u_{8x}(x,s) v_{8x}(x,s) ds dx
\end{aligned}$$

together with $\mathcal{H}$ form a Hilbert space. Moreover, the domain of the linear operator A is defined by

$$\mathcal{D}(A) := \left\{\begin{array}{l} W \in \mathcal{H} | \varphi \in \bar{H}_*^2(0,1), u \in \bar{H}_*^1(0,1), 3w - \psi \in H_*^2(0,1), q \in H_*^1(0,1), \\ \qquad w \in H_*^2(0,1), v \in H_*^1(0,1), \theta \in H_*^1(0,1), \xi \in \mathcal{D}(\mathcal{M}), \xi_x \in H^1(1,0) \\ \qquad \varphi_x(0,t) = \psi(1,t) = w_x(1,t) = \xi_x(1,s) = \theta_x(1,t) = 0 \end{array}\right\}.$$

3. Wellposedness

In this section, we state the existence and uniqueness result for our problem.

Theorem 3.1. *Assume* $(G1) - (G3)$ *hold and* $f_1, f_2 \in L^2(0,1)$. *If* $W_0 \in \mathcal{H}$, *then problem* (2.5)–(2.6) *has a unique weak solution*

$$W \in C(\mathbb{R}^+; \mathcal{H}).$$

Furthermore, if $W_0 \in \mathcal{D}(A)$, *then*

$$W \in C(\mathbb{R}^+; \mathcal{D}(A)) \cap C^1(\mathbb{R}^+; \mathcal{H}).$$

Proof. We show that the linear operator A is maximal monotone and that the function F is globally lipschitz. For the maximality and monotonicity of A, see [17]. For the lipschitzness of F, let $R > 0$ and set

$$B^R = \{U = (u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8) \in \mathcal{D}(A) : \|U\|_{\mathcal{H}} \le R\}.$$

Let $U, V \in B^R$, using the embedding of $H^1_*(0,1)$ in $L^\infty(0,1)$ and the fact that $h \in C^1(\mathbb{R})$, we have

$$\begin{aligned} \|F(U)-F(V)\|^2_{\mathcal{H}} &= 4\gamma \int_0^1 |h(u_6)-h(v_6)|^2 dx \\ &\leq 4\gamma \|h'(y)\|_{L^\infty(0,1)} \|u_6 - v_6\|^2_{L^2(0,1)} \\ &\leq C(R)\|U-V\|^2_{\mathcal{H}}, \end{aligned} \tag{3.1}$$

where $y = \alpha u_6 + (1-\alpha)v_6,\ \alpha \in (0,1)$. Therefore, F is locally lipschitz. Thus, by Hille-Yosida Theorem we obtain the existence of a local unique weak solution, that is

$$W \in C\left([0,T_m);\mathcal{H}\right), T_m > 0.$$

To obtain global existence, it is enough to show that $\|W(t)\|_{\mathcal{H}}$ is uniformly bounded independent of time. To this end, first, we multiply $(2.5)_1$ by φ_t and integrate over $(0,1)$, then using integration by parts and the boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho\|\varphi_t\|^2 + G\|(\psi-\varphi_x)\|^2\right) = G\left((\psi-\varphi_x),\psi_t\right) - (\theta_x,\varphi_t). \tag{3.2}$$

Secondly, we multiply $(2.5)_2$ by $(3w-\psi)_t$ and integrate over $(0,1)$, then using $(2.5)_5$, integration by parts and the boundary conditions, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[I_\rho\|(3w_t-\psi_t)\|^2 + l_0\|(3w_x-\psi_x)\|^2 + \|\xi_x\|^2_{\mathcal{M}} - 2\left((3w-\psi),f_1\right)\right] \\ &= G\left((\psi-\varphi_x),(3w_t-\psi_t)\right) + (\theta,(3w_t-\psi_t)) + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\xi_x(x,s)|^2 ds dx. \end{aligned} \tag{3.3}$$

Next, we multiply $(2.5)_3$ by $3w_t$ and integrate over $(0,1)$, then using integration by parts and the boundary conditions, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[3I_\rho\|w_t\|^2 + 3D\|w_x\|^2 + 8\gamma\left(H(w),1\right) - 2\left(w,f_2\right)\right] \\ &= -3G\left((\psi-\varphi_x),w_t\right) - 4\beta\|w_t\|^2. \end{aligned} \tag{3.4}$$

Finally, we multiply $(2.5)_4$ by θ, integrate over $(0,1)$, using integration by parts and the boundary conditions, we infer that

$$\frac{1}{2}\frac{d}{dt}\left(k\|\theta\|^2\right) = -\tau\|\theta_x\|^2 + (\theta_x,\varphi_t) - (\theta,(3w-\psi)_t). \tag{3.5}$$

Adding (3.2)–(3.5), we obtain

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\xi_x(x,s)|^2 ds dx - 4\beta\|w_t\|^2 - \tau\|\theta_x\|^2 \leq 0, \quad \forall t \geq 0, \tag{3.6}$$

where

$$\begin{aligned} E(t) =&\frac{1}{2}\left[\rho\|\varphi_t\|^2 + 3I_\rho\|w_t\|^2 + I_\rho\|(3w_t-\psi_t)\|^2 + 3D\|w_x\|^2 + G\|(\psi-\varphi_x)\|^2\right] \\ &+ \frac{1}{2}\left[l_0\|(3w_x-\psi_x)\|^2 + \|\xi_x\|^2_{\mathcal{M}} + 8\gamma\left(H(w),1\right) + k\|\theta\|^2\right] \\ &- \left((3w-\psi),f_1\right) - \left(w,f_2\right). \end{aligned} \tag{3.7}$$

Integration of (3.6) over $(0, t)$ gives

$$E(t) + 4\beta \int_0^t \|w_t(s)\|^2 ds \leq E(0), \forall t \geq 0. \tag{3.8}$$

From assumption (2.3), Young's inequality and the embedding of $H_*^1(0,1)$ in $L^2(0,1)$, we get

$$\begin{aligned}
E(t) \geq & \frac{1}{2}\left[\rho\|\varphi_t\|^2 + 3I_\rho\|w_t\|^2 + I_\rho\|(3w_t - \psi_t)\|^2 + 3D\|w_x\|^2 + G\|(\psi - \varphi_x)\|^2\right] \\
& + \frac{1}{2}\left[\|\xi_x\|_{\mathcal{M}}^2 + k\|\theta\|^2\right] + \frac{l_0}{4}\|(3w_x - \psi_x)\|^2 - 4\eta\gamma\lambda_1\|w_x\|^2 - 4\gamma C_\eta \\
& - \frac{\lambda_1^2}{l_0}\|f_1\|^2 - \frac{3D}{4}\|w_x\|^2 - \frac{\lambda_1^2}{3D}\|f_2\|^2 \\
= & \frac{\rho}{2}\|\varphi_t\|^2 + \frac{3I_\rho}{2}\|w_t\|^2 + \frac{I_\rho}{2}\|(3w_t - \psi_t)\|^2 + \left(\frac{3D}{4} - 4\eta\gamma\lambda_1\right)\|w_x\|^2 \\
& + \frac{G}{2}\|(\psi - \varphi_x)\|^2 + \frac{l_0}{4}\|(3w_x - \psi_x)\|^2 + \frac{1}{2}\|\xi_x\|_{\mathcal{M}}^2 + \frac{k}{2}\|\theta^2\| - C,
\end{aligned} \tag{3.9}$$

where λ_1 is the Poincaré's constant. We then choose η such that

$$\left(\frac{3D}{4} - 4\eta\gamma\lambda_1\right) = \frac{D}{2}$$

and obtain

$$E(t) \geq C_0\|(\varphi, \varphi_t, 3w - \psi, 3w_t - \psi_t, w, w_t, \theta, \xi)\|_{\mathcal{H}}^2 - C, \tag{3.10}$$

where $C_0 = \min\{\frac{\rho}{2}, \frac{G}{2}, \frac{I_\rho}{2}, \frac{l_0}{4}, \frac{1}{4}, \frac{k}{2}\}$. Combining (3.8) and (3.10), we get

$$\|W\|_{\mathcal{H}}^2 + \frac{4\beta}{C_0}\int_0^t \|w_t(s)\|^2 ds \leq \frac{1}{C_0}(E(0) + C) \leq \bar{C}, \forall t \geq 0. \tag{3.11}$$

Therefore, $\|W(t)\|_{\mathcal{H}}^2$ is uniformly bounded independent of time. Hence the solution is global. The computations above are done for regular solutions. However, the result remains true for weak solutions by density argument. This completes the proof. □

4. Global attractor

In this section, we establish the existence of the global attractor for system (2.5)–(2.6). The existence and uniqueness result in Theorem (3.1) guarantees the existence of solution semi-group

$$S(t) : \mathcal{H} \longrightarrow \mathcal{H}$$

defined by

$$S(t)W_0 = W(t), \ \forall t \geq 0,$$

where W is the unique solution of system (2.5)–(2.6).

Lemma 4.1. *The semigroup $S(t)$ is strongly continuous in $\mathcal{H}$.*

Proof. Let $W^j = (\varphi^j, \varphi_t^j, (3w-\psi)^j, (3w_t-\psi_t)^j, w^j, w_t^j, \theta^j, \xi^j)^T$, $j = 1,2$ be two solutions of system (2.5)–(2.6). Then $W = W^1 - W^2$ satisfies

$$\begin{cases} \rho\varphi_{tt} + G(\psi-\varphi_x)_x + \theta_x = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_\rho(3w-\psi)_{tt} - l_0(3w-\psi)_{xx} - \displaystyle\int_0^{+\infty} g(s)\xi_{xx}(x,s)ds & \\ \qquad\qquad - G(\psi-\varphi_x) - \theta = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_\rho w_{tt} - Dw_{xx} + G(\psi-\varphi_x) + \dfrac{4}{3}\gamma(h(w^1)-h(w^2)) + \dfrac{4}{3}\beta w_t = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ k\theta_t - \tau\theta_{xx} + \varphi_{xt} + (3w-\psi)_t = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ \xi_t + \xi_s - (3w-\psi)_t = 0, & (x,t)\in(0,1)\times(0,+\infty), \end{cases} \tag{4.1}$$

with initial data $W_0 = W_0^1 - W_0^2$. Now, multiplying $(4.1)_1$ by φ_t, $(4.1)_2$ by $(3w_t - \psi_t)$, $(4.1)_3$ by $3w_t$, $(4.1)_4$ by θ in $L^2(0,1)$, using integration by parts and adding the outcomes, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|W(t)\|_{\mathcal{H}}^2\right) + 4\beta\|w_t\|^2 + \frac{4\gamma}{3}(h(w^1)-h(w^2), 3w_t)\\ &= -\tau\|\theta_x\|^2 + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\xi_x(x,s)|^2dsdx. \end{aligned} \tag{4.2}$$

Due to assumption (G1), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|W(t)\|_{\mathcal{H}}^2\right) + 4\beta\|w_t\|^2 + \frac{4\gamma}{3}(h(w^1)-h(w^2), 3w_t) \le 0.$$

Using the same justification as in (3.1), we have on account of (3.11) that

$$\frac{1}{2}\frac{d}{dt}\left(\|W(t)\|_{\mathcal{H}}^2\right) + 4\beta\|w_t\|^2 \le C\|W(t)\|_{\mathcal{H}}^2, \tag{4.3}$$

where C is positive constant depending on W_0^1 and W_0^2. Application of Gronwall's lemma to (4.3) leads to

$$\|W(t)\|_{\mathcal{H}}^2 \le e^{Ct}\|W_0\|_{\mathcal{H}}^2, \forall t \ge 0,$$

and the desired result follows. □

Let us recall some basic definitions and theorems related to the theory of global attractor.

Definition 4.1. *Let X be a Banach space. A set $\mathcal{B} \subset X$ is an absorbing set for the semigroup $S(t)$: $X \to X$ if given any bounded set $B \subset X$ there exist a time $t_0(B)$ such that $S(t)B \subset \mathcal{B}$, for every $t \ge t_0(B)$.*

Definition 4.2. *The global attractor for a semigroup $S(t)$ acting on a Hilbert space H is a compact subset $\mathcal{A}$ of H satisfying the following conditions.*
(i) $\mathcal{A}$ is invariant for $S(t)$; i.e,

$$S(t)\mathcal{A} = \mathcal{A}, \quad \forall\ t \ge 0.$$

(ii) $\mathcal{A}$ attracts bounded sets; this means, for any bounded set $B \subset H$, we have

$$\lim_{t\longrightarrow\infty} d_H(S(t)B, \mathcal{A}) = 0,$$

where d_H is the Hausdorff semi-distance defined by

$$d_H(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_H.$$

Theorem 4.1. *[10] Let $S(t)$ be a dissipative semigroup on a metric space H. Then, $S(t)$ has a compact global attractor in H if and only if $S(t)$ is asymptotically smooth in H.*

4.1. Absorbing set

Next, we prove the existence of an absorbing set for system (2.5)-(2.6). To do this, we first state and prove some useful lemmas.

Lemma 4.2. *The functional J_1 defined by*

$$J_1(t) = -I_\rho \int_0^1 (3w_t - \psi_t) \int_0^{+\infty} g(s)\xi(x, s)dsdx$$

satisfies, along the solution of system (2.5)-(2.6)*, the estimate*

$$\begin{aligned} J_1'(t) \le &-\frac{I_\rho(D - l_0)}{2}\|3w_t - \psi_t\|^2 + \epsilon_1\|3w_x - \psi_x\|^2 + \epsilon_1\|\psi - \varphi_x\|^2 + \epsilon_1\lambda_1\|\theta_x\|^2 \\ &+ C\left(1 + \frac{1}{\epsilon_1}\right)\|\xi_x\|^2_{\mathcal{M}} - C\int_0^1\int_0^{+\infty} g'(s)|\xi_x(x, s)|^2dsdx + \frac{1}{2}\|f_1\|^2, \end{aligned} \tag{4.4}$$

for any $\epsilon_1 > 0$.

Proof. Differentiation of J_1 then using $(2.5)_2$ and $(2.5)_5$ along side integration by parts give

$$\begin{aligned} J_1'(t) = &- I_\rho(D - l_0)\|3w_t - \psi_t\|^2 \underbrace{- I_\rho \int_0^1 (3w_t - \psi_t) \int_0^{+\infty} g'(s)\xi(x, s)dsdx}_{I_1} \\ &\underbrace{+ D\int_0^1 (3w_x - \psi_x) \int_0^{+\infty} g(s)\xi_x(x, s)dsdx}_{I_2} + \underbrace{\int_0^1 \left(\int_0^{+\infty} g'(s)\xi(x, s)dsd\right)^2 dx}_{I_3} \\ &\underbrace{-G\int_0^1 (\psi - \varphi_x) \int_0^{+\infty} g(s)\xi(x, s)dsdx}_{I_4} \underbrace{- \int_0^1 \theta \int_0^{+\infty} g(s)\xi(x, s)dsdx}_{I_5} \\ &\underbrace{- \int_0^1 f_1(x) \int_0^{+\infty} g(s)\xi(x, s)dsdx.}_{I_6} \end{aligned} \tag{4.5}$$

Using Hölder's, Young's and Poincaré's equalities, we estimate the terms in (4.5) as follows:

$$\begin{aligned}
I_1 &\leq \frac{I_\rho(D-l_0)}{2}\|3w_t-\psi_t\|^2 - \frac{I_\rho\lambda_1 g(0)}{2(D-l_0)}\int_0^1\int_0^{+\infty} g'(s)|\xi_x(x,s)|^2 dsdx,\\
I_2 &\leq \epsilon_1\|3w_x-\psi_x\|^2 + \frac{D^2(D-l_0)}{\epsilon_1}\|\xi_x\|^2_{\mathcal{M}},\\
I_3 &\leq (D-l_0)\|\xi_x\|^2_{\mathcal{M}},\\
I_4 &\leq \epsilon_1\|\psi-\varphi_x\|^2 + \frac{G\lambda_1^2(D-l_0)}{\epsilon_1}\|\xi_x\|^2_{\mathcal{M}},\\
I_5 &\leq \epsilon_1\lambda_1^2\|\theta_x\|^2 + \frac{(D-l_0)\lambda_1^2}{2\epsilon_1}\|\xi_x\|^2_{\mathcal{M}},\\
I_6 &\leq \frac{1}{2}\|f_1\|^2 + \frac{D-l_0}{2}\|\xi_x\|^2_{\mathcal{M}}.
\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5), we obtain the result. This completes the proof. □

Lemma 4.3. *The functional*

$$J_2(t) = -k\rho\int_0^1 \theta\int_0^x \varphi_t(y)dydx$$

satisfies, along the solution of system (2.5)-(2.6)*, the estimate*

$$J_2'(t) \leq -\frac{\rho}{2}\|\varphi_t\|^2 + \epsilon_2\|\psi-\varphi_x\|^2 + C\left(1+\frac{1}{\epsilon_2}\right)\|\theta_x\|^2 + \rho\|3w_t-\psi_t\|^2, \tag{4.7}$$

for any $\epsilon_2 > 0$.

Proof. Direct differentiation of J_2, then making use of $(2.5)_1$ and $(2.5)_4$ with integration by parts give

$$\begin{aligned}
J_2'(t) =& \tau\rho\int_0^1 \theta_x\varphi_t dx - \rho\int_0^1 \varphi_t^2 dx + \rho\int_0^1(3w_t-\psi_t)\int_0^x\varphi_t(y)dydx\\
&+ kG\int_0^1\theta(\psi-\varphi_x)\,dx + k\int_0^1\theta^2 dx.
\end{aligned} \tag{4.8}$$

Using Young's and Poincaré's inequalities, we obtain

$$\begin{aligned}
J_2'(t) \leq& \rho\tau^2\|\theta_x\|^2 + \frac{\rho}{4}\|\varphi_t\|^2 - \rho\|\varphi_t\|^2 + \rho\|3w_t-\psi_t\|^2 + \frac{\rho}{4}\|\varphi_t\|^2 + \epsilon_2\|\psi-\varphi_x\|^2\\
&+ \frac{(Gk)^2\lambda_1}{4\epsilon_2}\|\theta_x\|^2 + k\lambda_1\|\theta_x\|^2\\
\leq& -\frac{\rho}{2}\|\varphi_t\|^2 + \epsilon_2\|\psi-\varphi_x\|^2 + C\left(1+\frac{1}{\epsilon_2}\right)\|\theta_x\|^2 + \rho\|3w_t-\psi_t\|^2,
\end{aligned} \tag{4.9}$$

for any $\epsilon_2 > 0$. This completes the proof. □

Lemma 4.4. *The functional*

$$J_3(t) = \rho\int_0^1 \varphi\varphi_t dx + \rho\int_0^1 \psi\int_0^x \varphi_t(y)dydx$$

satisfies, along the solution of (2.5)-(2.6)*, the estimate*

$$J_3'(t) \leq -\frac{G}{2}\|\psi - \varphi_x\|^2 + \frac{3\rho}{2}\|\varphi_t\|^2 + \rho\|3w_t - \psi_t\|^2 + 9\rho\|w_t\|^2 + \frac{\lambda_1^2}{2G}\|\theta_x\|^2. \tag{4.10}$$

Proof. We differentiate J_3 and make use of $(2.5)_1$ with integration by parts to get

$$J_3'(t) = \rho\int_0^1 \varphi_t^2 dx - G\int_0^1 (\psi - \varphi_x)^2 dx + \rho\int_0^1 \psi_t \int_0^x \varphi_t(y)dydx - \int_0^1 \theta(\psi - \varphi_x)dx.$$

Using Cauchy-Schwarz, Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} J_3'(t) &\leq \rho\|\varphi_t\|^2 - G\|\psi - \varphi_x\|^2 + \frac{\rho}{2}\|\varphi_t\|^2 + \frac{\rho}{2}\|\psi_t\|^2 + \frac{\lambda_1}{2G}\|\theta_x\|^2 + \frac{G}{2}\|\psi - \varphi_x\|^2 \\ &= -\frac{G}{2}\|\psi - \varphi_x\|^2 + \frac{3\rho}{2}\|\varphi_t\|^2 + \frac{\rho}{2}\|\psi_t\|^2 + \frac{\lambda_1^2}{2G}\|\theta_x\|^2. \end{aligned} \tag{4.11}$$

We observe that

$$\|\psi_t\|^2 = \| - (3w_t - \psi_t) + 3w_t\|^2 \leq 2\|3w_t - \psi_t\|^2 + 18\|w_t\|^2. \tag{4.12}$$

Substituting (4.12) into (4.11), we obtain the desired result. This completes the proof. □

Lemma 4.5. *The functional*

$$J_4(t) = I_\rho \int_0^1 (3w - \psi)(3w - \psi)_t dx$$

satisfies, along the solution of (2.5)-(2.6)*, the estimate*

$$\begin{aligned} J_4'(t) \leq & -\frac{l_0}{4}\|3w_x - \psi_x\|^2 + I_\rho\|3w_t - \psi_t\|^2 + \frac{3(D - l_0)}{2l_0}\|\xi_x\|^2_{\mathcal{M}} \\ & + \frac{3(G\lambda_1)^2}{2l_0}\|\psi - \varphi_x\|^2 + \frac{3\lambda_1^4}{2l_0}\|\theta_x\|^2 + C. \end{aligned} \tag{4.13}$$

Proof. By differentiating J_4, then using $(2.5)_2$ and integration by parts, we get

$$\begin{aligned} J_4'(t) =& I_\rho\|3w_t - \psi_t\|^2 - l_0\|3w_x - \psi_x\|^2 - \int_0^1 (3w_x - \psi_x)\int_0^{+\infty} g(s)\xi_x(x,s)dsdx \\ & + G\int_0^1 (3w - \psi)(\psi - \varphi_x)dx + \int_0^1 (3w - \psi)\theta dx + \int_0^1 (3w - \psi)f_1(x)dx. \end{aligned} \tag{4.14}$$

Applying Young's, Poincaré's and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} J_4'(t) \leq & I_\rho\|3w_t - \psi_t\|^2 - l_0\|3w_x - \psi_x\|^2 + \frac{3}{2l_0}\int_0^1 \left(\int_0^{+\infty} g(s)\xi_x(x,s)ds\right)^2 dx \\ & + \frac{l_0}{6}\|3w_x - \psi_x\|^2 + G\lambda_1\|3w_x - \psi_x\|\|\psi - \varphi_x\| + \lambda_1^2\|3w_x - \psi_x\|\|\theta_x\| \\ & + \frac{l_0}{4}\|3w_x - \psi_x\|^2 + C\|f_1\|^2 \\ \leq & -\frac{l_0}{4}\|3w_x - \psi_x\|^2 + I_\rho\|3w_t - \psi_t\|^2 + \frac{3(D - l_0)}{2l_0}\|\xi_x\|^2_{\mathcal{M}} \\ & + \frac{3(G\lambda_1)^2}{2l_0}\|\psi - \varphi_x\|^2 + \frac{3\lambda_1^4}{2l_0}\|\theta_x\|^2 + C. \end{aligned} \tag{4.15}$$

This completes the proof. □

Lemma 4.6. *The functional*

$$J_5(t) = I_\rho \int_0^1 w w_t dx$$

satisfies, along the solution of (2.5)-(2.6)*, the estimate*

$$J_5'(t) \leq -\frac{D}{4}\|w_x\|^2 + C\|w_t\|^2 + C\|\psi - \varphi_x\|^2 + C. \tag{4.16}$$

Proof. We differentiate J_5, then use $(2.5)_3$ and integration by parts to get

$$\begin{aligned} J_5'(t) = I_\rho\|w_t\|^2 - D\|w_x\|^2 - G\int_0^1 w(\psi - \varphi_x)dx - \frac{4\gamma}{3}\int_0^1 wh(w)dx \\ - \frac{4\beta}{3}\int_0^1 w w_t dx + \int_0^1 w f_2(x)dx. \end{aligned} \tag{4.17}$$

From assumption (2.3), Hölder's and Poincaré's inequalities, we get

$$\begin{aligned} J_5'(t) &\leq I_\rho\|w_t\|^2 - D\|w_x\|^2 + G\lambda_1\|w_x\|\|\psi - \varphi_x\| + \frac{4(C_1+1)\eta\gamma\lambda_1^2}{3}\|w_x\|^2 \\ &\quad + \frac{4\gamma}{3}(C_\eta' + C_\eta) + \frac{4\beta\lambda_1}{3}\|w_x\|\|w_t\| \\ &\leq -\left(\frac{D}{2} - \frac{4(C_1+1)\eta\gamma\lambda_1^2}{3}\right)\|w_x\|^2 + C\|w_t\|^2 + C\|\psi - \varphi_x\|^2 + C. \end{aligned} \tag{4.18}$$

Choose η small enough such that $\left(\frac{D}{2} - \frac{4(C_1+1)\eta\gamma\lambda_1^2}{3}\right) \geq \frac{D}{4}$, thus the desired result follows. □

Define the functional L by

$$L(t) = nE(t) + n_1 J_1(t) + n_2 J_2(t) + n_3 J_3(t) + J_4(t) + J_5(t),$$

where n, n_1, n_2, n_3 are positive constants to be determined later.

Lemma 4.7. *The functional L satisfies, along the solution of* (2.5)-(2.6)*, the estimate*

$$\mu_1\|W(t)\|_{\mathcal{H}}^2 - \bar{C}_1 \leq L(t) \leq \mu_2\|W(t)\|_{\mathcal{H}}^2 + \bar{C}_2, \ \forall t \geq 0, \tag{4.19}$$

for some positive constants $\mu_1, \mu_2, \bar{C}_1, \bar{C}_2$.

Proof. Using (2.3), Hölder's and Poincaré's inequalities, we have on one hand

$$\begin{aligned} L(t) \geq & \frac{n}{2}\left[\rho\|\varphi_t\|^2 + 3I_\rho\|w_t\|^2 + I_\rho\|(3w_t - \psi_t)\|^2 + 3D\|w_x\|^2 + G\|(\psi - \varphi_x)\|^2\right] \\ & + \frac{n}{2}\left[l_0\|(3w_x - \psi_x)\|^2 + \|\xi_x\|_{\mathcal{M}}^2 + k\|\theta\|^2 - 8\gamma\eta\lambda_1^2\|w_x\|^2 - 8\gamma C_\eta\right] \\ & - \frac{n}{2}\left[\lambda_1^2\delta_1\|(3w_x - \psi_x)\|^2 + C_{\delta_1}\|f_1\|^2 + \lambda_1^2\delta_2\|w_x\|^2 + C_{\delta_2}\|f_2\|^2\right] \\ & - n_1 I_\rho\left[\frac{1}{2}\|(3w_t - \psi_t)\|^2 + \frac{(D - l_0)\lambda_1^2}{2}\|\xi_x\|_{\mathcal{M}}^2\right] - n_2 k\rho\left[\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\theta\|^2\right] \\ & - n_3\rho\left[\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|(\psi - \varphi_x)\|^2\right] - I_\rho\left[\frac{\lambda_1^2}{2}\|w_x\|^2 + \frac{1}{2}\|w_t\|^2\right] \\ & - I_\rho\left[\frac{\lambda_1^2}{2}\|(3w_x - \psi_x)\|^2 + \frac{1}{2}\|(3w_t - \psi_t)\|^2\right]. \end{aligned}$$

This implies

$$
\begin{aligned}
L(t) \geq & \left(\frac{n}{2} - \frac{n_2 k}{2} - \frac{n_3}{2}\right)\rho\|\varphi_t\|^2 + \left(\frac{n}{2} - \frac{n_1}{2} - \frac{1}{2}\right) I_\rho \|(3w_t - \psi_t)\|^2 \\
& + 3I_\rho\left(\frac{n}{2} - \frac{1}{6}\right)\|w_t\|^2 + 3D\left(n\left(\frac{1}{2} - \frac{4\lambda_1^2\gamma\eta}{3D} - \frac{\lambda_1^2\delta_2}{6D}\right) - \frac{\lambda_1^2 I_\rho}{6D}\right)\|w_x\|^2 \\
& + G\left(\frac{n}{2} - \frac{n_3\rho}{2G}\right)\|(\psi - \varphi_x)\|^2 + l_0\left(n\left(\frac{1}{2} - \frac{\lambda_1^2\delta_1}{2l_0}\right) - \frac{\lambda_1^2 I_\rho}{2l_0}\right)\|(3w_x - \psi_x)\|^2 \\
& + k\left(\frac{n}{2} - \frac{\rho n_2}{2}\right)\|\theta\|^2 + \left(\frac{n}{2} - \frac{n_1(D-l_0)I_\rho\lambda_1^2}{2}\right)\|\xi_x\|^2_{\mathcal{M}} \\
& - \left(C_{\delta_1}\frac{n}{2}\|f_1\|^2 + C_{\delta_2}\frac{n}{2}\|f_2\|^2 + 4\gamma C_\eta n\right).
\end{aligned}
\tag{4.20}
$$

Now, we first choose δ_1, δ_2, η small enough such that

$$
\left(\frac{1}{2} - \frac{4\lambda_1^2\gamma\eta}{3D} - \frac{\lambda_1^2\delta_2}{6D}\right) > 0, \ \left(\frac{1}{2} - \frac{\lambda_1^2\delta_1}{2l_0}\right) > 0,
$$

then choose n large enough so that

$$
\left(\frac{n}{2} - \frac{n_2 k}{2} - \frac{n_3}{2}\right) > 0, \left(\frac{n}{2} - \frac{n_1}{2} - \frac{1}{2}\right) > 0,
$$

$$
\left(\frac{n}{2} - \frac{1}{6}\right) > 0, \left(n\left(\frac{1}{2} - \frac{4\lambda_1^2\gamma\eta}{3D} - \frac{\lambda_1^2\delta_2}{6D}\right) - \frac{\lambda_1^2 I_\rho}{6D}\right) > 0,
$$

$$
\left(\frac{n}{2} - \frac{n_3\rho}{2G}\right) > 0, \left(n\left(\frac{1}{2} - \frac{\lambda_1^2\delta_1}{2l_0}\right) - \frac{\lambda_1^2 I_\rho}{2l_0}\right) > 0,
$$

$$
\left(\frac{n}{2} - \frac{\rho n_2}{2}\right) > 0, \left(\frac{n}{2} - \frac{n_1(D-l_0)I_\rho\lambda_1^2}{2}\right) > 0
$$

and obtain

$$
L(t) \geq \mu_1\|(\varphi, \varphi_t, 3w - \psi, 3w_t - \psi_t, w, w_t, \theta, \xi)\|^2_{\mathcal{H}} - \bar{C}_1. \tag{4.21}
$$

On the other hand, again using assumption (2.3), Hölder's and Poincaré;s inequalities, we get

$$
\begin{aligned}
L(t) \leq & \alpha_1\|\varphi_t\|^2 + \alpha_2\|w_t\|^2 + \alpha_3\|(3w_t - \psi_t)\|^2 + \alpha_4\|w_x\|^2 + \alpha_5\|(\psi - \varphi_x)\|^2 \\
& + \alpha_6\|(3w_x - \psi_x)\|^2 + \alpha_7\|\theta\|^2 + \alpha_8\|\xi_x\|^2_{\mathcal{M}} + 4n\gamma C_1' \int_0^1 |wh(w)|dx \\
& + \left(C\|f_1\|^2 + C\|f_2\|^2 + 4\gamma n C_\eta\right),
\end{aligned}
\tag{4.22}
$$

for some positive constants $\alpha_i, i = 1, 2, ..., 8$. We observe that

$$
\begin{aligned}
\int_0^1 |wh(w)|dx &\leq \int_0^1 |w|(|h(w) - h(0)|)dx + \int_0^1 |w||h(0)|dx \\
&\leq \left(\lambda_1^2\|h'(\alpha w)\|^2_{L^\infty(0,1)} + \frac{\lambda_1^2}{2}\right)\|w_x\|^2 + C.
\end{aligned}
\tag{4.23}
$$

Substituting (4.23) into (4.22), we get

$$L(t) \leq \mu_2 \|(\varphi, \varphi_t, 3w - \psi, 3w_t - \psi_t, w, w_t, \theta, \xi)\|_{\mathcal{H}}^2 + \bar{C}_2. \tag{4.24}$$

Combining (4.21) and (4.24), we obtain the result in (4.19). □

Lemma 4.8. *The functional L satisfies, along the solution of* (2.5)-(2.6)*, the estimate*

$$L'(t) + C_0 L(t) \leq C, \ \forall t \geq 0, \tag{4.25}$$

for some positive constants C, C_0.

Proof. Combining (3.6), Lemmas 4.2, 4.3, 4.4,4.5, 4.6, we have

$$\begin{aligned}
L'(t) \leq & -\left(\frac{n_2\rho}{2} - \frac{n_3\rho}{2}\right)\|\varphi_t\|^2 - (4\beta n - 9\rho n_3 - C)\|w_t\|^2 - \left(\frac{l_0}{4} - n_1\epsilon_1\right)\|(3w_x - \psi_x)\|^2 \\
& - \left(\frac{n_1 I_\rho (D - l_0)}{2} - n_2\rho - n_3\rho - I_\rho\right)\|(3w_t - \psi_t)\|^2 \\
& - \left(\frac{Gn_3}{2} - n_1\epsilon_1 - n_2\epsilon_2 - C\right)\|(\psi - \varphi_x)\|^2 - \frac{D}{4}\|w_x\|^2 \\
& - \left(\tau n - \epsilon_1 n_1 \lambda_1^2 - n_2 C\left(1 + \frac{1}{\epsilon_2}\right) - n_3 C - C\right)\|\theta_x\|^2 + \frac{n_1}{2}\|f_1\|^2 + C \\
& + \left(n_1 C\left(1 + \frac{1}{\epsilon_1}\right) + \frac{3(D - l_0)}{2}\right)\|\xi_x\|_{\mathcal{M}}^2 \\
& + \left(\frac{n}{2} - n_1 C\right)\int_0^1 \int_0^\infty g'(s)|\xi_x(x,s)|^2 ds dx.
\end{aligned}$$

Applying condition (2.2) and choosing

$$\epsilon_1 = \frac{l_0}{8n_1}, \epsilon_2 = \frac{Gn_3}{4n_2},$$

we arrive at

$$\begin{aligned}
L'(t) \leq & -\left(\frac{n_2\rho}{2} - \frac{n_3\rho}{2}\right)\|\varphi_t\|^2 - (4\beta n - 9\rho n_3 - C)\|w_t\|^2 - \frac{l_0}{8}\|(3w_x - \psi_x)\|^2 \\
& - \left(\frac{n_1 I_\rho (D - l_0)}{2} - n_2\rho - n_3\rho - I_\rho\right)\|(3w_t - \psi_t)\|^2 \\
& - \left(\frac{Gn_3}{4} - C\right)\|(\psi - \varphi_x)\|^2 - \frac{D}{4}\|w_x\|^2 \\
& - \left(\tau n - \frac{\lambda_1^2 l_0}{8} - n_2 C\left(1 + \frac{4n_2}{Gn_3}\right) - n_3 C - C\right)\|\theta_x\|^2 + \frac{n_1}{2}\|f_1\|^2 + C \\
& - \left(\lambda\left(\frac{n}{2} - n_1 C\right) - n_1 C\left(1 + \frac{8n_1}{l_0}\right) - \frac{3(D - l_0)}{2}\right)\|\xi_x\|_{\mathcal{M}}^2.
\end{aligned}$$

Now, we choose n_3 large enough so that

$$\left(\frac{Gn_3}{4} - C\right) > 0, \tag{4.26}$$

then we select n_2 large enough such that

$$\left(\frac{n_2\rho}{2} - \frac{n_3\rho}{2}\right) > 0. \tag{4.27}$$

Next, we select n_1 large enough so that

$$\left(\frac{n_1 I_\rho (D - l_0)}{2} - n_2\rho - n_3\rho - I_\rho\right) > 0, \tag{4.28}$$

and finally, we select n large such that (4.19) remains valid and

$$\begin{aligned}
&(4\beta n - 9\rho n_3 - C) > 0,\\
&\left(\tau n - \frac{\lambda_1^2 l_0}{8} - n_2 C\left(1 + \frac{8n_1}{l_0}\right) - n_3 C - C\right) > 0,\\
&\left(\lambda\left(\frac{n}{2} - n_1 C\right) - n_1 C\left(1 + \frac{1}{\epsilon_1}\right) - \frac{3(D - l_0)}{2}\right) > 0.
\end{aligned} \tag{4.29}$$

From (4.26)–(4.29), there exist a constant $\tilde{C}_0 > 0$ such that

$$L'(t) \leq -\tilde{C}_0 \|(\varphi, \varphi_t, 3w - \psi, 3w_t - \psi_t, w, w_t, \theta, \xi)\|_{\mathcal{H}}^2 + C, \tag{4.30}$$

By using Lemma 4.7, we obtain

$$L'(t) \leq -C_0 L(t) + C, \quad \forall t \geq 0. \tag{4.31}$$

This completes the proof. □

Theorem 4.2. *Under the assumptions of Theorem 3.1, the semigroup $S(t)$ of system* (2.5)-(2.6)*, possesses a bounded absorbing set $\mathcal{B}_1$ in $\mathcal{H}$.*

Proof. Integration of (4.25) over $(0, t)$ leads to

$$L(t) \leq L(0)e^{-C_0 t} + C\left(1 - e^{-C_0 t}\right) \leq L(0)e^{-C_0 t} + C. \tag{4.32}$$

Using (4.19), we get

$$\begin{aligned}
&\|(\varphi, \varphi_t, 3w - \psi, 3w_t - \psi_t, w, w_t, \theta, \xi)\|_{\mathcal{H}}^2\\
&\leq \frac{1}{\mu_1} L(0)e^{-C_0 t} + \frac{1}{\mu_1}\left(C + \bar{C}_1\right)\\
&\leq \frac{\mu_2}{\mu_1}\|(\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0, \xi_0)\|_{\mathcal{H}}^2 + \frac{1}{\mu_1}\left(C + \bar{C}_1 + \bar{C}_2\right).
\end{aligned} \tag{4.33}$$

Therefore, for $R > \sqrt{\frac{1}{\mu_1}\left(C + \bar{C}_1 + \bar{C}_2\right)}$, the ball $\mathcal{B}_1(0, R)$ is a bounded absorbing set in $(\mathcal{H}, S(t))$. This completes the proof. □

4.2. Asymptotic smoothness of the semigroup S(t)

Here, we establish the asymptotic smoothness of the semigroup $S(t)$ generated by system (2.5) – (2.6) in $\mathcal{H}$. We shall make use of the following lemma:

Lemma 4.9. *[10] Let H be a Banach space. Let $S(t)$ be a semigroup on H. Assume that for any $\mathcal{B} \subset H$ bounded and positively invariant and for any $t \geq t_0 = t_0(\mathcal{B}) \geq 0$, there exists a function $\Psi_{\mathcal{B}}(t)$ on $[t_0, \infty)$ and a pseudometric $\chi^t_{\mathcal{B}}$ on $C([0,t], H)$ such that*

(i) $\Psi_{\mathcal{B}}(t) \geq 0$ *and* $\lim_{t\to\infty} \Psi_{\mathcal{B}}(t) = 0$;

(ii) *the pseudometric $\chi^t_{\mathcal{B}}$ is precompact (with respect to the norm of H) in the sense that: any sequence $\{y_n\} \subset \mathcal{B}$ has a subsequence $\{y_{n_k}\}$ such that the sequence $\{z_k\} \in C([0,t], H)$ where $z_k = S(\tau)y_{n_k}$ is Cauchy with respect to $\chi^t_{\mathcal{B}}$;*

(iii) *there holds the estimate*

$$\|S(t)y_1 - S(t)y_2\|_H \leq \Psi_{\mathcal{B}}(t)\|y_1 - y_2\|_H + \chi^t_{\mathcal{B}}(\{S(\tau)y_1\}, \{S(\tau)y_2\}),\ \forall y_1, y_2 \in \mathcal{B},\ \forall\, t \geq t_0,$$

where $\{S(\tau)y_1\}$ is a function in $C([0,t], H)$ given by $y_i(\tau) = S(\tau)y_i$.

Then $S(t)$ is asymptotically smooth in H.

In what follows, we establish the asymptotic smoothness of the semigroup $S(t)$ generated by the system (2.5)–(2.6). Let

$$W^j = (\varphi^j, \varphi^j_t, (3w-\psi)^j, (3w_t - \psi_t)^j, w^j, w^j_t, \theta^j, \xi^j)^T,\ j = 1,2$$

be solutions of system (2.5)-(2.6) with corresponding initial data

$$W^j_0 = (\varphi^j_0, \varphi^j_1, (3w_0 - \psi_0)^j, (3w_1 - \psi_1)^j, w^j_0, w^j_1, \theta^j_0, \xi^j)^T \in \mathcal{B}, j = 1,2,$$

where $\mathcal{B} \subset \mathcal{H}$ is a bounded and positive invariant set for the semigroup $S(t)$. We set $W = W^1 - W^2$ and $W_0 = W^1_0 - W^2_0$. Then W satisfies

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x + \theta_x = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ I_\rho(3w-\psi)_{tt} - l_0(3w-\psi)_{xx} - \displaystyle\int_0^{+\infty} g(s)\xi_{xx}(x,s)ds & \\ \qquad\qquad - G(\psi-\varphi_x) - \theta = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ I_\rho w_{tt} - Dw_{xx} + G(\psi-\varphi_x) + \dfrac{4}{3}\gamma(h(w^1) - h(w^2)) + \dfrac{4}{3}\beta w_t = 0, & \\ & (x,t) \in (0,1)\times(0,+\infty), \\ k\theta_t - \tau\theta_{xx} + \varphi_{xt} + (3w-\psi)_t = 0, & (x,t) \in (0,1)\times(0,+\infty), \\ \xi_t + \xi_s - (3w-\psi)_t = 0, & (x,t) \in (0,1)\times(0,+\infty), \end{cases} \tag{4.34}$$

with boundary conditions

$$\begin{cases} \varphi_x(0,t) = \psi(0,t) = w(0,t) = \theta(0,t) = \xi(0,s) = 0, & t \in [0,+\infty), \\ \varphi(1,t) = \psi_x(1,t) = w_x(1,t) = \theta_x(1,t) = \xi_x(1,s) = 0, & t \in [0,+\infty). \end{cases} \tag{4.35}$$

The associated energy functional to system (4.34)-(4.35) is given by

$$\begin{aligned} E_0(t) =&\frac{1}{2}\left(\rho\|\varphi_t\|^2 + I_\rho\|(3w_t-\psi_t)\|^2 + l_0\|(3w_x-\psi_x)\|^2 + 3I_\rho\|w_t\|^2\right)\\ &+\frac{1}{2}\left(3D\|w_x\|^2 + G\|(\psi-\varphi_x)\|^2 + k\|\theta\|^2 + \|\xi_x\|^2_{\mathcal{M}}\right)\\ =&\frac{1}{2}\|(\varphi,\varphi_t,3w-\psi,3w_t-\psi_t,w,w_t,\theta,\xi)\|^2_{\mathcal{H}}. \end{aligned} \tag{4.36}$$

Moreover, $E_0(t)$ satisfies

$$\frac{d}{dt}E_0(t)\le -3\beta\|w_t\|^2 + C_{\mathcal{B}}\|w\|^2 - \tau\|\theta_x\|^2 + \frac{1}{2}\int_0^1\int_0^\infty g'(s)|\xi_x(x,s)|^2dsdx. \tag{4.37}$$

To see this, we multiply $(4.34)_1$ by φ_t, $(4.34)_2$ by $(3w_t-\psi_t)$, $(4.34)_3$ by $3w_t$, $(4.34)_4$ by θ in $L^2(0,1)$, using integration by parts and adding the outcomes, we get

$$\begin{aligned} \frac{dE_0(t)}{dt} =&-4\beta\|w_t\|^2 - 4\gamma(h(w^1)-h(w^2),w_t) - \tau\|\theta_x\|^2\\ &+\frac{1}{2}\int_0^1\int_0^\infty g'(s)|\xi_x(x,s)|^2dsdx\\ \le&-4\beta\|w_t\|^2 + 4\gamma\int_0^1|h(w^1)-h(w^2)||w_t|dx - \tau\|\theta_x\|^2\\ &+\frac{1}{2}\int_0^1\int_0^\infty g'(s)|\xi_x(x,s)|^2dsdx\\ \le&-3\beta\|w_t\|^2 + C_\beta\int_0^1|h(w^1)-h(w^2)|^2dx - \tau\|\theta_x\|^2\\ &+\frac{1}{2}\int_0^1\int_0^\infty g'(s)|\xi_x(x,s)|^2dsdx. \end{aligned} \tag{4.38}$$

We have

$$\int_0^1|h(w^1)-h(w^2)|^2dx \le \|h'(y)\|^2_{L^\infty(0,1)}\int_0^1|w|^2dx \le C_{\mathcal{B}}\|w\|^2. \tag{4.39}$$

Substituting (4.39) into (4.38), we obtain (4.37). Now, we define the functional L_0 by

$$L_0(t) = mE(t) + m_1K_1(t) + m_2K_2(t) + m_3K_3(t) + K_4(t) + K_5(t),$$

where m, m_1, m_2, m_3 are positive constants to be determined and

$$\begin{cases} K_1(t) = -I_\rho\displaystyle\int_0^1(3w_t-\psi_t)\int_0^{+\infty}g(s)\xi(x,s)dsdx,\\ K_2(t) = -k\rho\displaystyle\int_0^1\theta\int_0^x\varphi_t(y)dydx,\\ K_3(t) = \rho\displaystyle\int_0^1\varphi\varphi_tdx + \rho\int_0^1\psi\int_0^x\varphi_t(y)dydx,\\ K_4(t) = I_\rho\displaystyle\int_0^1(3w-\psi)(3w-\psi)_tdx,\\ K_5(t) = I_\rho\displaystyle\int_0^1ww_tdx. \end{cases}$$

Lemma 4.10. *There exist* $\beta_1 > 0, \beta_2 > 0$ *such that*

$$\beta_1 E_0(t) \le L_0(t) \le \beta_2 E_0(t), \ \forall t \ge 0. \tag{4.40}$$

Proof. Using Cauchy-Schwarz, Young's and Poincaré's inqualities, we see that

$$\begin{aligned} |L_0(t) - mE_0(t)| &\le m_1|K_1(t)| + m_2|K_2(t)| + m_3|K_3(t)| + |K_4(t)| + |K_5(t)| \\ &\le \gamma_1\|\varphi_t\|^2 + \gamma_2\|w_t\|^2 + \gamma_3\|(3w_t - \psi_t)\|^2 + \gamma_4\|w_x\|^2 \\ &\quad + \gamma_5\|(\psi - \varphi_x)\|^2 + \gamma_6\|(3w_x - \psi_x)\|^2 + \gamma_7\|\theta\|^2 + \gamma_8\|\xi_x\|^2_{\mathcal{M}}, \end{aligned} \tag{4.41}$$

where $\gamma_i, \ i = 1, 2, ..., 8$ are positive constants. From (4.36) and (4.41), we can find a positive constant $\bar{\gamma_0}$ such that

$$|L_0(t) - mE_0(t)| \le \bar{\gamma_0} E_0(t).$$

It follows that

$$(m - \bar{\gamma_0})E_0(t) \le L_0(t) \le (m + \bar{\gamma_0})E_0(t). \tag{4.42}$$

By choosing m large enough such that $(m - \bar{\gamma_0}) > 0$, we obtain the result. This completes the proof. □

Lemma 4.11. *There exists* $\nu > 0$ *such that*

$$L_0'(t) + \nu L_0(t) \le C_{\mathcal{B}}\|w(t)\|^2, \ \forall t \ge 0. \tag{4.43}$$

Proof. Using similar computations as in Lemmas 4.2-4.6, we have

$$\begin{cases} K_1'(t) \le -\dfrac{I_\rho(D - l_0)}{2}\|3w_t - \psi_t\|^2 + \epsilon_1\|3w_x - \psi_x\| + \epsilon_1\|\psi - \varphi_x\|^2 + \epsilon_1\lambda_1\|\theta_x\|^2 \\ \qquad + C\left(1 + \dfrac{1}{\epsilon_1}\right)\|\xi_x\|^2_{\mathcal{M}} - C\displaystyle\int_0^1\int_0^{+\infty} g'(s)|\xi_x(x, s)|^2 ds dx, \\ K_2'(t) \le -\dfrac{\rho}{2}\|\varphi_t\|^2 + \epsilon_2\|\psi - \varphi_x\|^2 + C\left(1 + \dfrac{1}{\epsilon_2}\right)\|\theta_x\|^2 + \rho\|3w_t - \psi_t\|^2, \\ K_3'(t) \le -\dfrac{G}{2}\|\psi - \varphi_x\|^2 + \dfrac{3\rho}{2}\|\varphi_t\|^2 + \rho\|3w_t - \psi_t\|^2 + 9\rho\|w_t\|^2 + C\|\theta_x\|^2, \\ K_4'(t) \le -\dfrac{l_0}{4}\|3w_x - \psi_x\|^2 + I_\rho\|3w_t - \psi_t\|^2 + C\|\xi_x\|^2_{\mathcal{M}} + C\|\psi - \varphi_x\|^2 + C\|\theta_x\|^2, \\ K_5'(t) \le -\dfrac{D}{2}\|w_x\|^2 + C\|w_t\|^2 + C\|\psi - \varphi_x\|^2 + \tilde{C}_{\mathcal{B}}\|w(t)\|^2. \end{cases} \tag{4.44}$$

Combining (4.37) and (4.44), we obtain

$$\begin{aligned}
L_0'(t) \leq & -\left(\frac{m_2\rho}{2}-\frac{m_3\rho}{2}\right)\|\varphi_t\|^2-(3\beta m-9\rho m_3-C)\|w_t\|^2 \\
& -\left(\frac{l_0}{4}-m_1\varepsilon_1\right)\|(3w_x-\psi_x)\|^2+(mC_{\mathcal{B}}+\tilde{C}_{\mathcal{B}})\|w(t)\|^2 \\
& -\left(\frac{m_1 I_\rho(D-l_0)}{2}-m_2\rho-m_3\rho-I_\rho\right)\|(3w_t-\psi_t)\|^2 \\
& -\left(\frac{Gm_3}{2}-m_1\varepsilon_1-m_2\varepsilon_2-C\right)\|(\psi-\varphi_x)\|^2-\frac{D}{2}\|w_x\|^2 \\
& -\left(m\tau-\varepsilon_1 m_1\lambda_1^2-m_2C\left(1+\frac{1}{\varepsilon_2}\right)-m_3C-C\right)\|\theta_x\|^2 \\
& +\left(m_1C\left(1+\frac{1}{\varepsilon_1}\right)+C\right)\|\xi_x\|_{\mathcal{M}}^2+\left(\frac{m}{2}-m_1C\right)\int_0^1\int_0^\infty g'(s)|\xi_x(x,s)|^2dsdx.
\end{aligned}$$

By appropriate choices of $m, m_1, m_2, m_3, \varepsilon_1, \varepsilon_2$ in a similar manner as done in Lemma 4.8, we obtain

$$L_0'(t) \leq CE_0(t)+C_{\mathcal{B}}\|w(t)\|^2. \tag{4.45}$$

Making use of (4.40) and (4.45), we get (4.43). □

Theorem 4.3. *Under the assumptions of Theorem 3.1, the semigroup* $S(t)$ *of system* (2.5)-(2.6) *is asymptotically smooth in* $\mathcal{H}$.

Proof. Integrating (4.43) over $(0,t)$ and making use of Lemma 4.10, we arrive at

$$E_0(t) \leq \frac{\beta_2}{\beta_1}E_0(0)e^{-\nu t}+C_{\mathcal{B}}\int_0^t\|w(s)\|^2ds.$$

It follows that

$$\begin{aligned}
&\|(\varphi,\varphi_t,3w-\psi,3w_t-\psi_t,w,w_t,\theta,\xi)\|_{\mathcal{H}}^2 \\
&\qquad \leq Ce^{-\nu t}\|(\varphi_0,\varphi_1,3w_0-\psi_0,3w_1-\psi_1,w_0,w_1,\theta_0,\xi_0)\|_{\mathcal{H}}^2+C_{\mathcal{B}}\int_0^t\|w(s)\|^2ds.
\end{aligned}$$

That is

$$\|S(t)W_0^1-S(t)W_0^2\|_{\mathcal{H}}^2 \leq Ce^{-\nu t}\|W_0^1-W_0^2\|_{\mathcal{H}}^2+C_{\mathcal{B}}\int_0^t\|w^1(s)-w^2(s)\|^2ds, \tag{4.46}$$

for every $W_0^1, W_0^2 \in \mathcal{B}$. Thus in other to apply Lemma 4.9, we set $\Psi_{\mathcal{B}}(t) := Ce^{-\nu t}$ and $\chi_{\mathcal{B}}^t : \mathcal{H}\times\mathcal{H} \longrightarrow \mathbb{R}$ is defined by

$$\chi_{\mathcal{B}}^t\left(S(\tau)W_0^1, S(\tau)W_0^2\right) = C_{\mathcal{B}}\sup_{0\leq\tau\leq t}\int_0^\tau\|w^1(s)-w^2(s)\|^2ds. \tag{4.47}$$

Now, we show that $\chi_{\mathcal{B}}^t$ is precompact. Consider the sequence

$$\{(\varphi_{0n},\varphi_{1n},3w_{0n}-\psi_{0n},3w_{1n}-\psi_{1n},w_{0n},w_{1n},\theta_{0n},\xi_{0n})\}\subset\mathcal{B}.$$

We have that $\mathcal{B} \subset \mathcal{H}$ is a bounded and positive invariant set, thus the corresponding solutions

$$\{((\varphi_n(t), (\varphi_t)_n(t), 3w_n(t) - \psi_n(t), 3(w_t)_n(t) - (\psi_t)_n(t), w_n(t), (w_t)_n(t), \theta_n(t), \xi_n(s))\}$$

is bounded uniformly in $\mathcal{H}$. Therefore

$$\{((\varphi_n(t), (\varphi_t)_n(t), 3w_n(t) - \psi_n(t), 3(w_t)_n(t) - (\psi_t)_n(t), w_n(t), (w_t)_n(t), \theta_n(t), \xi_n(s))\}$$

is a bounded sequence in

$$C([0,t], \mathcal{H}).$$

This implies $\{w_n(t)\}$ is bounded in

$$C\left([0,t], H^1_*(0,1)\right).$$

Using the compact embedding of

$$C\left([0,t], H^1_*(0,1)\right) \cap C^1\left([0,t], L^2(0,1)\right)$$

into

$$C\left([0,t], L^2(0,1)\right),$$

we can extract a subsequence $\{w_{n_j}(t)\}$ which converges strongly in

$$C\left([0,t], L^2(0,1)\right), \forall t > 0.$$

Therefore

$$\lim_{j\to\infty} \lim_{l\to\infty} \sup_{0\le\tau\le t} \int_0^\tau \|w_{n_j}(s) - w_{n_l}(s)\|^2 ds = 0,$$

hence

$$\lim_{j\to\infty} \lim_{l\to\infty} \chi^t_{\mathcal{B}}\left(S(\tau)W_{0n_j}, S(\tau)W_{0n_l}\right) = 0. \tag{4.48}$$

In addition, we have $\lim_{t\to\infty} \Psi_{\mathcal{B}}(t) = 0$. By applying Lemma 4.9, we obtain the asymptotic smoothness of the semigroup $S(t)$ in $\mathcal{H}$. □

Theorem 4.4. *Under the assumptions of Theorem* (3.1)*, the semigroup $S(t)$ of system* (2.5)-(2.6) *possesses the global attractor $\mathcal{A}$ in $\mathcal{H}$, which is compact.*

Proof. In Theorem 4.2, we showed that the semigroup $S(t)$ of system (2.5)-(2.6) possesses a bounded absorbing set $\mathcal{B}_1$ in $\mathcal{H}$, and in Theorem 4.3 we showed that the semigroup $S(t)$ of system (2.5)-(2.6) is asymptotically smooth in $\mathcal{H}$. By applying Theorem 4.1, we obtain the result. □

5. Finite-fractal dimensional attractor

In this section, we show that the global attractor $\mathcal{A}$ obtained in section 4 has a finite-fractal dimension. To do this, we recall some basic concepts and results. We refer the reader to [10] and references therein for more details.

Let X be a metric space and $K \subset X$ be a compact set, then the fractal dimension of K is given by

$$\dim_f^X K = \limsup_{\epsilon\to 0} \frac{\text{In}(n(K,\epsilon))}{\text{In}(1/\epsilon)},$$

where $n(K, \epsilon)$ is the minimal number of closed balls with radius ϵ that cover K.
Given a seminorm $n_X(\cdot)$ on a Banach space X, it is known that n_X is compact whenever for any sequence $x_j \to 0$ weakly in X we have that $n_X(x_j) \to 0$.
We state the following result [10, Theorem 2.10], which guarantees that the global attractor $\mathcal{A}$ has a finite fractal dimension.

Theorem 5.1. [10, Theorem 2.10] *Let X be a separable Hilbert space and $\mathcal{U} \subset X$ be bounded and closed. Assume that there exists a mapping $\mathcal{J} : \mathcal{U} \to X$ such that $\mathcal{U} \subseteq \mathcal{J}(\mathcal{U})$ and*

(i) *$\mathcal{J}$ is Lipschitz on $\mathcal{U}$, i.e., there exists $L > 0$ such that*

$$\|\mathcal{J}y_1 - \mathcal{J}y_2\|_X \leq L\|y_1 - y_2\|_X, \ \forall\, y_1, y_2 \in \mathcal{U}; \tag{5.1}$$

(ii) *there exist compact seminorms $n_X(\cdot)$ and $m_X(\cdot)$ on X such that*

$$\|\mathcal{J}y_1 - \mathcal{J}y_2\|_X \leq \gamma\|y_1 - y_2\|_X + \lambda[n_X(y_1 - y_2) + m_X(\mathcal{J}y_1 - \mathcal{J}y_2)], \tag{5.2}$$

for any $y_1, y_2 \in \mathcal{U}$ and some constants $0 < \gamma < 1$ and $\lambda > 0$.

Then $\mathcal{U}$ is compact in X and has a finite fractal dimension.

The following is the main result of this section:

Theorem 5.2. *Suppose the dynamical system $(H, S(t))$ generated by (2.5)-(2.6) possesses a global attractor $\mathcal{A}$ and if there exist nonnegative scalar functions $\alpha(t)$ and $\phi(t)$ which are locally bounded in $[0, +\infty)$, and $\beta(t) \in L^1((0, +\infty))$ with $\lim_{t\to+\infty} \beta(t) = 0$, such that*

$$\|S(t)W_0^1 - S(t)W_0^2\|_H^2 \leq \alpha(t)\|W_0^1 - W_0^2\|_H^2, \tag{5.3}$$

and

$$\|S(t)W_0^1 - S(t)W_0^2\|_H^2 \leq \beta(t)\|W_0^1 - W_0^2\|_H^2 + \phi(t) \sup_{0<s<t} \|w^1(s) - w^2(s)\|^2, \tag{5.4}$$

for any $t > 0$ and $W_0^i = (\varphi_0^i, \varphi_1^i, (3w_0 - \psi_0)^i, (3w_1 - \psi_1)^i, w_0^i, w_1^i, \theta_0^i, \xi_0^i) \in \mathcal{A}$, $i = 1, 2$, where $S(t)W_0 = W(t)$. Then the global attractor $\mathcal{A}$ has a finite fractal dimension.

Proof. We adopt the method of [10, Theorem 3.11], by applying Theorem 5.1. Let

$$W_0^i = (\varphi_0^i, \varphi_1^i, (3w_0 - \psi_0)^i, (3w_1 - \psi_1)^i, w_0^i, w_1^i, \theta_0^i, \xi_0^i) \in \mathcal{A},\ i = 1, 2.$$

We observe that $W = W^1 - W^2$ satisfies (4.1). Thus, performing similar computations as in Lemma 4.1, we get

$$\|S(t)W_0^1 - S(t)W_0^2\|_{\mathcal{H}}^2 \leq e^{\kappa t}\|W_0^1 - W_0^2\|_{\mathcal{H}}^2, \ \forall t \geq 0. \tag{5.5}$$

Thus, we take $\alpha(t) = e^{\kappa t}$ in (5.3) and we easily see that $\alpha(t)$ is locally bounded in $[0, +\infty)$. Next, we show that (5.4) is satisfied. Indeed, integrating (4.43) over $(0, t)$ and making use of (4.40), we get

$$\|S(t)W_0^1 - S(t)W_0^2\|_{\mathcal{H}}^2 \leq \frac{\beta_2}{\beta_1} E_0(0)e^{-\nu t} + \frac{C}{\beta_1}\int_0^t e^{-\nu(t-s)}\|w(s)\|^2 ds$$

$$\leq C_0 e^{-\nu t}\|W_0^1 - W_0^2\|_{\mathcal{H}}^2 + C\int_0^t e^{-\nu(t-s)}ds \sup_{0<s<t}\|w^1(s) - w^2(s)\|^2$$
$$= \beta(t)\|W_0^1 - W_0^2\|_{\mathcal{H}}^2 + \phi(t)\sup_{0<s<t}\|w^1(s) - w^2(s)\|^2, \tag{5.6}$$

where
$$\beta(t) = C_0 e^{-\nu t},\ \phi(t) = C\int_0^t e^{-\nu(t-s)}ds,\ t \geq 0.$$

It's easy to see that
$$\beta(t) \in L^1(\mathbb{R}^+) \ \ and \ \ \lim_{t\to 0}\beta(t) = 0,$$

and that $\phi(t)$ is locally bounded in $[0,\infty)$.
We take $X = \mathcal{H} \times L^2(0,T;\mathcal{H})$ for some $T > 1$, where

$$L^2(0,T;\mathcal{H}) = \left\{ z(t) : \|z\|_{L^2(0,T;\mathcal{H})}^2 \equiv \int_0^T \|z(t)\|_{\mathcal{H}}^2 dt < \infty \right\}.$$

We denote the norm in X as
$$\|V\|_X^2 = \|W_0\|_{\mathcal{H}}^2 + \|z\|_{L^2(0,T;\mathcal{H})}^2,$$

where $V = (W_0, z)$, $W_0 = (\varphi_0, \varphi_1, 3w_0 - \psi_0, 3w_1 - \psi_1, w_0, w_1, \theta_0, \xi_0)$.
Let $w(t) = w^1(t) - w^2(t)$, then integrating (5.6) over $[T, 2T]$ with respect to t, we get

$$\int_T^{2T} \|S(t)W_0^1 - S(t)W_0^2\|_{\mathcal{H}}^2 dt \leq \beta_T\|W_0^1 - W_0^2\|^2 + \phi_T \sup_{0<s<2T}\|w^1(s) - w^2(s)\|^2, \tag{5.7}$$

where $\beta_T = \int_T^{2T}\beta(t)dt$ and $\phi_T = \int_T^{2T}\phi(t)dt$. Also, from (5.6) we deduce

$$\|S(T)W_0^1 - S(T)W_0^2\|_{\mathcal{H}}^2 \leq \beta(T)\|W_0^1 - W_0^2\|_{\mathcal{H}}^2 + \phi(T)\sup_{0<s<2T}\|w^1(s) - w^2(s)\|^2. \tag{5.8}$$

Addition of (5.7) and (5.8) yield

$$\|S(T)W_0^1 - S(T)W_0^2\|_{\mathcal{H}}^2 + \int_T^{2T}\|S(t)W_0^1 - S(t)W_0^2\|_{\mathcal{H}}^2 dt$$
$$\leq \tilde{\beta}_T\|W_0^1 - W_0^2\|_{\mathcal{H}}^2 + \tilde{\phi}_T \sup_{0<s<2T}\|w^1(s) - w^2(s)\|^2, \tag{5.9}$$

where $\tilde{\beta}_T = \beta_T + \beta(T)$ and $\tilde{\phi}_T = \phi_T + \phi(T)$.
Now, we consider in the space X the subset

$$\mathcal{A}_T := \{V \equiv (W(0), W(t)); t \in [0,T] : W(0) \in \mathcal{A}\},$$

where $W(0) = (\varphi, \varphi_t, (3w-\psi), (3w-\psi)_t, w, w_t, \theta, \xi)(0)$ and $W(t)$ is the solution to (2.5)-(2.6) with initial data $W_0 = W(0)$. In addition, we define the operator

$$\mathcal{J} : \mathcal{A}_T \to X$$
$$(W(0); W(t)) \mapsto (W(T), W(t+T)),$$

$\mathcal{J}$ is clearly Lipschitz on $\mathcal{A}_T$ with Lipschitz constant $\|\mathcal{J}\|$, moreover, $\mathcal{J}\mathcal{A}_T = \mathcal{A}_T$. It follows from (5.9) that

$$\|\mathcal{J}V^1 - \mathcal{J}V^2\|_X^2 \leq \tilde{\beta}_T\|V^1 - V^2\|_X^2 + \tilde{\phi}_T([n_X(V^1 - V^2)]^2 + [n_X(\mathcal{J}V^1 - \mathcal{J}V^2)]^2),$$

for any $V^1, V^2 \in \mathcal{A}_T$, where $n_X(V) = \sup_{0\leq s\leq T} \|w(s)\|$.
We have that $L^2(0,T;\mathcal{H})$ is compactly embedded in $C(0,T;\mathcal{H})$, thus n_X is a compact seminorm on X. We choose T such that $\tilde{\beta}_T \leq \frac{1}{2}$, then application of Theorem 5.1 implies that $\mathcal{A}_T$ is a compact set in X with finite fractal dimension.
We define the following projection operator by

$$\mathcal{P} : X \to \mathcal{H}$$
$$(W(0), W(t)) \mapsto W(0).$$

Clearly, $\mathcal{P}$ is Lipschitz continuous and $\mathcal{P}\mathcal{A}_T = \mathcal{A}$, it follows that

$$\dim_f^{\mathcal{H}}\mathcal{A} \leq \dim_f^X\mathcal{A}_T < \infty.$$

The proof is complete. □

6. Conclusions

In this article, we have established the well-posedness and finite fractal-dimensional global attractor for a non-linear thermoelastic laminated beam, where the heat conduction is given by Fourier's law. The presence of the nonlinear source term in system (1.1) gives an obvious challenge to establish that the system is dissipative. We are able to achieve this by defining and estimating several Lyapunov functionals.

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Conflict of interest

The authors declare that there is no conflict of interest.

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