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Research article

Convergence analysis and error estimate of finite element method of a nonlinear fluid-structure interaction problem

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Abstract: In this paper, a semi-discrete finite element method for the nonlinear fluid-structure interaction problem interacts between the Navier-Stokes fluids and linear elastic solids, is studied and developed. A classical mixed variational principle of the weak formulation is given, and the corresponding finite element method is defined. As for the nonlinearity arising from the nonlinear interaction problem, we consider in time of a solution for suitably small data, and uniqueness hypothesis. This approach is fairly robust and adapts to the important case of interface with fractures or cracks. Convergence and estimate of the finite element method are also obtained for the nonlinear fluid-structure interaction problem. Finally, numerical experiments are presented to show the performance of the proposed method.

Keywords: fluid-structure interaction; Navier-Stokes equations; finite element method; convergence analysis; error estimate; numerical analysis
Mathematics Subject Classification: Primary 35J20; Secondary 65N08, 76D05

1. Introduction

Fluid-structure interactions are interactions of some movable or deformable structure with an internal or surrounding fluid flow. The variety of fluid-structure occurrences are abundant and ranges from tent-roofs to micropumps, from parachutes via airbags to blood flow in arteries. It is the most important on both modelling, computational issues and applications, the most challenging multi-physics problems for engineers, mathematicians and physicists. The topic of fluid-structure has recently attracted more and more attention in the scientific community. Different methods have been developed and analyzed such as mathematical model [11, 26], mathematical theory [4, 13], the weak solution [20, 26], spectrum asymptotics [25], Lagrange multiplier method [3] and other method [27]. The literature regarding finite element methods can be found in [8, 9, 15, 23].

In this paper we describe a semi-discrete finite element scheme for the nonlinear fluid-structure interaction problem, which interact between the Navier-Stokes fluids and linear elastic solids. There are lots of literatures on fluid-structure interactions for which the fluid is modeled by viscous fluid models [2, 10, 19, 21]. However, the majority of them applies solid models of lower spatial dimensions or linear interaction problem. Especially, in this paper, we consider the interaction of a nonlinear viscous fluid with elastic body motion in bounder domain. We retain the condition: the interface Γ_0 between the fluid structure with continuous velocities and stresses. To some extend, numerical analysis of the fluid-structure interaction problem is more difficult than that of the fluid-fluid interaction problem. Here, we assume that the solid displacements of the linear elastic problem are infinitesimal size is of practical interest. Therefore, the approach provided in fluid-fluid interaction problem can be adapted to the fluid-structure case.

In the past several decades, their motivation, development and theoretical foundations have been presented in lots of literatures [3, 5, 15, 27]. This method can be considered of the extend of the reference [8, 9], in the sense that it intends to use the same Galerkin finite element method to analyze the nonlinear fluid-structure interaction problem. Here, we discuss the analysis of finite element method for fluid-structure interaction problem, which couples with the Navier-Stokes equations and linear elastic equations. The analysis of this model is not straightforward even if the data is sufficiently smooth. We must take special care of the nonlinear discrete terms arising from the finite element discretization for the fluid-structure interaction problem; these nonlinear trilinear terms no longer satisfy the anti-symmetry properties. Therefore, compared to the finite element analysis of the linear interaction problem, the most challenging aspect rests in the treatment of the nonlinear convection terms [12, 14, 16–18, 22, 24, 28], which has a significant impact on the analysis. In this paper, we analyze the discrete methods in time of a solution for suitably small data, and uniqueness of a suitably small solution, without smooth solutions. Therefore, the approach presented here is fairly robust and adapts to the important case of interface with fractures or cracks. On the other hand, numerical experiments are also provided for the model presented to confirm the theoretical results.

The rest of paper is organized as follows. In section 2, we introduce the fluid-structure model using the Navier-Stokes equation with the linear elastic equation. In section 3, the finite element method of the fluid-structure model is defined and its existence and uniqueness are provided. The convergence and estimate of the presented method are obtained in sections 4 and 5. Finally, we present several numerical examples to illustrate the features of the proposed methods in section 6.

2. Preliminaries

Let the Lipschitz bounded domain $\Omega = \Omega_1 \bigcup \Omega_2$ consist of two subdomains Ω_1 and Ω_2 of R^d, $d = 2, 3$, coupled across an interface $I_0 = \partial\Omega_1 \bigcap \partial\Omega_2$. $\Gamma_1 = \partial\Omega_1 \backslash I_0$, $\Gamma_2 = \partial\Omega_2 \backslash I_0$. Moreover, $\mathbf{n}_1$ and $\mathbf{n}_2$ denote the outward unit normal vectors for Ω_1 and Ω_2, respectively. The coupled fluid-structure problem is stated as follows: In the fluid region, the governer problem is

$$\begin{aligned}\rho_1\mathbf{v}_t + \nabla p - \mu_1\nabla\cdot(\nabla\mathbf{v} + \nabla\mathbf{v}^T) + (\mathbf{v}\cdot\nabla)\mathbf{v} &= \rho_1\mathbf{f}_1, \text{ in } \Omega_1,\\ \text{div}\mathbf{v} &= 0, \text{ in } \Omega_1,\\ \mathbf{v} &= 0, \text{ on } \Gamma_1,\\ \mathbf{v}(0,x) &= \mathbf{v}_0, \text{ in } \Omega_1, \end{aligned}\tag{2.1}$$

where the viscosity $\mu_1 > 0$, the density $\rho_1 > 0$, the body force $\mathbf{f}_1 : [0,T]\to H^1(\Omega_1)$, $\mathbf{v}_0$ is the initial value on $t = 0$. $\mathbf{v} : \Omega_1 \times [0,T]\to R^d$ and $p : \Omega_1 \times [0,T]\to R$ denote the velocity and pressure, respectively.

In the solid region, the solid is assumed to be governed by the following linear elasticity

$$\begin{aligned}\rho_2\mathbf{u}_{tt} - \mu_2\nabla\cdot(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - \lambda_2\nabla(\nabla\cdot\mathbf{u}) &= \rho_2\mathbf{f}_2, \text{ in } \Omega_2,\\ \mathbf{u} &= 0, \text{ on } \Gamma_2,\\ \mathbf{u}(0,x) = \mathbf{u}_0,\ \mathbf{u}_t(0,x) &= \mathbf{u}_1, \text{ in } \Omega_2,\end{aligned}\tag{2.2}$$

where μ_2 and λ_2 denote the Lame constants, ρ_2 the constant solid density, $\mathbf{u} : \Omega_2 \times [0,T]\to R^d$ the displacement of the solid, $\mathbf{f}_2 : [0,T]\to H^1(\Omega_2)$ the given loading force per unit mass, and $\mathbf{u}_0$ and $\mathbf{u}_1$ the given initial data.

Here, we begin as in the case with a fixed interface: the motion of the solid is wholly due to infinitesimal displacements. Again, we assume that the fluid-solid interface is stationary. Although the displacement $\mathbf{u}$ is small, the velocity $\mathbf{u}_t$ is not. Thus, we cannot impose the no-slip condition on the fluid velocity and must retain the interface condition $\mathbf{v} = \mathbf{u}_t$, along a fixed boundary. Then, across the fixed interface I_0 between the fluid and solid, the velocity and stress vector are continuous:

$$\begin{aligned}\mathbf{u}_t &= \mathbf{v} \text{ on } I_0, \qquad (2.3)\\ \mu_2(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\cdot\mathbf{n}_2 + \lambda_2(\nabla\cdot\mathbf{u})\mathbf{n}_2 &= p\mathbf{n}_1 - \mu_1(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\cdot\mathbf{n}_1 \text{ on } I_0.\end{aligned}$$

For the mathematical setting of problem (2.1)-(2.2), the following Hilbert spaces are introduced [1]:

$$\mathbf{X}_i = [H_0^1(\Omega_i)]^d = \{\mathbf{v} \in [H^1(\Omega_i)]^d : \mathbf{v}|_{\Gamma_i} = 0\},\ i = 1,2,$$

$$Q = L^2(\Omega_1),$$

$$\mathbf{\Psi} = \{\mathbf{w} \in [H_0^1(\Omega)]^d : \text{div}\mathbf{w} = 0 \text{ in } \Omega_1\}.$$

The fluid-structure interaction problem can be rewritten in variational form as follows: Given

$$\mathbf{f}_i \in C([0,T];\mathbf{L}^2(\Omega_i)),$$

$$\mathbf{v}_0 \in \mathbf{X}_1,\ \text{div}\mathbf{v}_0 = 0 \text{ in } \Omega_1,$$

$$\mathbf{u}_0 \in \mathbf{X}_2,\ \mathbf{u}_1 \in \mathbf{X}_2,\ \mathbf{v}_0|_{\Gamma_0} = \mathbf{u}_1|_{\Gamma_0},$$

such that $(\mathbf{v}, p, \mathbf{u}) \in L^2([0,T];\mathbf{X}_1) \times L^2([0,T];Q) \times L^2([0,T];\mathbf{X}_2)$

$$\rho_1[\mathbf{v}_t, \mathbf{w}]_{\Omega_1} + a_1[\mathbf{v}, \mathbf{w}] + b[\mathbf{w}, p] + \rho_2[\mathbf{u}_{tt}, \mathbf{w}]_{\Omega_2} + a_2[\mathbf{u}, \mathbf{w}] + c[\mathbf{v}, \mathbf{v}, \mathbf{w}]$$

$$= \rho_1[\mathbf{f}_1, \mathbf{w}]_{\Omega_1} + \rho_2[\mathbf{f}_2, \mathbf{w}]_{\Omega_2},\ \forall \mathbf{w} \in \mathbf{X} \in [H_0^1(\Omega)]^d, \tag{2.4}$$

$$b[\mathbf{v}, q] = 0,\ \forall q \in Q, \tag{2.5}$$

$$\mathbf{v}(0, x) = \mathbf{v}_0,\ \mathbf{u}(0, x) = \mathbf{u}_0,\ \mathbf{u}_t(0, x) = \mathbf{u}_1, \tag{2.6}$$

$$\int_0^t \mathbf{v}(s)|_{\Gamma_0} ds = \mathbf{u}(t)|_{\Gamma_0} - \mathbf{u}_0|_{\Gamma_0} \text{ a.e. t.} \tag{2.7}$$

Next, the divergence-free weak formulation for (2.4)-(2.7) is defined as follows: Given

$$\begin{aligned} &\mathbf{f}_i \in C([0, T]; \mathbf{L}^2(\Omega_i)), \\ &\mathbf{v}_0 \in \mathbf{X}_1,\ \text{div}\mathbf{v}_0 = 0 \text{ in } \Omega_1, \\ &\mathbf{u}_0 \in \mathbf{X}_2,\ \ \mathbf{u}_1 \in\ \ X_2,\ \mathbf{v}_0|_{\Gamma_0} = \mathbf{u}_1|_{\Gamma_0}, \end{aligned} \tag{2.8}$$

seek a pair $(\mathbf{v}, \mathbf{u}) \in L^2([0, T]; \mathbf{X}_1) \times L^2([0, T]; \mathbf{X}_2)$, $\text{div}\mathbf{v} = 0$ such that

$$\rho_1[\mathbf{v}_t, \mathbf{w}]_{\Omega_1} + \rho_2[\mathbf{u}_{tt}, \mathbf{w}]_{\Omega_2} + a_1[\mathbf{v}, \mathbf{w}] + a_2[\mathbf{u}, \mathbf{w}] + c[\mathbf{v}, \mathbf{v}, \mathbf{w}]$$
$$= \rho_1[\mathbf{f}_1, \mathbf{w}]_{\Omega_1} + \rho_2[\mathbf{f}_2, \mathbf{w}]_{\Omega_2},\ \forall \mathbf{w} \in \Psi, \tag{2.9}$$

$$\mathbf{v}(0, x) = \mathbf{v}_0,\ \mathbf{u}(0, x) = \mathbf{u}_0,\ \mathbf{u}_t(0, x) = \mathbf{u}_1, \tag{2.10}$$

$$\int_0^t \mathbf{v}(s)|_{\Gamma_0} ds = \mathbf{u}(t)|_{\Gamma_0} - \mathbf{u}_0|_{\Gamma_0} \text{ a.e. t,} \tag{2.11}$$

where the continuous bilinear forms $[\cdot, \cdot]_{\Omega_i}$, $a_i[\cdot, \cdot]$ and $b[\cdot, \cdot]$ are defined on $\mathbf{X}_i \times \mathbf{X}_i$ and $\mathbf{X}_1 \times Q$, respectively, by

$$[\mathbf{w}_1, \mathbf{w}]_{\Omega_i} = \int_{\Omega_i} \mathbf{w}_1 \mathbf{w} d\Omega_i,\ \ \mathbf{w}_1, \mathbf{w} \in \mathbf{X}_i,$$

$$a_1[\mathbf{v}, \mathbf{w}] = \frac{\mu_1}{2} \int_{\Omega_1} (\nabla \mathbf{v} + \nabla \mathbf{v}^T) : (\nabla \mathbf{w} + \nabla \mathbf{w}^T) d\Omega,\ \ \mathbf{v}, \mathbf{w} \in \mathbf{X}_1,$$

$$a_2[\mathbf{u}, \mathbf{w}] = \frac{\mu_2}{2} \int_{\Omega_2} \Big((\nabla \mathbf{u} + \nabla \mathbf{u}^T) : (\nabla \mathbf{w} + \nabla \mathbf{w}^T) + \lambda_2 (\text{div}\mathbf{u})(\text{div}\mathbf{w}) \Big) d\Omega,\ \ \mathbf{u}, \mathbf{w} \in \mathbf{X}_2,$$

$$b[\mathbf{v}, q] = -\int_{\Omega_1} \text{div}\mathbf{v} q d\Omega,\ \ \forall \mathbf{v} \in \mathbf{X}_1,\ q \in Q.$$

Then, the following inequalities hold

$$a_1[\mathbf{v}, \mathbf{v}] \geq \mu_1 ||\nabla \mathbf{v}||^2_{0,\Omega_1},\ \forall \mathbf{v} \in \mathbf{X}_1, a_2[\mathbf{u}, \mathbf{u}] \geq \frac{1}{2} |\mathbf{u}|^2_{0,\Omega_2},\ \forall \mathbf{u} \in \mathbf{X}_2, \tag{2.12}$$

where

$$|\mathbf{v}|_{0,\Omega_2} \equiv \left(\mu ||\nabla \mathbf{v}||^2_{0,\Omega_2} + ||\text{div}\mathbf{v}||_{0,\Omega_2} \right)^{1/2}$$

is equivalent to the classical H^1-norm. Moreover, the bilinear term $b[\cdot, \cdot]$ satisfy the inf-sup condition for the whole system (2.4)-(2.7) and the Navier-Stokes equations:

$$\inf_{q \in Q} \sup_{\mathbf{w} \in X_1} \frac{b[\mathbf{w}, q]}{||\nabla \mathbf{w}||_{0,\Omega} ||q||_{0,\Omega_1}} \ \ \geq\ \ \beta, \tag{2.13}$$

where the positive constant β is dependent of Ω_1. Similarly, the trilinear term $c[\cdot,\cdot,\cdot]$ is defined as follows [28]:

$$c[\mathbf{v}_1,\mathbf{v}_2,\mathbf{w}] \;=\; (\mathbf{v}_1\cdot\nabla\mathbf{v}_2,\mathbf{w}),\ \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{w}\in\mathbf{X}_1.$$

Also, the following inequality is valid:

$$|c[\mathbf{u},\mathbf{v},\mathbf{w}]| \le C_0\|\nabla\mathbf{u}\|_{0,\Omega_1}\|\nabla\mathbf{v}\|_{0,\Omega_1}\|\nabla\mathbf{w}\|_{0,\Omega_1},\ \mathbf{u}_1,\ \mathbf{v}_2,\ \mathbf{w}\in\mathbf{X}_1. \tag{2.14}$$

Using the auxiliary problem and the results in [8, 28], we yield the following existence and uniqueness of the divergence-free weak formulation for (2.9)-(2.11). For convenience, we set $\mathbf{f}_{i,t}\equiv\partial_t\mathbf{f}_i, i=1,2$ in the following.

In order to deal with the nonlinear terms, we have the following lemma.

Lemma 2.1. *Assume that both* $\mathbf{v}$ *satisfy the following smallness condition*

$$\|\nabla\mathbf{v}\|_{0,\Omega_1}\le\frac{\mu_1}{4C_0}, \tag{2.15}$$

for all $t\in[0,T]$. *Then, we have the estimate*

$$|((\mathbf{v}\cdot\nabla)\mathbf{v}_1,\mathbf{w})|\le\frac{\mu_1}{4}\|\nabla\mathbf{v}_1\|_0\|\nabla\mathbf{w}\|_0\ \forall\ \mathbf{v}_1,\mathbf{w}\in\mathbf{X}_1. \tag{2.16}$$

Lemma 2.2. *Under the hypothesis of (2.8) and* (2.15) *below, the solution* $(\mathbf{v},\mathbf{u})\in L^2([0,T];\mathbf{X}_1)\times L^2([0,T];\mathbf{X}_2)$ *for (2.9)-(2.11) has the following error estimates:*

$$\begin{aligned}
&\|\nabla\mathbf{v}\|_{L^\infty([0,T],L^2(\Omega_1))}<\frac{\mu_1}{4C_0},\\
&\|\mathbf{v}\|^2_{L^\infty([0,T],L^2(\Omega_1))}+\|\mathbf{u}_t\|^2_{L^\infty([0,T],L^2(\Omega_2))}+\|\mathbf{v}\|^2_{L^2([0,T];X_1)}\le\kappa_0,\\
&\|\mathbf{v}_t\|^2_{L^\infty([0,T],L^2(\Omega_1))}+\|\mathbf{u}_{tt}\|^2_{L^\infty([0,T],L^2(\Omega_2))}\le\kappa_1,
\end{aligned} \tag{2.17}$$

where $\kappa_i, i=0,1$ *are defined in* (2.21) *and* (2.27), *respectively.*

Proof. Let the positive constant $\gamma>0$ only depend on the Ω. Then, the following inequalities hold true

$$\|\mathbf{v}\|_{0,\Omega_1}\le\gamma\|\nabla\mathbf{v}\|_{0,\Omega_1},\ \ \|\mathbf{v}\|_{0,\Omega_1}\le\gamma\|\mathbf{v}\|_{1,\Omega_1},\ \gamma>0. \tag{2.18}$$

Choosing $\mathbf{w}$ in (2.9) with

$$\mathbf{w}|_{\Omega_i}=\begin{cases}\mathbf{v} & \text{if } i=1,\\ \mathbf{u}_t & \text{if } i=2,\end{cases}$$

and using the Young inequality, get

$$\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\|\mathbf{v}\|^2_{0,\Omega_1}+\frac{\rho_2}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2_{0,\Omega_2}+\frac{1}{2}\frac{d}{dt}|\mathbf{u}|^2_{0,\Omega_2}+\mu_1\|\nabla\mathbf{v}\|^2_{0,\Omega_1}\\
\le\ & C_0\|\nabla\mathbf{v}\|^3_{0,\Omega_1}+\rho_1\|\mathbf{f}_1\|_{0,\Omega_1}\|\mathbf{v}\|_{0,\Omega_1}+\rho_2\|\mathbf{f}_2\|_{0,\Omega_2}\|\mathbf{u}_t\|_{0,\Omega_2}
\end{aligned}$$

$$
\begin{aligned}
&\le C_0 \times \frac{\mu_1}{4C_0}\|\nabla \mathbf{v}\|_{0,\Omega_1}^2 + \frac{\mu_1}{4}\|\nabla \mathbf{v}\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\|\mathbf{u}_t\|_{0,\Omega_1}^2 + \frac{\rho_1^2\gamma^2}{\mu_1}\|\mathbf{f}_1\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\|\mathbf{f}_2\|_{0,\Omega_2}^2 \\
&= \frac{\mu_1}{2}\|\nabla \mathbf{v}\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\|\mathbf{u}_t\|_{0,\Omega_2}^2 + \frac{\rho_1^2\gamma^2}{\mu_1}\|\mathbf{f}_1\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\|\mathbf{f}_2\|_{0,\Omega_2}^2. \qquad (2.19)
\end{aligned}
$$

Noting that $\frac{\mu_1}{2}\|\nabla \mathbf{v}\|_{0,\Omega_1}^2$ can be absorbed by the left hand of the above inequality and integrating the above inequality with respect to the time from 0 to $s \in (0, T]$, we have

$$
\begin{aligned}
& \frac{\rho_1}{2}\|\mathbf{v}\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\|\mathbf{u}_t\|_{0,\Omega_2}^2 + \frac{1}{2}|\mathbf{u}|_{0,\Omega_2}^2 + \frac{\mu_1}{2}\int_0^s \|\nabla \mathbf{v}\|_{0,\Omega_1}^2 dt \\
\le\ & \frac{\rho_1}{2}\|\mathbf{v}_0\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\|\mathbf{u}_1\|_{0,\Omega_2}^2 + \frac{1}{2}|\mathbf{u}_0|_{0,\Omega_2}^2 + \frac{\rho_2}{2}\int_0^s \|\mathbf{u}_t\|_{0,\Omega_2}^2 dt \\
& + \int_0^{T^*} \left(\frac{\rho_1^2\gamma^2}{\mu_1}\|\mathbf{f}_1\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\|\mathbf{f}_2\|_{0,\Omega_2}^2\right) dt \\
=\ & \kappa_0, \qquad (2.20)
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_0 =\ & \frac{\rho_1}{2}\|\mathbf{v}_0\|_{0,\Omega_1}^2 + \frac{\rho_2}{2}\|\mathbf{u}_1\|_{0,\Omega_2}^2 + \frac{1}{2}|\mathbf{u}_0|_{0,\Omega_2}^2 + \frac{\rho_1^2\gamma^2 T}{\mu_1}\|\mathbf{f}_1\|_{L^\infty([0,T],L^2(\Omega_1))}^2 \\
& + \frac{\rho_2 T}{2}\|\mathbf{f}_2\|_{L^\infty([0,T],L^2(\Omega_2))}^2. \qquad (2.21)
\end{aligned}
$$

Using the Gronwall inequality, yields that

$$
\begin{aligned}
& \|\mathbf{v}\|_{0,\Omega_1}^2 + \|\mathbf{u}_t\|_{0,\Omega_2}^2 + |\mathbf{u}|_{0,\Omega_2}^2 + \|\mathbf{v}\|_{L^2([0,T],\mathbf{X}_1)}^2 \\
\le\ & Ce^{CT}\kappa_0 \le C\kappa_0,\ \forall s \in (0, T]. \qquad (2.22)
\end{aligned}
$$

Using (2.15), the Young inequality, and choosing appropriate parameter, yields

$$
\begin{aligned}
& a_1[\mathbf{v}, \mathbf{v}] + a_2[\mathbf{u}, \mathbf{u}] \\
=\ & -\rho_1[\mathbf{v}_t, \mathbf{v}] - \rho_2[\mathbf{u}_{tt}, \mathbf{u}] - c[\mathbf{v}, \mathbf{v}, \mathbf{v}] + \rho_1[\mathbf{f}_1, \mathbf{v}] + \rho_2(\mathbf{f}_2, \mathbf{u}) \\
\le\ & \rho_1\|\mathbf{v}_t\|_{0,\Omega_1}\|\mathbf{v}\|_{0,\Omega_1} + \rho_2\|\mathbf{u}_{tt}\|_{0,\Omega_2}\|\mathbf{u}\|_{0,\Omega_2} + C_0\|\nabla \mathbf{v}\|_{0,\Omega_1}^3 \\
& + \rho_1\|\mathbf{f}\|_{0,\Omega_1}\|\mathbf{v}\|_{0,\Omega_1} + \rho_2\|\mathbf{f}_2\|_{0,\Omega_2}\|\mathbf{u}\|_{0,\Omega_2} \\
\le\ & \frac{2\rho_1^2\gamma^2}{\mu_1}\|\mathbf{v}_t\|_{0,\Omega_1}^2 + \frac{\mu_1}{8}\|\nabla \mathbf{v}\|_{0,\Omega_1}^2 + \rho_2^2\gamma^2\|\mathbf{u}_{tt}\|_{0,\Omega_2}^2 + \frac{1}{4}\|\nabla \mathbf{u}\|_{0,\Omega_2}^2 + \frac{\mu_1}{4}\|\nabla \mathbf{v}\|_{0,\Omega_1}^2 \\
& + \frac{2\rho_1^2\gamma^2}{\mu_1}\|\mathbf{f}_1\|_{0,\Omega_1}^2 + \frac{\mu_1}{8}\|\nabla \mathbf{v}\|_{0,\Omega_1}^2 + \rho_2^2\gamma^2\|\mathbf{f}_2\|_{0,\Omega_2}^2 + \frac{1}{4}\|\nabla \mathbf{u}\|_{0,\Omega_2}^2
\end{aligned}
$$

which implies

$$
\begin{aligned}
& \frac{\mu_1}{2}\|\nabla \mathbf{v}\|_{0,\Omega_1}^2 + \frac{1}{2}|\mathbf{u}|_{0,\Omega_2}^2 \\
\le\ & \frac{2\rho_1^2\gamma^2}{\mu_1}\|\mathbf{v}_t\|_{0,\Omega_1}^2 + \rho_2^2\gamma^2\|\mathbf{u}_{tt}\|_{0,\Omega_2}^2 + \frac{2\rho_1^2\gamma^2}{\mu_1}\|\mathbf{f}_1\|_{0,\Omega_1}^2 + \rho_2^2\gamma^2\|\mathbf{f}_2\|_{0,\Omega_2}^2. \qquad (2.23)
\end{aligned}
$$

In order to estimate the above inequality, we bound the two terms $\|\mathbf{v}_t\|_0$ and $\|\mathbf{u}_{tt}\|_0$. First, we differentiate the first equations of (2.1) and (2.2) with respect to the time, multiplying (2.1) and (2.2) by the respective test functions $\mathbf{v}_t$ and $\mathbf{u}_{tt}$, respectively, and integrating them on the respective domains Ω_1 and Ω_2 to obtain the following:

$$\begin{aligned}
&\frac{d}{dt}(\frac{\rho_1}{2}\|\mathbf{v}_t\|^2_{0,\Omega_1}+\frac{\rho_2}{2}\|\mathbf{u}_{tt}\|^2_{0,\Omega_2})+\frac{1}{2}\frac{d}{dt}|\mathbf{u}_t|^2_{0,\Omega_2}+\mu_1\|\nabla\mathbf{v}_t\|^2_{0,\Omega_1}\\
\leq\ & 2C_0\|\nabla\mathbf{v}\|_{0,\Omega_1}\|\nabla\mathbf{v}_t\|^2_{0,\Omega_1}+\rho_1\|\mathbf{f}_{t1}\|_{0,\Omega_1}\|\mathbf{v}_t\|_{0,\Omega_1}+\rho_2\|\mathbf{f}_{t2}\|_{0,\Omega_2}\|\mathbf{u}_{tt}\|_{0,\Omega_2}\\
\leq\ & \frac{\mu_1}{2}\|\nabla\mathbf{v}_t\|^2_{0,\Omega_1}+\frac{2\rho_1^2\gamma^2}{\mu_1}\|\mathbf{f}_{t1}\|^2_{0,\Omega_1}+\frac{\mu_1}{8}\|\nabla\mathbf{v}_t\|^2_{0,\Omega_1}\\
&+\frac{\rho_2^2\gamma^2}{2}\|\mathbf{f}_{t2}\|^2_{0,\Omega_2}+\frac{1}{2}|\mathbf{u}_{tt}|^2_{0,\Omega_2}.
\end{aligned}\tag{2.24}$$

Using (2.15), and integrating the above inequality from 0 to $s\in(0,T]$, we can get

$$\begin{aligned}
&\rho_1\|\mathbf{v}_t\|^2_{0,\Omega_1}+\rho_2\|\mathbf{u}_{tt}\|^2_{0,\Omega_2}+|\mathbf{u}_t|^2_{0,\Omega_2}+\mu_1\int_0^s\|\nabla\mathbf{v}_t\|^2_{0,\Omega_1}dt\\
\leq\ & C(\rho_1\|\mathbf{v}_t(0)\|^2_{0,\Omega_1}+\rho_2\|\mathbf{u}_{tt}(0)\|^2_{0,\Omega_2}+|\mathbf{u}_1|^2_{0,\Omega_2})\\
&+C\int_0^s(\|\mathbf{f}_{t1}\|^2_{0,\Omega_1}+\|\mathbf{f}_{t2}\|^2_{0,\Omega_2})dt+\int_0^s|\mathbf{u}_t|^2_{0,\Omega_2}dt.
\end{aligned}\tag{2.25}$$

Here, $\mathbf{v}_t(0)$ and $\mathbf{u}_{tt}(0)$ can be bounded by the following procedure. Taking $t=0$ and setting

$$\mathbf{w}|_{\Omega_i}=\begin{cases}\mathbf{v}_t(0) & \text{if } i=1,\\ \mathbf{u}_{tt}(0) & \text{if } i=2,\end{cases}$$

we have

$$\begin{aligned}
&\rho_1\|\mathbf{v}_t(0)\|^2_{0,\Omega_1}+\rho_2\|\mathbf{u}_{tt}(0)\|^2_{0,\Omega_2}+a_1[\mathbf{v}_0,\mathbf{v}_t(0)]+b[\mathbf{v}_t(0),p_0]\\
&\quad+a_2(\mathbf{u}_0,\mathbf{u}_{tt}(0))+c[\mathbf{v}_0,\mathbf{v}_0,\mathbf{v}_t(0)]\\
=\ & \rho_1[\mathbf{f}_1(0),\mathbf{v}_t(0)]+\rho_2[\mathbf{f}_2(0),\mathbf{u}_{tt}(0)],
\end{aligned}$$

which together with (2.3) implies

$$\begin{aligned}
&\rho_1\|\mathbf{v}_t(0)\|^2_0+\rho_2\|\mathbf{u}_{tt}(0)\|^2_0+(-\mu_2\mathrm{div}(\nabla\mathbf{u}_0+\mathbf{u}_0^T)-\lambda_2\nabla(\mathrm{div}\mathbf{u}_0),\mathbf{u}_{tt}(0))\\
&+(-\mu_1\mathrm{div}(\nabla\mathbf{u}_0+\mathbf{u}_0^T)+\nabla p_0+\mathbf{v}_0\cdot\nabla\mathbf{v}_0,\mathbf{v}_t(0))\\
=\ & \rho_1[\mathbf{f}_1(0),\mathbf{v}_t(0)]+\rho_2[\mathbf{f}_2(0),\mathbf{u}_{tt}(0)].
\end{aligned}$$

That is

$$\rho_1\|\mathbf{v}_t(0)\|^2_{0,\Omega_1}+\rho_2\|\mathbf{u}_{tt}(0)\|^2_{0,\Omega_2}\leq C(\rho_1\|\mathbf{f}_1(0)\|^2_{0,\Omega_1}+\rho_2\|\mathbf{f}_2(0)\|^2_{0,\Omega_2}).\tag{2.26}$$

Then, setting

$$\kappa_1=C\sum_{i=1}^2(\rho_i\|\mathbf{f}_i(0)\|^2_{0,\Omega_1}+\|\mathbf{f}_{ti}\|^2_{L^2([0,T],L^2(\Omega_i))}).\tag{2.27}$$

combining (2.25) with (2.26), and applying the Gronwall inequality, we obtain

$$
\begin{aligned}
\|\mathbf{v}_t\|^2_{0,\Omega_1} + \|\mathbf{u}_{tt}\|^2_{0,\Omega_2} \leq & \; C(\rho_1\|\mathbf{f}_1(0)\|^2_{0,\Omega_1} + \rho_2\|\mathbf{f}_2(0)\|^2_{0,\Omega_2}) \\
& + \int_0^s (\|\mathbf{f}_{t1}\|^2_{0,\Omega_1} + \|\mathbf{f}_{t2}\|^2_{0,\Omega_2})dt) < \kappa_1,
\end{aligned} \tag{2.28}
$$

□

Using the same approach as above for the solution of (2.9)-(2.11), we can obtain the following stability of the solution to (2.4)-(2.7).

Lemma 2.3. *Under the hypothesis of Lemma 2.1 and*

$$
\begin{aligned}
\mathbf{f}_{i,t} \in L^2([0,T];\mathbf{L}^2(\Omega_i)), i = 1,2, \; \mathbf{v}_0 \in [H^2(\Omega_1)]^d, \\
\mathbf{u}_1 \in [H^1(\Omega_2)]^d, \; \mathbf{u}_0 \in [H^2(\Omega_2)]^d,
\end{aligned} \tag{2.29}
$$

the solution $(\mathbf{v}, p, \mathbf{u})$ *to (2.4)-(2.7) satisfies*

$$
\begin{aligned}
\mathbf{v} \in L^\infty([0,T];\mathbf{L}^2(\Omega_1)) \cap L^2([0,T];\mathbf{X}_1), \; \mathbf{v}_t \in L^\infty([0,T];\mathbf{L}^2(\Omega_1)) \cap L^2([0,T];\mathbf{X}_1), \\
\mathbf{u} \in L^\infty([0,T];\mathbf{X}_2), \; \mathbf{u}_t \in L^\infty([0,T];\mathbf{X}_2), \; \mathbf{u}_{tt} \in L^\infty([0,T];\mathbf{L}^2(\Omega_2)),
\end{aligned} \tag{2.30}
$$

and

$$
\begin{aligned}
& \|\mathbf{v}_t\|^2_{L^\infty([0,T];\mathbf{L}^2(\Omega_1))} + \|\mathbf{u}_{tt}\|^2_{L^\infty([0,T];\mathbf{L}^2(\Omega_2))} + \|\mathbf{v}_t\|^2_{L^2([0,T];\mathbf{X}_1)} + \|\mathbf{u}_t\|^2_{L^\infty([0,T];\mathbf{X}_2)} \\
& \quad + \|p\|_{L^2([0,T];L^2(\Omega_1))} \\
\leq & \; Ce^{CT}(\|\mathbf{f}\|^2_{H^1([0,T];\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_1} + \|p_0\|_{1,\Omega_2} + \|\mathbf{u}_1\|^2_{1,\Omega_2}).
\end{aligned} \tag{2.31}
$$

Proof. Using the same approach as for the linear fluid-structure interaction problem [8, 9] and Lemma 2.1, we can obtain (2.31).

3. Finite element methods

Given two shape-regular, quasi-uniform triangulation $\mathcal{T}_{h_i}$ of $\Omega_i, i = 1, 2$, the finite element method is to solve (2.4)-(2.7) in a pair of finite dimensional spaces $(\mathbf{X}^h_1, Q^h, \mathbf{X}^h_2) \subset (\mathbf{X}_1, Q, \mathbf{X}_2)$: The triangulations $\mathcal{T}^h_i$ do not cross the interface Γ_0 and consist of triangular elements in two dimensions or tetrahedral elements in three dimensions [6, 7].

Moreover, we assume that the finite element spaces $(\mathbf{X}^h_1, Q^h, \mathbf{X}^h_2) \subset (\mathbf{X}_1, Q, \mathbf{X}_2)$ satisfies the following approximation properties: For each $\mathbf{w} \in [H^2(\Omega_i)]^d$ and $q \in H^1(\Omega) \cap Q$, there exist approximations $\mathbf{w}_h \in X^h_i$ and $q_h \in Q^h$ such that [28]

$$
\begin{aligned}
\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega_i} + h\|\nabla(\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega_i} & \leq Ch^{r+1}\|\mathbf{w}\|_{r+1,\Omega_i}, \\
\|q - q_h\|_{0,\Omega_1} & \leq Ch^r\|q\|_{r,\Omega_1},
\end{aligned} \tag{3.1}
$$

Here, r is the degree of piecewise polynomial of the finite elements. For each $\mathbf{w}_h \in X^h_i$, we have the inverse inequality

$$
\|\nabla\mathbf{w}_h\|_{0,\Omega_i} \leq Ch^{-1}\|\mathbf{w}_h\|_{0,\Omega_i}, \;\; \mathbf{w}_h \in X^h_i. \tag{3.2}
$$

Moreover, the so-called inf-sup condition is valid: for each $q_h \in Q^h$, there exists $\mathbf{w}_h \in X_h, \mathbf{v}_h \neq 0$, such that [9]

$$\inf_{q_h \in Q^h} \sup_{\mathbf{w}_h \in X_h} \frac{d(\mathbf{w}_h, q_h)}{\|\nabla \mathbf{w}_h\|_{0,\Omega} \|q_h\|_{0,\Omega_1}} \geq \beta, \tag{3.3}$$

where β is a positive constant depending on Ω. In the special case, (3.3) is valid for a general choice $\mathbf{X}_1^h$ and Q^h.

The corresponding semi-discrete fluid-structure interaction problem can be rewritten in variational form as follows: seek $(\mathbf{v}_h, p_h, \mathbf{u}_h) \in C^1([0,T]; \mathbf{X}_1^h) \times C([0,T]; Q^h) \times C^1([0,T]; \mathbf{X}_2^h)$ such that [12, 28]

$$\begin{aligned} \rho_1[\mathbf{v}_{ht}, \mathbf{w}_h]_{\Omega_1} + a_1[\mathbf{v}_h, \mathbf{w}_h] + b[\mathbf{w}_h, p_h] + \rho_2[\mathbf{u}_{htt}, \mathbf{w}_h]_{\Omega_2} + a_2[\mathbf{u}_h, \mathbf{w}_h] + c[\mathbf{u}_h, \mathbf{u}_h, \mathbf{w}_h] & \\ = \rho_1[\mathbf{f}_1, \mathbf{w}_h]_{\Omega_1} + \rho_2[\mathbf{f}_2, \mathbf{w}_h]_{\Omega_2}, \ \forall \mathbf{w}_h \in \mathbf{X}^h, & \quad (3.4) \\ b[\mathbf{v}_h, q_h] = 0, \ \forall q_h \in Q^h, & \quad (3.5) \\ \mathbf{v}_h(0,x) = \mathbf{v}_{0h}, \ \mathbf{u}_h(0,x) = \mathbf{u}_{0h}, \ \mathbf{u}_{ht}(0,x) = \mathbf{u}_{1h}, & \quad (3.6) \\ \mathbf{v}_h|_{\Gamma_0} = \mathbf{u}_{ht}|_{\Gamma_0}, \ \text{a.e. t}. & \quad (3.7) \end{aligned}$$

If we define divergence-free finite element space

$$\Psi^h = \{\mathbf{w}_h \in \mathbf{X}_1^h : b[\mathbf{w}_h, q_h] = 0, \ \forall q_h \in Q^h\},$$

then the corresponding weak formulation of semi-discrete finite element methods for (3.4)-(3.7) is defined as follows: seek a pair $(\mathbf{v}_h, \mathbf{u}_h) \in C^1([0,T]; \mathbf{\Psi}_h) \times C^1([0,T]; \mathbf{X}_2^h)$ such that

$$\begin{aligned} \rho_1[\mathbf{v}_{ht}, \mathbf{w}_h]_{\Omega_1} + \rho_2[\mathbf{u}_{htt}, \mathbf{w}_h]_{\Omega_2} + a_1[\mathbf{v}_h, \mathbf{w}_h] + a_2[\mathbf{u}_h, \mathbf{w}_h] + c[\mathbf{u}_h, \mathbf{u}_h, \mathbf{w}_h] & \\ = \rho_1[\mathbf{f}_1, \mathbf{w}_h]_{\Omega_1} + \rho_2[\mathbf{f}_2, \mathbf{w}_h]_{\Omega_2}, \ \forall \mathbf{w}_h \in \Psi^h, & \quad (3.8) \\ \mathbf{v}_h(0,x) = \mathbf{v}_{0h}, \ \mathbf{u}_h(0,x) = \mathbf{u}_{0h}, \ \mathbf{u}_{ht}(0,x) = \mathbf{u}_{1h}, & \quad (3.9) \\ \mathbf{v}_h|_{\Gamma_0} = \mathbf{u}_{ht}|_{\Gamma_0}, \ \text{a.e. t}. & \quad (3.10) \end{aligned}$$

Here, the approximation of the initial condition $(\mathbf{v}_{0h}, p_{0h}, \mathbf{u}_{1h}) \in (\mathbf{X}_1^h, Q^h, \mathbf{X}_2^h)$ of $(\mathbf{v}_0, p_0, \mathbf{u}_1)$ is defined as follows: for $\forall (\mathbf{w}_h, q_h) \in \mathbf{X}^h \times Q^h$

$$\begin{aligned} a_1[\mathbf{v}_{0h}, \mathbf{w}_h] + [\mathbf{u}_{1h}, \mathbf{w}_h] + b[\mathbf{w}_h, p_{0h}] &= a_1[\mathbf{v}_0, \mathbf{w}_h] + [\mathbf{u}_1, \mathbf{w}_h]_{\Omega_2} + b[\mathbf{w}_h, p_0], & (3.11) \\ b[\mathbf{v}_{0h}, q_h] &= 0, & (3.12) \\ \mathbf{v}_{0h}|_{\Gamma_0} &= \mathbf{u}_{1h}|_{\Gamma_0}, & (3.13) \end{aligned}$$

such that

$$\begin{aligned} & \|\nabla(\mathbf{v}_0 - \mathbf{v}_{0h})\|_{0,\Omega_1} + \|p_0 - p_{0h}\|_{0,\Omega_1} + \|\mathbf{u}_1 - \mathbf{u}_{1h}\|_{0,\Omega_2} \\ \leq \ & Ch^r(\|\mathbf{v}_0\|_{r+1,\Omega_1} + \|p_0\|_{r,\Omega_1} + \|\mathbf{u}_1\|_{r+1,\Omega_2}), \end{aligned} \tag{3.14}$$

with $\mathbf{v}_0 \in \mathbf{H}^{r+1}(\Omega_1), \ p_0 \in H^r(\Omega_1)$ and $\mathbf{u}_1 \in \mathbf{H}^{r+1}(\Omega_2), \ r \in [0,k]$. Moreover, we assume that $\mathbf{u}_{0h} = P_h u_h$ is defined by

$$a_2[\mathbf{u}_{0h}, \mathbf{w}_h] = a_2[\mathbf{u}_0, \mathbf{w}_h], \ \forall \mathbf{w}_h \in X_2^h. \tag{3.15}$$

Based on the results [8, 12, 28], we can obtain the following existence and uniqueness of the semi-discrete finite element approximation for the fluid-structure interaction.

Lemma 3.1 *Under the hypothesis of Lemmas 2.1-2.2, there exists a unique solution to (3.4)-(3.7) such that* $(\mathbf{v}_h, p_h, \mathbf{u}_h) \in C^1([0,T];\mathbf{X}_1^h)\times C([0,T];Q^h)\times C([0,T];\mathbf{X}_2^h)$ *and furthermore has the following bound*

$$\begin{aligned}&\|\mathbf{v}_h\|^2_{L^\infty([0,T];\mathbf{L}^2(\Omega_1))} + \|\mathbf{u}_{ht}\|^2_{L^\infty([0,T];\mathbf{L}^2(\Omega_2))} + \|\mathbf{v}_h\|^2_{L^2([0,T];\mathbf{X}_1)} + \|\mathbf{u}_h\|^2_{L^\infty([0,T];\mathbf{X}_2)}\\ &\leq Ce^{CT}(\|\mathbf{f}\|^2_{L^2([0,T];\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{1,\Omega_2} + \|\mathbf{v}_0\|^2_{1,\Omega_1} + \|p_0\|_{0,\Omega_1} + \|\mathbf{u}_1\|^2_{0,\Omega_2}),\end{aligned}$$

and

$$\begin{aligned}&\|\mathbf{v}_{ht}\|^2_{L^\infty([0,T];\mathbf{L}^2(\Omega_1))} + \|\mathbf{u}_{htt}\|^2_{L^\infty([0,T];\mathbf{L}^2(\Omega_2))} + \|\mathbf{v}_{ht}\|^2_{L^2([0,T];\mathbf{X}_1)} + \|\mathbf{u}_{ht}\|^2_{L^\infty([0,T];\mathbf{X}_2)}\\ &\quad+\|p_h\|_{L^2([0,T];L^2(\Omega_1))}\\ \leq\ & Ce^{CT}(\|\mathbf{f}\|^2_{H^1([0,T];\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{2,\Omega_2} + \|\mathbf{v}_0\|^2_{2,\Omega_2} + \|p_0\|^2_{1,\Omega_2} + \|\mathbf{u}_1\|^2_{2,\Omega_2}).\end{aligned}$$

Proof. Using the same approach as for Lemmas 2.1-2.2, we can obtain the proof of Lemma 3.1.

4. Convergence

In this section, we mainly consider the convergence of the semi-discrete finite element method for the nonlinear fluid-structure interaction problem. Then, we will provide the stability of the limit of the finite element approximation based on the results on the previous section.

Theorem 4.1 *Assume that the finite element meshes are nested and the data* $(v_0, u_0, u_1, f_1, f_2)$ *satisfies* (2.8) *and* (2.29). *Let* $(\mathbf{v}_h, p_h, \mathbf{u}_h) \in \mathbf{X}_1^h\times Q^h\times\mathbf{X}_2^h$ *be the solution to (3.4)-(3.7), then there exists a unique solution* $(\mathbf{v}, p, \mathbf{u}) \in \mathbf{X}_1 \times Q \times \mathbf{X}_2$ *satisfying*

$$\begin{aligned}\mathbf{v} &\in L^\infty([0,T];\mathbf{L}^2(\Omega_1))\bigcap L^2([0,T];\mathbf{X}_1),\\ \mathbf{v}_t &\in L^\infty([0,T];\mathbf{L}^2(\Omega_1))\bigcap L^2([0,T];\mathbf{X}_1),\ \ p\in L^2([0,T];L^2(\Omega_1)),\\ \mathbf{u} &\in L^\infty([0,T];\mathbf{X}_2), \mathbf{u}_t\in L^\infty([0,T];\mathbf{X}_2), \mathbf{u}_{tt}\in L^\infty([0,T];\mathbf{L}^2(\Omega_2)),\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\mathbf{v}_h \overset{*}{\rightharpoonup} \mathbf{v}\ \textit{in}\ L^\infty([0,T];\mathbf{L}^2(\Omega_1)), \qquad \mathbf{v}_h\to\mathbf{v},\ \textit{in}\ L^2([0,T];\mathbf{X}_1),\\ &\mathbf{v}_{ht} \overset{*}{\rightharpoonup} \mathbf{v}_t\ \textit{in}\ L^\infty([0,T];\mathbf{L}^2(\Omega_1)), \qquad \mathbf{v}_{ht}\to\mathbf{v}_t\ \textit{in}\ L^2([0,T];\mathbf{X}_1),\\ &\mathbf{u}_h \overset{*}{\rightharpoonup} \mathbf{u}\ \textit{in}\ L^\infty([0,T];\mathbf{X}_2), \qquad \mathbf{u}_{htt} \overset{*}{\rightharpoonup} \mathbf{u}_{tt}\ \textit{in}\ L^\infty([0,T];\mathbf{L}^2(\Omega_2)),\\ &\mathbf{u}_{ht} \overset{*}{\rightharpoonup} \mathbf{u}_t\ \textit{in}\ L^\infty([0,T];\mathbf{L}^2(\Omega_2)), \qquad \mathbf{u}_{ht} \overset{*}{\rightharpoonup} \mathbf{u}_t\ \textit{in}\ L^\infty([0,T];\mathbf{X}_2),\\ &p_h\to p\ \textit{weakly in}\ L^2([0,T];L^2(\Omega_1)).\end{aligned}\tag{4.2}$$

Furthermore, $(\mathbf{v}, p, \mathbf{u})$ *satisfies* (2.4)-(2.7).

Proof. Using the boundness of the finite element solution $(\mathbf{v}_h, p_h, \mathbf{u}_h)$, we may extract a subsequence $(\mathbf{v}_{h_\mu}, p_{h_\mu}, \mathbf{u}_{h_\mu})$ from it with the mesh scale decreasing to zero as $\mu\to\infty$, and satisfy (4.1)-(4.2). A proof of the error estimate can be found in [8, 9]. For completeness and to show the results of the nonlinear fluid-structure interaction, we will sketch the proof here.

As for the trilinear term, we have for $\mathbf{v} \in L^{\infty}([0,T];\mathbf{L}^2(\Omega_1)) \cap L^2([0,T];\mathbf{X}_1)$ and $\mathbf{w} \in C^1([0,T];\mathbf{X}_1^{h_\mu})$

$$\int_0^T c[\mathbf{v}_{h_\mu}, \mathbf{v}_{h_\mu}, \mathbf{w}]dt = \int_0^T dt \int_{\Omega_1} \sum_{i,j} \mathbf{v}^i_{h_\mu} \frac{\partial \mathbf{v}^j_{h_\mu}}{\partial x^i} \mathbf{w}^j d\Omega. \tag{4.3}$$

For all $\mu > N$, passing to the limit as $\mu \to \infty$, yields that

$$\int_0^T dt \int_{\Omega_1} \sum_{i,j} \mathbf{v}^i_{h_\mu} \frac{\partial \mathbf{v}^j_{h_\mu}}{\partial x^i} \mathbf{w}^j d\Omega \;\to\; \int_0^T dt \int_{\Omega_1} \sum_{i,j} \mathbf{v}^i \frac{\partial \mathbf{v}^j}{\partial x^i} \mathbf{w}^j d\Omega$$

since

$$\begin{aligned}
&\left| \int_0^T dt \int_{\Omega_1} \sum_{i,j} \mathbf{v}^i \frac{\partial \mathbf{v}^j}{\partial x^i} \mathbf{w}^j d\Omega - \int_0^T dt \int_{\Omega_1} \sum_{i,j} \mathbf{v}^i_{h_\mu} \frac{\partial \mathbf{v}^j_{h_\mu}}{\partial x^i} \mathbf{w}^j d\Omega \right| \\
\leq\ & \left| \int_0^T dt \int_{\Omega_1} \sum_{i,j} (\mathbf{v}^i - \mathbf{v}^i_{h_\mu}) \frac{\partial \mathbf{v}^j}{\partial x^i} \mathbf{w}^j d\Omega + \int_0^T dt \int_{\Omega_1} \sum_{i,j} \mathbf{v}^i_{h_\mu} \left(\frac{\partial \mathbf{v}^j_{h_\mu}}{\partial x^i} - \frac{\partial \mathbf{v}^j}{\partial x^i}\right) \mathbf{w}^j d\Omega \right| \\
\leq\ & \|(\mathbf{v}^i - \mathbf{v}^i_{h_\mu})\|_{L^\infty([0,T];\mathbf{L}^2(\Omega_1))} \|v^j\|_{L^2([0,T];\mathbf{X}_1)} \|w^j\|_{L^2([0,T];\mathbf{L}^2(\Omega_1))} \\
& + \|\mathbf{v}^i_{h_\mu}\|_{L^\infty([0,T];\mathbf{L}^2(\Omega_1))} \|v^j - v^j_{h_\mu}\|_{L^2([0,T];\mathbf{X}_1)} \|w^j\|_{L^2([0,T];\mathbf{L}^2(\Omega_1))}.
\end{aligned}$$

Recalling [8, 9], we can obtain the following results. Since $\mathbf{u}_{ht} \to \mathbf{u}_t$ weak start convergence in $L^\infty([0,T];\mathbf{L}^2(\Omega_2))$ and $L^\infty([0,T];\mathbf{L}^2(\Omega_2))$ is dense in $L^2([0,T];\mathbf{L}^2(\Omega_2))$, we can obtain the strong convergence for $\mathbf{u}_{ht} \to \mathbf{u}_t$ in $L^2([0,T];L^2(\Omega_2))$ with its norm. Similarly, we can also have the same result on the strong convergence for $\mathbf{v}_h \to \mathbf{v}$ in $L^2([0,T];\mathbf{L}^2(\Omega_1))$ with its norm. Thus, all these implies $\mathbf{v}|_{\Gamma_0} = \mathbf{u}_t|_{\Gamma_0}$. Applying the same approach as above, we can obtain (2.5) since $\bigcup_{\mu=N}^{\infty} L^2([0,T];Q^{h_\mu})$ is dense in $L^2([0,T];L^2(\Omega_1))$.

Using the estimates of semi-discrete finite element approximation in Lemma 3.1 and analysis above, we have for each fixed N with $\mu > N$

$$\begin{aligned}
&\int_0^T (\rho_1[\mathbf{v}_{h_\mu t}, \mathbf{w}]_{\Omega_1} + a_1[\mathbf{v}_{h_\mu}, \mathbf{w}] + b[\mathbf{w}, p_{h_\mu}] + c[\mathbf{v}_{h_\mu}, \mathbf{v}_{h_\mu}, \mathbf{w}] \\
&\qquad + \rho_2[\mathbf{u}_{tth_\mu}, \mathbf{w}]_{\Omega_2} + a_2[\mathbf{u}_{h_\mu}, \mathbf{w}])dt \\
&= \int_0^T (\rho_1[\mathbf{f}_1, \mathbf{w}]_{\Omega_1} + \rho_2[\mathbf{f}_2, \mathbf{w}]_{\Omega_2})dt, \ \ \mathbf{w} \in C^1([0,T];\mathbf{X}^{h_\mu}).
\end{aligned} \tag{4.4}$$

Thus, passing to the limit as $\mu \to \infty$, we conclude that

$$\begin{aligned}
&\int_0^T (\rho_1[\mathbf{v}_t, \mathbf{w}]_{\Omega_1} + a_1[\mathbf{v}, \mathbf{w}] + b[\mathbf{w}, p] + c[\mathbf{v}, \mathbf{v}, \mathbf{w}] + \rho_2[\mathbf{u}_{tt}, \mathbf{w}]_{\Omega_2} + a_2[\mathbf{u}, \mathbf{w}])dt \\
&\qquad = \int_0^T (\rho_1[\mathbf{f}_1, \mathbf{w}]_{\Omega_1} + \rho_2[\mathbf{f}_2, \mathbf{w}]_{\Omega_2})dt, \forall w \in C^1([0,T], X^{h_\mu})
\end{aligned} \tag{4.5}$$

and noting $\bigcup_{\mu=N}^{\infty} C^1([0,T];\mathbf{X}^{h_\mu})$ is dense in $L^2([0,T];\mathbf{X})$, yields that

$$\int_0^T (\rho_1[\mathbf{v}_t,\mathbf{w}]_{\Omega_1} + a_1[\mathbf{v},\mathbf{w}] + b[\mathbf{w},p] + c[\mathbf{v},\mathbf{v},\mathbf{w}] + \rho_2[\mathbf{u}_{tt},\mathbf{w}]_{\Omega_2} + a_2[\mathbf{u},\mathbf{w}])dt$$
$$= \int_0^T (\rho_1[\mathbf{f}_1,\mathbf{w}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\mathbf{w}]_{\Omega_2})dt,\ \ \mathbf{w}\in\mathbf{X}, a.e.t. \tag{4.6}$$

Moreover, we will consider the convergence of the finite element approximate on the initial value and interface boundary. First, setting $\mathbf{w}(T) = 0$ and noticing

$$\begin{aligned}\int_0^T [\mathbf{v}_t,\mathbf{w}]_{\Omega_1}dt &= \int_0^T \frac{d}{dt}[\mathbf{v},\mathbf{w}]_{\Omega_1}dt - \int_0^T [\mathbf{v},\mathbf{w}_t]_{\Omega_1}dt\\ &= [\mathbf{v}(T),\mathbf{w}(T)]_{\Omega_1} - [\mathbf{v}(0),\mathbf{w}(0)]_{\Omega_1} - \int_0^T [\mathbf{v},\mathbf{w}_t]_{\Omega_1}dt,\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\int_0^T [\mathbf{u}_{tt},\mathbf{w}]_{\Omega_2}dt &= \int_0^T \frac{d}{dt}[\mathbf{u}_t,\mathbf{w}]_{\Omega_2}dt - \int_0^T [\mathbf{u}_t,\mathbf{w}_t]_{\Omega_2}dt\\ &= [\mathbf{u}_t(T),\mathbf{w}(T)]_{\Omega_2} - [\mathbf{u}_t(0),\mathbf{w}(0)]_{\Omega_2} - \int_0^T [\mathbf{u}_t,\mathbf{w}_t]_{\Omega_2}dt,\end{aligned} \tag{4.8}$$

we obtain from (4.6) for $\mathbf{w}\in C^1([0,T];\mathbf{X})$

$$\int_0^T (-\rho_1[\mathbf{v},\mathbf{w}_t]_{\Omega_1} + a_1[\mathbf{v},\mathbf{w}] + b[\mathbf{w},p] + c[\mathbf{v},\mathbf{v},\mathbf{w}] - \rho_2[\mathbf{u}_t,\mathbf{w}_t]_{\Omega_2} + a_2[\mathbf{u},\mathbf{w}])dt$$
$$= \int_0^T (\rho_1[\mathbf{f}_1,\mathbf{w}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\mathbf{w}]_{\Omega_2})dt + \rho_1[\mathbf{v}(0),\mathbf{w}(0)]_{\Omega_1} + \rho_2[\mathbf{u}_t(0),\mathbf{w}(0)]_{\Omega_2}. \tag{4.9}$$

On the other hand, by the same reason of (4.4), and passing to the limit as $\mu\to\infty$, we infer for $\mathbf{w}\in C^1([0,T];\mathbf{X}^{h_\mu})$

$$\int_0^T (-\rho_1[\mathbf{v},\mathbf{w}_t]_{\Omega_1} + a_1[\mathbf{v},\mathbf{w}] + b[\mathbf{w},p] + c[\mathbf{v},\mathbf{v},\mathbf{w}] - \rho_2[\mathbf{u}_t,\mathbf{w}_t]_{\Omega_2} + a_2[\mathbf{u},\mathbf{w}])dt$$
$$= \int_0^T (\rho_1[\mathbf{f}_1,\mathbf{w}]_{\Omega_1} + \rho_2[\mathbf{f}_2,\mathbf{w}]_{\Omega_2})dt + \rho_1[\mathbf{v}_0,\mathbf{w}(0)]_{\Omega_1} + \rho_2[\mathbf{u}_1,\mathbf{w}(0)]_{\Omega_2}, \tag{4.10}$$

which together with (4.9) implies

$$\rho_1[\mathbf{v}(0)-\mathbf{v}_0,\mathbf{w}(0)]_{\Omega_1} + \rho_2[\mathbf{u}_t(0)-\mathbf{u}_1,\mathbf{w}(0)]_{\Omega_2} = 0,\ \ \mathbf{w}(0)\in\mathbf{X}^{h_\mu}. \tag{4.11}$$

Thus, using the bedding theorem that $\bigcup_{\mu=0}^{\infty}\mathbf{X}^{h_\mu}$ is dense in $\mathbf{L}^2(\Omega)$, we can infer that

$$\mathbf{v}(0) = \mathbf{v}_0 \text{ in } \mathbf{L}^2(\Omega_1),\ \ \mathbf{u}_t(0) = \mathbf{u}_1 \text{ in } \mathbf{L}^2(\Omega_2). \tag{4.12}$$

Using (4.1) and compact embedding theory, we can deduce the following strong convergence for a further subsequence h_μ

$$\begin{aligned} \mathbf{u}_{h_\mu,t} &\to \mathbf{u}_t \text{ in } L^2([0,T];\mathbf{L}^2(\Omega_2)),\\ \mathbf{u}_{h_\mu} &\to \mathbf{u} \text{ in } L^2([0,T];\mathbf{L}^2(\Omega_2)). \end{aligned} \tag{4.13}$$

Then, noticing

$$\mathbf{u}_{h_\mu} = \mathbf{u}_{0h_\mu} + \int_0^t \mathbf{u}_{h_\mu\ t}(s)ds, \tag{4.14}$$

and $||\mathbf{u}_{0h} - \mathbf{u}_0||_{0,\Omega_2} = 0$ as $h \to 0$, yields that

$$\mathbf{u} = \mathbf{u}_0 + \int_0^t \mathbf{u}_t ds. \tag{4.15}$$

Therefore, we have

$$\mathbf{u}_0 = \mathbf{u} - \int_0^t \mathbf{u}_t(s)ds = \mathbf{u}(0,x), \tag{4.16}$$

which is the second equation of (2.10). □

5. Error estimate

The goal of this section is to analyze the estimates of the discretization errors for the Galerkin finite element method for the nonlinear fluid-structure interaction problem. The estimates are based on the solution to (3.4)-(3.7). The approach is based on a priori estimates for the solution of Galerkin semi-discrete scheme in space [14, 16, 18, 28], the full set of estimates is obtained by differentiateting the discrete version with the respect to time. Furthermore, the analysis in Theorem 5.1 below, which provide a good global assessment of the discretization error under reasonable assumptions.

First, the projection operator $P_h : \mathbf{L}^2(\Omega) \to \Psi_h$ is introduced as follows:

$$\rho_1[P_h\mathbf{v},\mathbf{w}] + \rho_2[P_h\mathbf{u},\mathbf{w}] = \rho_1[\mathbf{v},\mathbf{w}] + \rho_2[\mathbf{u},\mathbf{w}],\ \forall \mathbf{w} \in \mathbf{\Psi}_h, \tag{5.1}$$

which implies

$$b[P_h\mathbf{w}, q_h] = 0,\ \forall q_h \in Q^h. \tag{5.2}$$

Under the hypotheses of angle condition on two domains Ω_1 and Ω_2, it holds for $\epsilon \in (0,1)$ and integer $k > 0$ that [9]

$$\begin{aligned} &(||\mathbf{v} - P_h\mathbf{v}||_{0,\Omega_1} + ||\mathbf{u} - P_h\mathbf{u}||_{0,\Omega_2}) + h(||\nabla(\mathbf{v} - P_h\mathbf{v})||_{0,\Omega_1} + ||\nabla(\mathbf{v} - P_h\mathbf{v})||_{0,\Omega_2})\\ \leq\ & Ch^{1+r-\epsilon}(||\mathbf{v}||_{r+1,\Omega_1} + ||\mathbf{u}||_{r+1,\Omega_2}),\\ &\forall \mathbf{v} \in \mathbf{H}^{r+1}(\Omega_1),\ \mathbf{u} \in \mathbf{H}^{r+1}(\Omega_2),\ i = 1,2, r \in [0,k]. \end{aligned} \tag{5.3}$$

Then, we have the following result of semi-discrete finite element approximations of the fluid-solid interaction problem.

Theorem 5.1 *Under the hypothesis of Theorem 4.1, let* $(\mathbf{v}, p, \mathbf{u}) \in \mathbf{X}_1 \times Q \times \mathbf{X}_2$ *and* $(\mathbf{v}_h, p_h, \mathbf{u}_h) \in \mathbf{X}_1^h \times Q^h \times \mathbf{X}_2^h$ *be the solution of (2.4)-(2.7) and (3.4)-(3.7), respectively. Then, it holds*

$$\begin{aligned}
&\|\mathbf{v}-\mathbf{v}_h\|^2_{L^\infty([0,T],L^2(\Omega_1))} + \|\mathbf{u}_t-\mathbf{u}_{ht}\|^2_{L^\infty([0,T],L^2(\Omega_2))}\\
&\quad+\|\nabla(\mathbf{v}-\mathbf{v}_h)\|^2_{L^2([0,T],L^2(\Omega_1))} + \|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2([0,T],L^2(\Omega_2))}\\
\le\ & Ch^{2r}(\|\mathbf{v}_0\|^2_{r+1,\Omega_1} + \|p_0\|^2_{r,\Omega_1} + \|\mathbf{u}_1\|^2_{r+1,\Omega_2}) + Ch^{2(r-\epsilon)}(\|\mathbf{v}\|^2_{L^2([0,T],H^{r+1}(\Omega_1))}\\
&\quad+\|\mathbf{u}\|^2_{L^2([0,T],H^{r+1}(\Omega_2))} + \|p\|^2_{L^2([0,T],H^{r}(\Omega_1))} + \|\mathbf{u}_t\|^2_{L^2([0,T],H^{r+1}(\Omega_2))})dt.
\end{aligned} \tag{5.4}$$

Proof. Subtracting (2.4) from (3.4), yields that

$$\begin{aligned}
&\rho_1[\mathbf{v}_t-\mathbf{v}_{ht},\mathbf{w}_h]_{\Omega_1} + a_1[\mathbf{v}-\mathbf{v}_h,\mathbf{w}_h] + b[\mathbf{w}_h, p-p_h] + c[\mathbf{v},\mathbf{v},\mathbf{w}_h] - c[\mathbf{v}_h,\mathbf{v}_h,\mathbf{w}_h]\\
&\qquad \rho_2[\mathbf{u}_{tt}-\mathbf{u}_{htt},\mathbf{w}_h]_{\Omega_2} + a_2[\mathbf{u}-\mathbf{u}_h,\mathbf{w}_h] = 0,\ \forall \mathbf{w}_h \in \mathbf{X}^h,\ a.e.t.
\end{aligned} \tag{5.5}$$

In order to uniform the variational formulation, we define $\mathbf{w}_h$ in two different domains. Setting $\mathbf{w}_h = \tilde{\xi} - \xi_h$ with

$$\xi_h|_{\Omega_i} = \begin{cases} \mathbf{v}_h & \text{if } i=1,\\ \mathbf{u}_{ht} & \text{if } i=2,\end{cases}$$

and

$$\tilde{\xi}|_{\Omega_i} = \begin{cases} P_h\mathbf{v} & \text{if } i=1,\\ P_h\mathbf{u}_t & \text{if } i=2,\end{cases}$$

using (5.1), then we can obtain the following result

$$\begin{aligned}
&\rho_1[\mathbf{v}_t-\mathbf{v}_{ht},\mathbf{w}_h]_{\Omega_1} + \rho_2[\mathbf{u}_{tt}-\mathbf{u}_{htt},\mathbf{w}_h]_{\Omega_2}\\
=\ &\rho_1[P_h\mathbf{v}_t-\mathbf{v}_{ht},\mathbf{w}_h]_{\Omega_1} + \rho_2[P_h\mathbf{u}_t-\mathbf{u}_{ht},\mathbf{w}_h]_{\Omega_2}\\
=\ &\frac{\rho_1}{2}\frac{d}{dt}\|P_h\mathbf{v}-\mathbf{v}_h\|^2_{0,\Omega_1} + \frac{\rho_2}{2}\frac{d}{dt}\|P_h\mathbf{u}_t-\mathbf{u}_{ht}\|^2_{0,\Omega_2}.
\end{aligned} \tag{5.6}$$

By (3.5) and (5.2),

$$b[P_h\mathbf{v}-\mathbf{v}_h, p-p_h] = b[P_h\mathbf{v}-\mathbf{v}_h, p-q_h],\ \forall q_h \in Q^h. \tag{5.7}$$

Similarly,

$$\begin{aligned}
&a_1[P_h\mathbf{v}-\mathbf{v}_h, P_h\mathbf{v}-\mathbf{v}_h] + a_2[P_h\mathbf{u}-\mathbf{u}_h, P_h\mathbf{u}_t-\mathbf{u}_{ht}]\\
=\ &\mu_1\|\nabla(P_h\mathbf{v}-\mathbf{v}_h)\|^2_{0,\Omega_1} + \frac{1}{2}\frac{d}{dt}|P_h\mathbf{u}-\mathbf{u}_h|^2_{0,\Omega_2}.
\end{aligned} \tag{5.8}$$

Using the estimates of the trilinear term $c[\cdot,\cdot,\cdot]$, leads to

$$\begin{aligned}
&c[\mathbf{v},\mathbf{v},\mathbf{w}_h] - c[\mathbf{v}_h,\mathbf{v}_h,\mathbf{w}_h]\\
=\ &c[\mathbf{v}-\mathbf{v}_h,\mathbf{v},\mathbf{w}_h] - c[\mathbf{v}_h,\mathbf{v}-\mathbf{v}_h,\mathbf{w}_h]\\
\le\ &c[\mathbf{v}-P_h\mathbf{v},\mathbf{v},\mathbf{w}_h] + c[P_h\mathbf{v}-\mathbf{v}_h,\mathbf{v},\mathbf{w}_h] + c[\mathbf{v}_h,\mathbf{v}-P_h\mathbf{v},\mathbf{w}_h]
\end{aligned}$$

$$
\begin{aligned}
&+c[\mathbf{v}_h, P_h\mathbf{v}-\mathbf{v}_h, \mathbf{w}_h] \\
&= I_1+I_2+I_3+I_4,
\end{aligned}
\tag{5.9}
$$

Substituting $\mathbf{w}_h = P_h\mathbf{v}-\mathbf{v}_h$ into the above inequalities and using (2.15), we have

$$
\begin{aligned}
|I_1+I_3| &\le C_0\|\nabla(\mathbf{v}-P_h\mathbf{v})\|_{0,\Omega_1}(\|\nabla\mathbf{v}\|_{0,\Omega_1}+\|\nabla\mathbf{v}_h\|_{0,\Omega_1})\|\nabla(\mathbf{v}_h-P_h\mathbf{v})\|_{0,\Omega_1} \\
&\le \frac{\mu_1}{2}\|\nabla(\mathbf{v}-P_h\mathbf{v})\|_{0,\Omega_1}\|\nabla(\mathbf{v}_h-P_h\mathbf{v})\|_{0,\Omega_1} \\
&\le \frac{\mu_1}{8}\|\nabla(\mathbf{v}_h-P_h\mathbf{v})\|_{0,\Omega_1}^2+\frac{\mu_1}{2}\|\nabla(\mathbf{v}-P_h\mathbf{v})\|_{0,\Omega_1}^2
\end{aligned}
\tag{5.10}
$$

and

$$
\begin{aligned}
|I_2+I_4| &\le C_0(\|\nabla\mathbf{v}\|_{0,\Omega_1}+\|\nabla\mathbf{v}_h\|_{0,\Omega_1})\|\nabla(\mathbf{v}_h-P_h\mathbf{v})\|_{0,\Omega_1}^2 \\
&\le \frac{\mu_1}{2}\|\nabla(\mathbf{v}_h-P_h\mathbf{v})\|_{0,\Omega_1}^2,
\end{aligned}
\tag{5.11}
$$

which together with (5.9) yields that

$$
\begin{aligned}
&|c[\mathbf{v},\mathbf{v},\mathbf{w}_h]-c[\mathbf{v},\mathbf{v},\mathbf{w}_h]| \\
\le\ &\frac{5\mu_1}{8}\|\nabla(\mathbf{v}_h-P_h\mathbf{v})\|_{0,\Omega_1}^2+\frac{\mu_1}{2}\|\nabla(\mathbf{v}-P_h\mathbf{v})\|_{0,\Omega_1}^2.
\end{aligned}
\tag{5.12}
$$

From the bounds of these inequality and the Young inequality, hence, after inserting (5.5) and noting that

$$
a_2[\mathbf{u}-P_h\mathbf{u}, P_h\mathbf{u}_t-\mathbf{u}_{ht}] = \frac{d}{dt}a_2[\mathbf{u}-P_h\mathbf{u}, P_h\mathbf{u}-\mathbf{u}_h] - a_2[\mathbf{u}_t-P_h\mathbf{u}_t, P_h\mathbf{u}-\mathbf{u}_h]
$$

we have

$$
\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\|P_h\mathbf{v}-\mathbf{v}_h\|_{0,\Omega}^2+\frac{\rho_2}{2}\frac{d}{dt}\|P_h\mathbf{u}_t-\mathbf{u}_{ht}\|_{0,\Omega}^2+\mu_1\|\nabla(P_h\mathbf{v}-\mathbf{v}_h)\|_{0,\Omega_1}^2+\frac{1}{2}\frac{d}{dt}|(P_h\mathbf{u}-\mathbf{u}_h)|_{0,\Omega_2}^2 \\
=\ &-a_1[\mathbf{v}-P_h\mathbf{v}, P_h\mathbf{v}-\mathbf{v}_h]-b[P_h\mathbf{v}-\mathbf{v}_h, p-q_h]-a_2[\mathbf{u}-P_h\mathbf{u}, P_h\mathbf{u}_t-\mathbf{u}_{ht}] \\
&-c[\mathbf{v},\mathbf{v},\mathbf{w}_h]+c[\mathbf{v}_h,\mathbf{v}_h,\mathbf{w}_h] \\
=\ &-a_1[\mathbf{v}-P_h\mathbf{v}, P_h\mathbf{v}-\mathbf{v}_h]-b[P_h\mathbf{v}-\mathbf{v}_h, p-q_h]-\frac{d}{dt}a_2[\mathbf{u}-P_h\mathbf{u}, P_h\mathbf{u}-\mathbf{u}_h] \\
&\quad +a_2[\mathbf{u}_t-P_h\mathbf{u}_t, P_h\mathbf{u}-\mathbf{u}_h]-c[\mathbf{v},\mathbf{v},\mathbf{w}_h]+c[\mathbf{v}_h,\mathbf{v}_h,\mathbf{w}_h] \\
\le\ &\frac{7\mu_1}{8}\|\nabla(P_h\mathbf{v}-\mathbf{v}_h)\|_{0,\Omega_1}^2+C(\|\nabla(\mathbf{v}-P_h\mathbf{v})\|_{0,\Omega_1}^2+\|p-q_h\|_{0,\Omega_1}^2+|\mathbf{u}_t-P_h\mathbf{u}_t|_{0,\Omega_2}^2) \\
&+\frac{1}{2}|(P_h\mathbf{u}-\mathbf{u}_h)|_{0,\Omega_2}^2-\frac{d}{dt}a_2[\mathbf{u}-P_h\mathbf{u}, P_h\mathbf{u}-\mathbf{u}_h],
\end{aligned}
\tag{5.13}
$$

where the last positive constant C determined by the bounds of the data $(\mu, \mathbf{f}, \mathbf{v}_0, \mathbf{u}_0, \mathbf{u}_1)$. Recalling the estimate in (3.14), the definition of initial value u_{0h} in (3.15), integrating the above equation from 0 to s, and using the Gronwall inequality, we can achieve the following result:

$$
\begin{aligned}
&\|P_h\mathbf{v}-\mathbf{v}_h\|_{0,\Omega}^2+\|P_h\mathbf{u}_t-\mathbf{u}_{ht}\|_{0,\Omega}^2+\|\nabla(P_h\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega_2}^2) \\
&+\int_0^s(\|\nabla(P_h\mathbf{v}-\mathbf{v}_h)\|_{0,\Omega_1}^2dt \\
\le\ &Ch^{2r}(\|\mathbf{v}_0\|_{r+1,\Omega_1}^2+\|p_0\|_{r,\Omega_1}^2+\|\mathbf{u}_1\|_{r+1,\Omega_2}^2) \\
&+Ch^{2(r-\epsilon)}\int_0^T(\|\mathbf{v}\|_{r+1,\Omega_1}^2+\|p\|_{r,\Omega_1}^2+\|\mathbf{u}\|_{r+1,\Omega_2}^2+\|\mathbf{u}_t\|_{r+1,\Omega_2}^2)dt,
\end{aligned}
$$

which together with a triangle inequality, yields the desired result. □

6. Numerical experiments

In this section, we present numerical experiment to examine the convergence of the fluid-structure interaction system. The computational domain Ω is designed as $[0, 2\pi] \times [-1, 1]$, where $\Omega_1 = [0, 2\pi] \times [0, 1]$, $\Omega_2 = [0, 2\pi] \times [-1, 0]$, $I_0 = [0, 2\pi] \times \{0\}$ and Γ_1, Γ_2 defined as before, i.e. Figure 1.

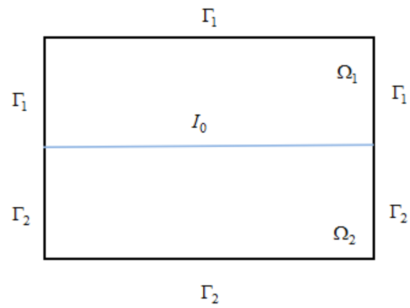

Figure 1. A sketch of the fluid domain Ω_1, structure domain Ω_2 and interface I_0.

In order to satisfy the conditions on coupled interface I_0, we set $\mu_1 = \mu_2 = \frac{1-2\nu}{4\sin(1)(1-\nu)}$ and $\lambda_2 = \frac{\nu}{2\sin(1)(1-\nu)}$ by Lame formulation, in which $0 < \nu < \frac{1}{2}$ is Poisson's ratio. Especially, let $\nu = \frac{1}{4}, \rho_1 = \rho_2 = 1$ and $T = 1$ in this example. Then the boundary data, initial data and the source terms are chosen such that the exact solution of the fluid-structure interaction system is given by

$$\begin{cases} \mathbf{v} = \{-\cos(x)\sin(y-1)e^t, \sin(x)(\cos(y-1)-1)e^t\}, \\ p = \sin(x)\cos(y)e^t, \\ \mathbf{u} = \{-\cos(x)\sin(y-1)e^t, \sin(x)(\cos(y+1)-1)e^t\}. \end{cases}$$

For the discretization in space we have considered Taylor-Hood element for the Navier-Stokes equations and P_2 for the Elasticity system. It should be noted that the finite element partition in Ω_1 and Ω_2 must match at the interaction interface. For the discretization in time we combine the central difference scheme for the second-order derivative with the Implicit Euler scheme for the first-order derivative. First, without consider the nonlinear term, we can express the discrete system (3.4) – (3.5) as the following linear algebraic systems

$$M_2 \frac{d^2\mathbf{w}}{dt^2} + M_1 \frac{d\mathbf{w}}{dt} + S_1 \mathbf{w} = \mathbf{F},$$

where the matrices M_2, M_1 and S_1 are deduced from the bilinear $a_1[\cdot,\cdot], a_2[\cdot,\cdot], b[\cdot,\cdot], [\frac{d\cdot}{dt},\cdot], [\frac{d^2\cdot}{dt^2},\cdot]$, $\mathbf{F}$ is the variation of the source term and $\mathbf{w} = \{v_1, v_2, p_h, u_1, u_2\}$ are the unknowns. In particular, the matrix M_2, M_1 and S_1, respectively, have the following form

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{21} & 0 \\ 0 & 0 & 0 & 0 & M_{21} \end{bmatrix},$$

$$M_1 = \begin{bmatrix} M_{11} & 0 & 0 & M_{12} & 0 \\ 0 & M_{11} & 0 & 0 & M_{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{13} & 0 \\ 0 & 0 & 0 & 0 & M_{13} \end{bmatrix}$$

and

$$S_1 = \begin{bmatrix} A_1 & A_2 & B_1 & * & * \\ A_2^T & A_3 & B_2 & * & * \\ B_1^T & B_2^T & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}.$$

Then as a result of treating trilinear by Oseen iteration, i.e. $(\mathbf{v}_h^n \cdot \nabla)\mathbf{v}_h^{n+1}$, the time-discrete problem can be read as:

$$(\frac{M_2}{\tau^2} + \frac{M_1}{\tau} + S_1 + S_2)\mathbf{w}^{n+1} = \mathbf{F}^{n+1} + (\frac{2M_2}{\tau^2} + \frac{M_1}{\tau})\mathbf{w}^n - \frac{M_2}{\tau^2}\mathbf{w}^{n-1},$$

where S_2 can be express as

$$\begin{bmatrix} A_4 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously, the foregoing matrix systems can be derived from the following equation:

$$\begin{aligned}
&\rho_1\left[\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t}, \mathbf{w}_{1h}\right]_{\Omega_1} + a_1[\mathbf{v}_h^{n+1}, \mathbf{w}_{1h}] + b[\mathbf{w}_{1h}, p_h^{n+1}] - b[\mathbf{v}_h^{n+1}, q_h] + (\mathbf{v}_h^n \cdot \nabla)\mathbf{v}_h^{n+1} \\
&\quad + \rho_2\left[\frac{\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}}{\Delta t^2}, \mathbf{w}_{2h}\right]_{\Omega_2} + a_2[\mathbf{u}_h^{n+1}, \mathbf{w}_{2h}] \\
&\quad + \int_{I_0} (\mathbf{v}_h^{n+1} - \mathbf{u}_{ht}^{n+1})\mathbf{w}_{1h} ds + \int_{I_0} [\mu_2(\nabla \mathbf{u}_h + \nabla \mathbf{u}_h^T)\mathbf{n}_2 + \lambda(\mathrm{div}\mathbf{u}_h \mathbf{n}_2)]^{n+1}\mathbf{w}_{1h} ds \\
&\quad + \int_{I_0} (\mathbf{u}_{ht}^{n+1} - \mathbf{v}_h^{n+1})\mathbf{w}_{2h} ds - \int_{I_0} [p\mathbf{n}_1 - \mu_1(\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^T)\mathbf{n}_1]^{n+1}\mathbf{w}_{2h} ds \\
&= \rho_1[\mathbf{f}_1^{n+1}, \mathbf{w}_{1h}]_{\Omega_1} + \rho_2[\mathbf{f}_2^{n+1}, \mathbf{w}_{2h}]_{\Omega_2}, n = 0, 1, \cdots N, \qquad (6.1)
\end{aligned}$$

where $\Delta t = T/N$ is the uniform time step size. Moreover, when treating the initial conditions, we use the Implicit Euler scheme as the method of difference.

We partition domain Ω into a uniform matching triangulation. The refined meshes are obtained by dividing primary meshes into four similar cells by connecting the edge midpoints. By matching the time step size Δt with the mesh size $O(8h^3)$, Table 1 gives the numerical results through using the monolithic scheme. We see that the convergence rates for the velocity, pressure and displacement of the solid are just about $O(h^3)$, $O(h^2)$ and $O(h^3)$, respectively, as the theoretical prediction.

In order to verify the order of time convergence, thought there is no theoretical analysis in this paper, we test the case in which space step size h can be chosen as small as enough. We can observe that the expected order of convergence in τ, i.e. $O(\tau)$ in Table 2.

Table 1. Error results for the triples $P_2 - P_1 - P_2$ at the end time.

h	$\|\|\mathbf{v}^n - \mathbf{v}_h^n\|\|_1$	$\|\|p^n - p_h^n\|\|_0$	$\|\|\mathbf{u}^n - \mathbf{u}_h^n\|\|_0$	$\|\|\mathbf{u}^n - \mathbf{u}_h^n\|\|_1$
2^{-3}	5.0313e-02	3.7129e-02	2.2842e-02	7.1309e-02
2^{-4}	1.1786e-02	7.4693e-03	2.8885e-03	1.3052e-02
2^{-5}	2.8876e-03	1.7159e-03	3.7417e-04	2.8635e-03
2^{-6}	7.1868e-04	4.1590e-04	5.3240e-05	6.8775e-04
Rate	2.0431	2.1601	2.9150	2.2320

Table 2. Error results for different time step size τ and $h = 2^{-6}$.

τ	$\|\|\mathbf{v}^n - \mathbf{v}_h^n\|\|_1$	$\|\|p^n - p_h^n\|\|_0$	$\|\|\mathbf{u}^n - \mathbf{u}_h^n\|\|_0$	$\|\|\mathbf{u}^n - \mathbf{u}_h^n\|\|_1$
1/5	2.7888e-01	2.9410e-01	2.8505e-01	7.0934e-01
1/10	1.3926e-01	1.5211e-01	1.4489e-01	3.6428e-01
1/20	6.9729e-02	7.7373e-02	7.3046e-02	1.8516e-01
1/40	3.4912e-02	3.9019e-02	3.6674e-02	9.3457e-02
Rate	0.9993	0.9714	0.9861	0.9747

Then in Table 3 and Table 4, we test the different decoupling order scheme for this interaction system. Table 3 shows the results by first solving Navier-Stokes equation, then solving Elasticity equation; Table 4 shows the results in reverse order. In both of them, we use the Nitsche's trick to treat the Dirichlet interface condition, i.e. the first equation of (2.3). That is to say, we use $\int_{I_0}(\mathbf{v}_h^n - \mathbf{u}_{ht}^{n-1})\mathbf{w}_{1h}ds$ or $\int_{I_0}(\mathbf{u}_{ht}^n - \mathbf{v}_h^n)\mathbf{w}_{2h}ds$ to weakly treat the Dirichlet interface condition.

Table 3. Error results for first solving Navier-Stokes equation.

h	$\|\|\mathbf{v}^n - \mathbf{v}_h^n\|\|_1$	$\|\|p^n - p_h^n\|\|_0$	$\|\|\mathbf{u}^n - \mathbf{u}_h^n\|\|_0$	$\|\|\mathbf{u}^n - \mathbf{u}_h^n\|\|_1$
2^{-3}	8.3133e-02	4.3960e-02	2.2426e-02	7.2486e-02
2^{-4}	1.4350e-02	7.8258e-03	2.8314e-03	1.3255e-02
2^{-5}	3.0401e-03	1.7028e-03	3.6418e-04	2.8794e-03
2^{-6}	7.2433e-04	4.7933e-04	5.1454e-05	6.8676e-04
Rate	2.2809	2.1730	2.9226	2.2406

Table 4. Error results for first solving Elasticity equation.

h	$\|\|\mathbf{v}^n - \mathbf{v}_h^n\|\|_1$	$\|\|p^n - p_h^n\|\|_0$	$\|\|\mathbf{u}^n - \mathbf{u}_h^n\|\|_0$	$\|\|\mathbf{u}^n - \mathbf{u}_h^n\|\|_1$
2^{-3}	5.0113e-02	4.1194e-02	2.6434e-02	9.3836e-02
2^{-4}	1.1725e-02	7.7002e-03	3.2815e-03	1.5731e-02
2^{-5}	3.5428e-03	2.0901e-03	4.1104e-04	3.6282e-03
2^{-6}	7.1862e-04	4.4337e-04	5.3622e-05	7.3375e-04
Rate	2.0413	2.1793	2.9818	2.3329

Concretely, we take the decoupling method used in Table 3 as an example, i.e. first solving Navier-Stokes equation then solving Elasticity equation, to explain the decoupling computational process. First, we give the following fully discrete scheme:

$$\rho_1[\mathbf{v}_{ht}^n, \mathbf{w}_{1h}]_{\Omega_1} + a_1[\mathbf{v}_h^n, \mathbf{w}_{1h}] + b[\mathbf{w}_{1h}, p_h^n] - b[\mathbf{v}_h^n, q_h] + (\mathbf{v}_h^{n-1} \cdot \nabla)\mathbf{v}_h^n$$
$$+ \int_{I_0} (\mathbf{v}_h^n - \mathbf{u}_{ht}^{n-1})\mathbf{w}_{1h} ds = \rho_1[\mathbf{f}_1^n, \mathbf{w}_{1h}]_{\Omega_1} - \int_{I_0} [\mu_2(\nabla \mathbf{u}_h + \nabla \mathbf{u}_h^T)\mathbf{n}_2 + \lambda(\mathrm{div}\mathbf{u}_h\mathbf{n}_2)]^{n-1}\mathbf{w}_{1h} ds$$

$$\rho_2[\mathbf{u}_{htt}^n, \mathbf{w}_{2h}]_{\Omega_2} + a_2[\mathbf{u}_h^n, \mathbf{w}_{2h}] + \int_{I_0} (\mathbf{u}_{ht}^n - \mathbf{v}_h^n)\mathbf{w}_{2h} ds = \rho_2[\mathbf{f}_2^n, \mathbf{w}_{2h}]_{\Omega_2}$$
$$+ \int_{I_0} [p\mathbf{n}_1 - \mu_1(\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^T)\mathbf{n}_1]^n \mathbf{w}_{2h} ds,$$

then the corresponding matrix representations of foregoing equations can be written by imitating the process before. What we need emphasize is that we substitute the coupled Neumann interface $\int_{I_0} [p\mathbf{n}_1 - \mu_1(\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^T)\mathbf{n}_1]^n \mathbf{w}_{1h} ds$ by the known value of $n-1$, i.e, $-\int_{I_0} [\mu_2(\nabla \mathbf{u}_h + \nabla \mathbf{u}_h^T)\mathbf{n}_2 + \lambda(\nabla \cdot \mathbf{u}_h\mathbf{n}_2)]^{n-1}\mathbf{w}_{1h} ds$ in Navier-Stokes equation; then we substitute the coupled Neumann interface $\int_{I_0} [\mu_2(\nabla \mathbf{u}_h + \nabla \mathbf{u}_h^T)\mathbf{n}_2 + \lambda(\nabla \cdot \mathbf{u}_h\mathbf{n}_2)]^n \mathbf{w}_{2h} ds$ by the updated value of Navier-Stokes equation, i.e, $\int_{I_0} [p\mathbf{n}_1 - \mu_1(\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^T)\mathbf{n}_1]^n \mathbf{w}_{2h} ds$ in the Elasticity equation.

In the end, we need to explain the second method in Table 4. When treating the coupled interface I_0 in the Elasticity equation, we substitute $\int_{I_0} [p\mathbf{n}_1 - \mu_1(\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^T)\mathbf{n}_1]^{n-1}\mathbf{w}_{2h} ds$ for corresponding term; When treating the coupled interface I_0 in the Navier-Stokes equation, we substitute $\int_{I_0} [\mu_2(\nabla \mathbf{u}_h + \nabla \mathbf{u}_h^T)\mathbf{n}_2 + \lambda(\nabla \cdot \mathbf{u}_h\mathbf{n}_2)]^n \mathbf{w}_{1h} ds$ for corresponding term.

It can be seen that the convergence rates for the velocity, pressure of the fluid and displacement of the solid are the same as previous coupled method.

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Conflict of interest

All authors contributed equally to the manuscript and read and approved the final manuscript.

The authors declare that they have no competing interests.

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