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Research article

Hermite–Jensen–Mercer type inequalities via Ψ–Riemann–Liouville *k*–fractional integrals

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Abstract: Integral inequalities involving various fractional integral operators are used to solve many fractional differential equations. In this paper, authors prove some Hermite–Jensen–Mercer type inequalities using Ψ–Riemann–Liouville *k*–Fractional integrals via convex functions. We established some new Ψ–Riemann–Liouville *k*-Fractional integral inequalities. We also give Ψ–Riemann–Liouville *k*–Fractional integrals identities for differentiable mapping, and these will be used to derive estimates for some fractional Hermite–Jensen–Mercer type inequalities. Some known results are recaptured from our results as special cases.

Keywords: convex function; Hermite–Hadamard inequality; Jensen inequality; Jensen–Mercer inequality; Hölder inequality; improved power mean integral inequality; ψ–Riemann–Liouville *k*-Fractional integrals
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1. Introduction and preliminaries

Inequalities have always proved to be useful in establishing mathematical models and their solutions in almost all branches of applied sciences, in particular, in physics and engineering. Convexity plays a very important role in the optimization of solutions of mathematical problems. Theory of convex functions has great importance in various fields of pure and applied sciences. It is known that theory of convex functions is closely related to theory of inequalities. It is now become a trending aspect of mathematical research to generalize classical known results via fractional integral operator.

Fractional calculus which is calculus of integrals and derivatives of any arbitrary real or complex order has gained remarkable popularity and importance during the last four decades or so, due mainly to its demonstrated applications in diverse and widespread fields ranging from natural sciences to social sciences, see [4–7, 15] and references therein. Many authors have established a variety of inequalities for those fractional integral and derivative operators, some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations.

Definition 1.1. The function $\Upsilon : [a, b] \to \mathbb{R}$ is said to be convex, if we have

$$\Upsilon\left(\lambda y_1 + (1-\lambda) y_2\right) \leq \lambda \Upsilon(y_1) + (1-\lambda)\, \Upsilon(y_2)$$

for all $y_1, y_2 \in [a, b]$ and $\lambda \in [0, 1]$.

In 1883, Hermite–Hadamard's (H–H) inequality has been considered the most useful inequality in mathematical analysis. It is also known as classical H-H inequality.

The Hermite–Hadamard inequality assert that, if $\Upsilon : J \subseteq R \to R$ is a convex function in J and $\ell_1, \ell_2 \in J$, where $\ell_1 < \ell_2$, then

$$\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right) \leq \frac{1}{\ell_2-\ell_1} \int_{\ell_1}^{\ell_2} \Upsilon(\lambda) d\lambda \leq \frac{\Upsilon(\ell_1)+\Upsilon(\ell_2)}{2}.$$

Let $0 < y_1 \leq y_2 \leq \ldots \leq y_n$ and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be non-negative weights such that $\sum_{i=1}^{n} \mu_i = 1$.

The famous Jensen inequality, see [24] in the literature states that, if Υ is convex function on the interval $[\ell_1, \ell_2]$, then

$$\Upsilon\left(\sum_{i=1}^{n} \mu_i y_i\right) \leq \left(\sum_{i=1}^{n} \mu_i \Upsilon(y_i)\right) \tag{1.1}$$

for all $y_i \in [\ell_1, \ell_2]$ and $\mu_i \in [0, 1]$, $(i = 1, 2, \ldots, n)$.

In (2003) Mercer gave a variant of Jensen's inequality, see [19] as:

Theorem 1.2. *If Υ is a convex function on $[\ell_1, \ell_2]$, then*

$$\Upsilon\left(\ell_1 + \ell_2 - \sum_{i=1}^{n} \mu_i\, y_i\right) \leq \Upsilon(\ell_1) + \Upsilon(\ell_2) - \sum_{i=1}^{n} \mu_i \Upsilon(y_i) \tag{1.2}$$

for all $y_i \in [\ell_1, \ell_2]$ and $\mu_i \in [0, 1]$, $(i = 1, 2, \ldots, n)$.

In [18], Matković et al. have proved Jensen's inequality of Mercer's type for operators with applications in 2006. Later, in 2009 Mercer's result was generalized to higher dimensions by M. Niezgoda [22]. In recent years, notable contributions have been made on Jensen–Mercer's type inequality. In 2014 M. Kian gave concept of Jensen inequality for superquadratic functions [13]. Further, in 2016 to 2018 E. Anjidani worked on Reverse Jensen–Mercer type operator inequalities and Jensen–Mercer operator inequalities for superquadratic functions [2, 3]. M. M. Ali and A. R. Khan generalized integral Mercer's inequality and integral means in 2019 [1]. Some improvements have been made for Jensen–Mercer–Type inequalities by H. R. Moradi and S. Furuichi in 2019 [20].

Now we give necessary definitions of fractional calculus theory which is used throughout this paper.

Definition 1.3. [14] Let $(\ell_1, \ell_2)(-\infty \leq \ell_1 < \ell_2 \leq \infty)$ and $\alpha > 0$. Also let Ψ be an increasing and positive monotone function on $(\ell_1, \ell_2]$, having a continuous derivative Ψ' on (ℓ_1, ℓ_2). Then the left-sided and right-sided Ψ-Riemann-Liouville Fractional integrals of a function Υ with respect to another function Ψ on $[\ell_1, \ell_2]$ are defined as follows:

$$(I_{\ell_1+}^{\alpha:\Psi})\Upsilon(y_1) = \frac{1}{\Gamma(\alpha)} \int_{\ell_1}^{y_1} \Psi'(\lambda)(\Psi(y_1) - \Psi(\lambda))^{\alpha-1}\Upsilon(\lambda)d\lambda, \qquad \ell_1 < y_1 \tag{1.3}$$

and

$$(I_{\ell_2-}^{\alpha:\Psi})\Upsilon(y_1) = \frac{1}{\Gamma(\alpha)} \int_{y_1}^{\ell_2} \Psi'(\lambda)(\Psi(\lambda) - \Psi(y_1))^{\alpha-1}\Upsilon(\lambda)d\lambda, \qquad y_1 < \ell_2, \tag{1.4}$$

respectively.

Definition 1.4. [8] Diaz and Parigun have defined the k–gamma function $\Gamma_k(\cdot)$, a generalization of the classical gamma function, which is given by the following formula

$$\Gamma_k(y) = \lim_{n\to+\infty} \frac{n!k^n(nk)^{\frac{y}{k}-1}}{(y)_{n,k}}, \quad k > 0.$$

It is shown that Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k–gamma function clearly given by:

$$\Gamma_k(\alpha) = \int_0^\infty e^{-\frac{t^k}{k}}\, t^{\alpha-1}\, dt.$$

Obviously, $\Gamma_k(y + k) = y\Gamma_k(y)$, $\Gamma(y) = \lim_{k\to1} \Gamma_k(y)$ and $\Gamma_k(y) = k^{\frac{y}{k}-1}\Gamma(\frac{y}{k})$.

Many researchers have generalized the classical and fractional operators by introducing a parameter $k > 0$ about a decade ago. Mubeen and Habibullah [21] used special k-functions theory in fractional calculus for the first time in literature in the form of k–Riemann–Liouville integral. Recently, many researchers are presenting new fractional differential and integral operators and they generalized by iteration procedure and by introducing a new parameter $k > 0$. They also found relationships of these generalized fractional operators with existing fractional and classical operators under the special values of the parameter k. Many k–fractional operators, their properties, related identities, and inequalities are proved during the past years.

In [16], the author define a more general form of Riemann–Liouville k–fractional integrals with respect to an increasing function and use them to obtain Ostrowski-type inequalities. Utilizing a simple inequality via an increasing function and assumptions of Ostrowski inequality several fractional integral inequalities are obtained. These results provide the Ostrowski type inequalities for Riemann–Liouville fractional integrals which are published in [9].

Definition 1.5. [16] Let $(\ell_1, \ell_2)(-\infty \leq \ell_1 < \ell_2 \leq \infty)$ and $\alpha, k > 0$. Also let Ψ be an increasing and positive monotone function on $[\ell_1, \ell_2]$, having a continuous derivative Ψ' on (ℓ_1, ℓ_2). Then the left-sided and right-sided Ψ–Riemann–Liouville k–Fractional integrals of a function Υ with respect to another function Ψ on $[\ell_1, \ell_2]$ are defined as follows:

$$({}_kI^{\alpha:\Psi}_{\ell_1+})\Upsilon(y_1) = \frac{1}{k\Gamma_k(\alpha)} \int_{\ell_1}^{y_1} \Psi'(\lambda)(\Psi(y_1) - \Psi(\lambda))^{\frac{\alpha}{k}-1}\Upsilon(\lambda)d\lambda, \qquad \ell_1 < y_1 \tag{1.5}$$

and

$$({}_kI^{\alpha:\Psi}_{\ell_2-})\Upsilon(y_1) = \frac{1}{k\Gamma_k(\alpha)} \int_{y_1}^{\ell_2} \Psi'(\lambda)(\Psi(\lambda) - \Psi(y_1))^{\frac{\alpha}{k}-1}\Upsilon(\lambda)d\lambda, \qquad y_1 < \ell_2, \tag{1.6}$$

respectively.

In this article, by using the Jensen–Mercer's inequality, we proved Hermite–Hadamard's inequalities for fractional integrals and we established some new Ψ–Riemann–Liouville k–Fractional integrals connected with the left and right sides of Hermite–Hadamard type inequalities for differentiable mappings whose derivatives in absolute value are convex. From our results some known results will be taken. At the end, a briefly conclusion is given as well.

2. Hermite–Jensen–Mercer type inequalities

Throughout the paper, we need the following assumption:

(A_1) : Let $0 \leq \ell_1 < \ell_2$, $\Upsilon : [\ell_1, \ell_2] \to \Re$ be a positive function and $\Upsilon \in L_1[\ell_1, \ell_2]$. Also suppose that Υ is a convex function on $[\ell_1, \ell_2]$, $\Psi(\cdot)$ is an increasing and positive monotone function on $(\ell_1, \ell_2]$, having a continuous derivative Ψ' on (ℓ_1, ℓ_2) and $\alpha, k > 0$.

Theorem 2.1. *Let (A_1) satisfied, then the following fractional integral inequalities hold:*

$$\begin{aligned}
\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right) &\leq [\Upsilon(\ell_1) + \Upsilon(\ell_2)] - \frac{\Gamma_k(\alpha + k)}{2(y_2 - y_1)^{\frac{\alpha}{k}}} \\
&\times \left\{\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(y_1)^+}\right)(\Upsilon \circ \Psi)(\Psi^{-1}(y_2)) + \left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(y_2)^-}\right)(\Upsilon \circ \Psi)(\Psi^{-1}(y_1))\right\} \\
&\leq [\Upsilon(\ell_1) + \Upsilon(\ell_2)] - \Upsilon\left(\frac{y_1 + y_2}{2}\right)
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right) &\leq \frac{\Gamma_k(\alpha + k)}{2(y_2 - y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^+}\right)(\Upsilon \circ \Psi)(\Psi^{-1}(\ell_1 + \ell_2 - y_2)) \\
&+ \left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^-}\right)(\Upsilon \circ \Psi)(\Psi^{-1}(\ell_1 + \ell_2 - y_1))\Big] \\
&\leq \frac{\Upsilon(\ell_1 + \ell_2 - y_1) + \Upsilon(\ell_1 + \ell_2 - y_2)}{2} \leq \Upsilon(\ell_1) + \Upsilon(\ell_2) - \Upsilon\left(\frac{y_1 + y_2}{2}\right)
\end{aligned} \tag{2.2}$$

for all $y_1, y_2 \in [\ell_1, \ell_2]$ and $\Gamma_k(\cdot)$ is the k-gamma function.

Proof. Using the Jensen–Mercer inequality, we have

$$\Upsilon\left(\ell_1 + \ell_2 - \frac{u + v}{2}\right) \leq \Upsilon(\ell_1) + \Upsilon(\ell_2) - \frac{\Upsilon(u) + \Upsilon(v)}{2}$$

for all $u, v \in [\ell_1, \ell_2]$.

Now by change of variables $u = \lambda y_1 + (1-\lambda)y_2$ and $v = (1-\lambda)y_1 + \lambda y_2, \quad \forall\, y_1, y_2 \in [\ell_1, \ell_2]$ and $\lambda \in [0,1]$, we get

$$\Upsilon\left(\ell_1 + \ell_1 - \frac{y_1+y_2}{2}\right) \leq \Upsilon(\ell_1) + \Upsilon(\ell_2) - \frac{\Upsilon(\lambda y_1 + (1-\lambda)y_2) + \Upsilon((1-\lambda)y_1 + \lambda y_2)}{2}.$$

Multiplying both sides of above inequality by $\lambda^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we obtain

$$\begin{aligned}\frac{k}{\alpha}\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1+y_2}{2}\right) &\leq \frac{k}{\alpha}\{\Upsilon(\ell_1) + \Upsilon(\ell_2)\} \\ &\quad - \frac{1}{2}\left\{\int_0^1 \lambda^{\frac{\alpha}{k}-1}\left(\Upsilon(\lambda y_1 + (1-\lambda)y_2) + \Upsilon((1-\lambda)y_1 + \lambda y_2)\right)d\lambda\right\},\end{aligned}$$

where

$$\begin{aligned}&\frac{\alpha}{2k}\left\{\int_0^1 \lambda^{\frac{\alpha}{k}-1}\left(\Upsilon(\lambda y_1 + (1-\lambda)y_2) + \Upsilon((1-\lambda)y_1 + \lambda y_2)\right)d\lambda\right\} \\ &= \frac{\alpha}{2k}\int_0^1 \lambda^{\frac{\alpha}{k}-1}\Upsilon(\lambda y_1 + (1-\lambda)y_2)\,d\lambda + \frac{\alpha}{2k}\int_0^1 \lambda^{\frac{\alpha}{k}-1}\Upsilon((1-\lambda)y_1 + \lambda y_2)\,d\lambda \\ &= \frac{\alpha}{2k}\int_{\Psi^{-1}(y_1)}^{\Psi^{-1}(y_2)}\left(\frac{y_2 - \Psi(\gamma)}{y_2 - y_1}\right)^{\frac{\alpha}{k}-1}\Upsilon(\Psi(\gamma))\frac{\Psi'(\gamma)}{y_2 - y_1}d\gamma \\ &+ \frac{\alpha}{2k}\int_{\Psi^{-1}(y_1)}^{\Psi^{-1}(y_2)}\left(\frac{\Psi(\gamma) - y_1}{y_2 - y_1}\right)^{\frac{\alpha}{k}-1}\Upsilon(\Psi(\gamma))\frac{\Psi'(\gamma)}{y_2 - y_1}d\gamma.\end{aligned}$$

So, final form will be of this type

$$\begin{aligned}&\frac{\alpha}{2k}\left\{\int_0^1 \lambda^{\frac{\alpha}{k}-1}\left(\Upsilon(\lambda y_1 + (1-\lambda)y_2) + \Upsilon((1-\lambda)y_1 + \lambda y_2)\right)d\lambda\right\} \\ &= \frac{\Gamma_k(\alpha+k)}{2(y_2 - y_1)^{\frac{\alpha}{k}}}\left\{\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(y_1)^+}\right)\left(\Upsilon\circ\Psi)(\Psi^{-1}(y_2)\right) + \left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(y_2)^-}\right)\left(\Upsilon\circ\Psi)(\Psi^{-1}(y_1)\right)\right\}\end{aligned}$$

and so the first inequality of (2.1) proved.

Now for the proof of second inequality of (2.1), we first note that, if Υ is convex function, then for $\lambda \in [0,1]$, it gives

$$\Upsilon\left(\frac{y_1+y_2}{2}\right) = \Upsilon\left(\frac{\lambda y_1 + (1-\lambda)y_2 + (1-\lambda)y_1 + \lambda y_2}{2}\right) \leq \frac{\Upsilon(\lambda y_1 + (1-\lambda)y_2) + \Upsilon((1-\lambda)y_1 + \lambda y_2)}{2}.$$

Multiplying both sides of above inequality by $\lambda^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, and let $\Psi(\gamma) = \lambda y_1 + (1-\lambda)y_2$, and $\Psi(\beta) = (1-\lambda)y_1 + \lambda y_2$, we have

$$\Upsilon\left(\frac{y_1+y_2}{2}\right) \leq \frac{\alpha}{2k}\left\{\int_0^1 \lambda^{\frac{\alpha}{k}-1}\left(\Upsilon(\lambda y_1 + (1-\lambda)y_2) + \Upsilon((1-\lambda)y_1 + \lambda y_2)\right)d\lambda\right\}$$

$$= \frac{\Gamma_k(\alpha+k)}{2(y_2-y_1)^{\frac{\alpha}{k}}}\left\{\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(y_1)^+}\right)\left(\Upsilon\circ\Psi)(\Psi^{-1}(y_2)\right)+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(y_2)^-}\right)\left(\Upsilon\circ\Psi)(\Psi^{-1}(y_1)\right)\right\}.$$

Multiplying by (-1) and adding $\Upsilon(\ell_1)+\Upsilon(\ell_2)$ both sides, we get the second inequality of (2.1).

Now for the proof of inequality of (2.2), we first note that, if Υ is convex function, then for $\lambda \in [0,1]$, it gives

$$\Upsilon\left(\ell_1+\ell_2-\frac{u_1+u_2}{2}\right)=\Upsilon\left(\frac{\ell_1+\ell_2-u_1+\ell_1+\ell_2-u_2}{2}\right)\le\frac{\Upsilon(\ell_1+\ell_2-u_1)+\Upsilon(\ell_1+\ell_2-u_2)}{2}$$

for all $u_1,u_2\in[\ell_1,\ell_2]$.

Now by change of variables $u_1=\lambda(\ell_1+\ell_2-y_1)+(1-\lambda)(\ell_1+\ell_2-y_2)$ and $u_2=(1-\lambda)(\ell_1+\ell_2-y_1)+\lambda(\ell_1+\ell_2-y_2)$, $\forall\, y_1,y_2\in[\ell_1,\ell_2]$ and $\lambda\in[0,1]$, we have

$$\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\le \frac{1}{2}\Big[\Upsilon\left(\lambda(\ell_1+\ell_2-y_1)+(1-\lambda)(\ell_1+\ell_2-y_2)\right)$$
$$+\Upsilon\left((1-\lambda)(\ell_1+\ell_2-y_1)+\lambda(\ell_1+\ell_2-y_2)\right)\Big].$$

Multiplying both sides of above inequality by $\lambda^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, and let $\Psi(\gamma)=\lambda(\ell_1+\ell_2-y_1)+(1-\lambda)(\ell_1+\ell_2-y_2)$, and $\Psi(\beta)=(1-\lambda)(\ell_1+\ell_2-y_1)+\lambda(\ell_1+\ell_2-y_2)$, we get

$$\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\le\frac{\alpha}{2k}\Big[\int_0^1\lambda^{\frac{\alpha}{k}-1}\Upsilon\left(\lambda(\ell_1+\ell_2-y_1)+(1-\lambda)(\ell_1+\ell_2-y_2)\right)d\lambda$$
$$+\int_0^1\lambda^{\frac{\alpha}{k}-1}\Upsilon\left((1-\lambda)(\ell_1+\ell_2-y_1)+\lambda(\ell_1+\ell_2-y_2)\right)d\lambda\Big]$$
$$=\frac{\alpha}{2k}\int_{\Psi^{-1}(\ell_1+\ell_2-y_1)}^{\Psi^{-1}(\ell_1+\ell_2-y_2)}\left(\frac{(\ell_1+\ell_2-y_2)-\Psi(\gamma)}{y_2-y_1}\right)^{\frac{\alpha}{k}-1}\Upsilon(\Psi(\gamma))\frac{\Psi'(\gamma)}{y_2-y_1}d\gamma$$
$$+\frac{\alpha}{2k}\int_{\Psi^{-1}(\ell_1+\ell_2-y_1)}^{\Psi^{-1}(\ell_1+\ell_2-y_2)}\left(\frac{\Psi(\gamma)-(\ell_1+\ell_2-y_1)}{y_2-y_1}\right)^{\frac{\alpha}{k}-1}\Upsilon(\Psi(\gamma))\frac{\Psi'(\gamma)}{y_2-y_1}d\gamma.$$

Hence

$$\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\le\frac{\Gamma_k(\alpha+k)}{2(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^+}\right)\left(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_2)\right)$$
$$+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^-}\right)\left(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_1)\right)\Big]$$

and so the first inequality of (2.2) proved.

Now for the proof of second inequality of (2.2), we first note that, if Υ is convex function, then for $\lambda\in[0,1]$, it gives

$$\Upsilon\left(\lambda(\ell_1+\ell_2-y_1)+(1-\lambda)(\ell_1+\ell_2-y_2)\right)\le\lambda\Upsilon(\ell_1+\ell_2-y_1)+(1-\lambda)\Upsilon(\ell_1+\ell_2-y_2);$$
$$\Upsilon\left((1-\lambda)(\ell_1+\ell_2-y_1)+\lambda(\ell_1+\ell_2-y_2)\right)\le(1-\lambda)\Upsilon(\ell_1+\ell_2-y_1)+\lambda\Upsilon(\ell_1+\ell_2-y_2).$$

By adding these inequalities and using the Jensen–Mercer inequality, we have

$$\begin{aligned}&\Upsilon\left(\lambda(\ell_1+\ell_2-y_1)+(1-\lambda)(\ell_1+\ell_2-y_2)\right)+\Upsilon\left((1-\lambda)(\ell_1+\ell_2-y_1)+\lambda(\ell_1+\ell_2-y_2)\right)\\&\qquad\leq \Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)\leq 2[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-[\Upsilon(y_1)+\Upsilon(y_2)].\end{aligned}$$

Multiplying both sides of above inequality by $\lambda^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we obtain second and third inequalities of (2.2). □

Corollary 1. *Under the assumption of Theorem 2.1, choosing $\Psi(\gamma)=\gamma$ we get the following inequalities*

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\frac{\Gamma_k(\alpha+k)}{2(y_2-y_1)^{\frac{\alpha}{k}}}\left\{{}_kJ^{\alpha}_{(y_2)-}\Upsilon(y_1)\;+\;{}_kJ^{\alpha}_{(y_1)+}\Upsilon(y_2)\right\}\\&\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\Upsilon\left(\frac{y_1+y_2}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\\&\leq\frac{\Gamma_k(\alpha+k)}{2(y_2-y_1)^{\frac{\alpha}{k}}}\left[\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_1)^+}\right)\Upsilon(\ell_1+\ell_2-y_2)+\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_2)^-}\right)\Upsilon(\ell_1+\ell_2-y_1)\right]\\&\leq\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2}\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\Upsilon\left(\frac{y_1+y_2}{2}\right).\end{aligned}$$

Remark 1. For $k=1$ and taking $\Psi(\gamma)=\gamma$, we have the following inequalities proved in [23].

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\frac{\Gamma(\alpha+1)}{2(y_2-y_1)^{\alpha}}\left\{J^{\alpha}_{(y_2)-}\Upsilon(y_1)+J^{\alpha}_{(y_1)+}\Upsilon(y_2)\right\}\\&\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\Upsilon\left(\frac{y_1+y_2}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(y_2-y_1)^{\alpha}}\left[\left(J^{\alpha}_{(\ell_1+\ell_2-y_2)^+}\right)\Upsilon(\ell_1+\ell_2-y_1)^+ +\left(J^{\alpha}_{(\ell_1+\ell_2-y_1)^-}\right)\Upsilon(\ell_1+\ell_2-y_2)^-\right]\\&\leq\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2}\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\Upsilon\left(\frac{y_1+y_2}{2}\right).\end{aligned}$$

Remark 2. For $\Psi(\gamma)=\gamma$ and $\alpha=k=1$ in Theorem 2.1, we will get the following inequalities proved in [12] and [23].

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\int_0^1\Upsilon(\lambda y_1+(1-\lambda)y_2)\,d\lambda\\&\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\Upsilon\left(\frac{y_1+y_2}{2}\right)\end{aligned}$$

and

$$\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\leq\frac{1}{y_2-y_1}\int_{y_1}^{y_2}\Upsilon(\ell_1+\ell_2-\lambda)\,d\lambda\leq[\Upsilon(\ell_1)+\Upsilon(\ell_2)]-\frac{\Upsilon(y_1)+\Upsilon(y_2)}{2}.$$

Theorem 2.2. *Let* (A_1) *holds, then the following fractional integral inequalities will be of the form:*

$$\begin{aligned}\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right) &\leq \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_1)\big)\\ &+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_2)\big)\Big]\\ &\leq \Upsilon(\ell_1)+\Upsilon(\ell_2)-\left(\frac{\Upsilon(y_1)\ +\ \Upsilon(y_2)}{2}\right)\end{aligned} \tag{2.3}$$

for all $y_1, y_2 \in [\ell_1, \ell_2]$.

Proof. To prove the first part of inequality (2.3), we have

$$2\Upsilon\left(\ell_1+\ell_2-\frac{u+v}{2}\right)\leq\Upsilon(\ell_1+\ell_2-u)+\Upsilon(\ell_1+\ell_2-v)$$

for all $u, v \in [\ell_1, \ell_2]$.

By change of variables $u = \frac{\lambda}{2}y_1 + \frac{2-\lambda}{2}y_2$ and $v = \frac{2-\lambda}{2}y_1 + \frac{\lambda}{2}y_2$, $\lambda \in [0,1]$, we get

$$2\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\leq\Upsilon\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)+\Upsilon\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right).$$

Multiplying both sides of above inequality by $\lambda^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, and let $\Psi(\gamma) = \left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)$ and $\Psi(\beta) = \left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{t}{2}y_2\right)\right)$, we have

$$\begin{aligned}2\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)&\int_0^1\lambda^{\frac{\alpha}{k}-1}d\lambda\\ &\leq\int_0^1\lambda^{\frac{\alpha}{k}-1}\times\left[\Upsilon\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)+\Upsilon\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\right]d\lambda\end{aligned}$$

$$\begin{aligned}\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)&\leq\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\Big\{\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_1)\big)\\ &+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_2)\big)\Big\}\end{aligned}$$

and so the first inequality of (2.3) is proved.

Now for the proof of second inequality of (2.3), we first note that, if Υ is convex function, then for $\lambda \in [0,1]$, it gives

$$\Upsilon\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)\leq\Upsilon(\ell_1)+\Upsilon(\ell_2)-\left[\frac{\lambda}{2}\Upsilon(y_1)+\frac{2-\lambda}{2}\Upsilon(y_2)\right] \tag{2.4}$$

and

$$\Upsilon\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\leq\Upsilon(\ell_1)+\Upsilon(\ell_2)-\left[\frac{2-\lambda}{2}\Upsilon(y_1)+\frac{\lambda}{2}\Upsilon(y_2)\right]. \tag{2.5}$$

By adding the inequalities of (2.4) and (2.5), we have

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)+\Upsilon\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\\&\le\ 2\left(\Upsilon\left(\ell_1\right)+\Upsilon\left(\ell_2\right)\right)-\left(\Upsilon\left(y_1\right)+\Upsilon\left(y_2\right)\right).\end{aligned}$$

Multiplying both sides of above inequality by $\lambda^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we get

$$\begin{aligned}&\int_0^1\lambda^{\frac{\alpha}{k}-1}\left[\Upsilon\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)+\Upsilon\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\right]d\lambda\\&\le\ \left\{2\left(\Upsilon\left(\ell_1\right)+\Upsilon\left(\ell_2\right)\right)-\left(\Upsilon\left(y_1\right)+\Upsilon\left(y_2\right)\right)\right\}\int_0^1\lambda^{\frac{\alpha}{k}-1}d\lambda.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\frac{2^{\frac{\alpha}{k}}\ k\Gamma_k\left(\alpha\right)}{\left(y_2-y_1\right)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\left(\Upsilon\circ\Psi\right)(\Psi^{-1}(\ell_1+\ell_2-y_1)\Big)\\&+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\Big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_2)\Big)\Big]\le\ \left\{2\left(\Upsilon\left(\ell_1\right)+\Upsilon\left(\ell_2\right)\right)-\left(\Upsilon\left(y_1\right)+\Upsilon\left(y_2\right)\right)\right\}\frac{k}{\alpha}.\end{aligned}$$

Multiplying by $\frac{\alpha}{2k}$, we get

$$\begin{aligned}&\frac{2^{\frac{\alpha}{k}-1}\ \Gamma_k\left(\alpha+k\right)}{\left(y_2-y_1\right)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\Big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_1)\Big)\\&+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\Big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_2)\Big)\Big]\le\ \left(\Upsilon\left(\ell_1\right)+\Upsilon\left(\ell_2\right)\right)-\frac{\Upsilon\left(y_1\right)+\Upsilon\left(y_2\right)}{2}\end{aligned}$$

and so the second inequality of (2.3) is proved. □

Corollary 2. *Choosing $\Psi(\gamma)=\gamma$ in Theorem 2.2, we will get the following inequalities*

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\\&\le\frac{2^{\frac{\alpha}{k}-1}\Gamma_k\left(\alpha+k\right)}{\left(y_2-y_1\right)^{\frac{\alpha}{k}}}\left\{{}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})-}\Upsilon(\ell_1+\ell_2-y_2)\ \ +{}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})+}\Upsilon(\ell_1+\ell_2-y_1)\right\}\\&\le\Upsilon\left(\ell_1\right)+\Upsilon\left(\ell_2\right)-\left(\frac{\Upsilon\left(y_1\right)\ +\ \Upsilon\left(y_2\right)}{2}\right).\end{aligned}$$

Remark 3. For $k=1$ and taking $\Psi(\gamma)=\gamma$, we will get the following inequalities proved in [23].

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\\&\le\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(y_2-y_1\right)^{\alpha}}\left\{J^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})-}\Upsilon(\ell_1+\ell_2-y_2)+J^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})+}\Upsilon(\ell_1+\ell_2-y_1)\right\}\\&\le\Upsilon\left(\ell_1\right)+\Upsilon\left(\ell_2\right)-\left(\frac{\Upsilon\left(y_1\right)\ +\ \Upsilon\left(y_2\right)}{2}\right).\end{aligned}$$

Remark 4. For $\Psi(\gamma) = \gamma$ and $\alpha = k = 1$ in Theorem 2.2, we obtain

$$\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right) \leq \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} \Upsilon(\ell_1 + \ell_2 - \lambda)\, d\lambda \leq \Upsilon(\ell_1) + \Upsilon(\ell_2) - \left(\frac{\Upsilon(y_1) + \Upsilon(y_2)}{2}\right)$$

for all $y_1, y_2 \in [\ell_1, \ell_2]$. This inequality was proved by Kian and Moslehian in [12].

Theorem 2.3. *Let (A_1) holds, then the following fractional integral inequalities will be of the form:*

$$\begin{aligned}
\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right) &\leq \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha + k)}{(y_2 - y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)(\Upsilon \circ \Psi)\left(\Psi^{-1}(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2})\right) \\
&+ \left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)(\Upsilon \circ \Psi)\left(\Psi^{-1}(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2})\right)\Big] \\
&\leq \Upsilon(\ell_1) + \Upsilon(\ell_2) - \left(\frac{\Upsilon(y_1) + \Upsilon(y_2)}{2}\right)
\end{aligned} \tag{2.6}$$

for all $y_1, y_2 \in [\ell_1, \ell_2]$.

Proof. To prove the first part of inequality (2.6), we have

$$2\Upsilon\left(\ell_1 + \ell_2 - \frac{u + v}{2}\right) \leq \Upsilon(\ell_1 + \ell_2 - u) + \Upsilon(\ell_1 + \ell_2 - v)$$

$\forall\, u, v \in [\ell_1, \ell_2]$.
By change of variables $u = \frac{1+\lambda}{2}y_1 + \frac{1-\lambda}{2}y_2$ and $v = \frac{1-\lambda}{2}y_1 + \frac{1+\lambda}{2}y_2$, $\lambda \in [0, 1]$, we get

$$2\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right) \leq \Upsilon\left(\ell_1 + \ell_2 - \left(\frac{1 + \lambda}{2}y_1 + \frac{1 - \lambda}{2}y_2\right)\right) + \Upsilon\left(\ell_1 + \ell_2 - \left(\frac{1 - \lambda}{2}y_1 + \frac{1 + \lambda}{2}y_2\right)\right).$$

Multiplying both sides of above inequality by $\lambda^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to λ over $[0, 1]$, and let $\Psi(\gamma) = \left(\ell_1 + \ell_2 - \left(\frac{1+\lambda}{2}y_1 + \frac{1-\lambda}{2}y_2\right)\right)$ and $\Psi(\beta) = \left(\ell_1 + \ell_2 - \left(\frac{1-\lambda}{2}y_1 + \frac{1+\lambda}{2}y_2\right)\right)$, we obtain

$$\begin{aligned}
&2\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right)\int_0^1 \lambda^{\frac{\alpha}{k}-1} d\lambda \\
&\quad \leq \int_0^1 \lambda^{\frac{\alpha}{k}-1}\left[\Upsilon\left(\ell_1 + \ell_2 - \left(\frac{1 + \lambda}{2}y_1 + \frac{1 - \lambda}{2}y_2\right)\right) + \Upsilon\left(\ell_1 + \ell_2 - \left(\frac{1 - \lambda}{2}y_1 + \frac{1 + \lambda}{2}y_2\right)\right)\right] d\lambda.
\end{aligned}$$

So, we have

$$\begin{aligned}
\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right) &\leq \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha + k)}{(y_2 - y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left((\Upsilon \circ \Psi)\left(\Psi^{-1}(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2})\right)\right) \\
&+ \left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left((\Upsilon \circ \Psi)\left(\Psi^{-1}(\ell_1 + \ell_1 - \frac{y_1 + y_2}{2})\right)\right)\Big]
\end{aligned} \tag{2.7}$$

and so the first inequality of (2.6) is proved.

Now for the proof of second inequality of (2.6), we first note that, if Υ is convex function, then for $\lambda \in [0,1]$, it gives

$$\Upsilon\left(\ell_1+\ell_2-\left(\frac{1+\lambda}{2}y_1+\frac{1-\lambda}{2}y_2\right)\right)\le \Upsilon(\ell_1)+\Upsilon(\ell_2)-\left[\frac{1+\lambda}{2}\Upsilon(y_1)+\frac{1-\lambda}{2}\Upsilon(y_2)\right] \tag{2.8}$$

and

$$\Upsilon\left(\ell_1+\ell_2-\left(\frac{1-\lambda}{2}y_1+\frac{1+\lambda}{2}y_2\right)\right)\le \Upsilon(\ell_1)+\Upsilon(\ell_2)-\left[\frac{1-\lambda}{2}\Upsilon(y_1)+\frac{1+\lambda}{2}\Upsilon(y_2)\right]. \tag{2.9}$$

By adding the inequalities of (2.8) and (2.9), we have

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\left(\frac{1+\lambda}{2}y_1+\frac{1-\lambda}{2}y_2\right)\right)+\Upsilon\left(\ell_1+\ell_2-\left(\frac{1-\lambda}{2}y_1+\frac{1+\lambda}{2}y_2\right)\right)\\ &\le\ 2(\Upsilon(\ell_1)+\Upsilon(\ell_2))-(\Upsilon(y_1)+\Upsilon(y_2)).\end{aligned}$$

Multiplying both sides by $\lambda^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 \lambda^{\frac{\alpha}{k}-1}\left[\Upsilon\left(\ell_1+\ell_2-\left(\frac{1+\lambda}{2}y_1+\frac{1-\lambda}{2}y_2\right)\right)+\Upsilon\left(\ell_1+\ell_2-\left(\frac{1-\lambda}{2}y_1+\frac{1+\lambda}{2}y_2\right)\right)\right]d\lambda\\ &\le \left(2(\Upsilon(\ell_1)+\Upsilon(\ell_2))-(\Upsilon(y_1)+\Upsilon(y_2))\right)\int_0^1 \lambda^{\frac{\alpha}{k}-1}d\lambda.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\frac{2^{\frac{\alpha}{k}}\,k\Gamma_k(\alpha)}{(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)\\ &+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)\Big]\\ &\le \left(2(\Upsilon(\ell_1)+\Upsilon(\ell_2))-(\Upsilon(y_1)+\Upsilon(y_2))\right)\frac{k}{\alpha}.\end{aligned}$$

Multiplying by $\frac{\alpha}{2k}$, we get

$$\begin{aligned}&\frac{2^{\frac{\alpha}{k}-1}\,\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\right)\right)\\ &+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\right)\right)\Big]\\ &\le\ (\Upsilon(\ell_1)+\Upsilon(\ell_2))-\frac{\Upsilon(y_1)+\Upsilon(y_2)}{2}.\end{aligned} \tag{2.10}$$

From (2.7) and (2.10), we obtain (2.6). □

Corollary 3. *Choosing $\Psi(\gamma) = \gamma$ in Theorem 2.3, we get the following inequalities*

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\\&\le \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\left\{{}_kJ^{\alpha}_{(\ell_1+\ell_2-y_1)-}\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})+{}_kJ^{\alpha}_{(\ell_1+\ell_2-y_2)+}\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right\}\\&\le \Upsilon(\ell_1)+\Upsilon(\ell_2)-\left(\frac{\Upsilon(y_1)\ +\ \Upsilon(y_2)}{2}\right).\end{aligned}$$

Remark 5. For $k = 1$ and taking $\Psi(\gamma) = \gamma$, we have

$$\begin{aligned}&\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\\&\le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y_2-y_1)^{\alpha}}\left\{J^{\alpha}_{(\ell_1+\ell_2-y_1)-}\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})+J^{\alpha}_{(\ell_1+\ell_2-y_2)+}\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right\}\\&\le \Upsilon(\ell_1)+\Upsilon(\ell_2)-\left(\frac{\Upsilon(y_1)\ +\ \Upsilon(y_2)}{2}\right).\end{aligned}$$

3. New identities and related results

We start this section with the following important lemma:

Lemma 3.1. *Let $0 \le \ell_1 < \ell_2$, $\Upsilon : [\ell_1, \ell_2] \to \Re$ be a positive function and $\Upsilon \in L_1[\ell_1, \ell_2]$. Also suppose that Υ is a convex function on $[\ell_1, \ell_2]$,$\Psi(\gamma)$ is an increasing and positive monotone function on $(\ell_1, \ell_2]$, having a continuous derivative Ψ' on (ℓ_1, ℓ_2) and $\alpha, k > 0$. Then the following identity holds:*

$$\begin{aligned}&\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon\ (\ell_1+\ell_2-y_2)}{2}-\frac{\Gamma_k(\alpha+k)}{2(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_1-y_1)\right)\\&+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_2)\right)\Big]=\frac{1}{2(y_2-y_1)^{\frac{\alpha}{k}}}\\&\times\int_{\Psi^{-1}(\ell_1+\ell_2-y_2)}^{\Psi^{-1}(\ell_1+\ell_2-y_1)}\left((\Psi(\gamma)-(\ell_1+\ell_2-y_2))^{\frac{\alpha}{k}}-((\ell_1+\ell_2-y_1)-\Psi(\gamma))^{\frac{\alpha}{k}}\right)(\Upsilon'\circ\Psi)(\gamma)\Psi'(\gamma)d\gamma \qquad (3.1)\end{aligned}$$

for all $y_1, y_2 \in [\ell_1, \ell_2]$.

Proof. It suffices to note that

$$I=\frac{\Upsilon(\ell_1+\ell_2-y_1)-\Upsilon(\ell_1+\ell_2-y_2)}{2}-\{I_1+I_2\}, \qquad (3.2)$$

where

$$\begin{aligned}I_1&=\frac{\Gamma_k(\alpha+k)}{2(y_2-y_1)^{\frac{\alpha}{k}}}\left[{}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\left(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_1)\right)\right]\\&=\frac{\alpha}{2k(y_2-y_1)^{\frac{\alpha}{k}}}\int_{\Psi^{-1}(\ell_1+\ell_2-y_2)}^{\Psi^{-1}(\ell_1+\ell_2-y_1)}\Psi'(\gamma)\left((\ell_1+\ell_2-y_1)-\Psi(\gamma)\right)^{\frac{\alpha}{k}-1}(\Upsilon\circ\Psi)(\gamma)d\gamma\end{aligned}$$

$$= \frac{1}{2(y_2 - y_1)^{\frac{\alpha}{k}}} \left[\Upsilon(\ell_1 + \ell_2 - y_2)(y_2 - y_1)^{\frac{\alpha}{k}} \right]$$
$$+ \int_{\Psi^{-1}(\ell_1+\ell_2-y_2)}^{\Psi^{-1}(\ell_1+\ell_2-y_1)} \Psi'(\gamma) \left((\ell_1 + \ell_2 - y_1) - \Psi(\gamma)\right)^{\frac{\alpha}{k}} (\Upsilon' \circ \Psi)(\gamma) d\gamma \tag{3.3}$$

and

$$I_2 = \frac{\Gamma_k(\alpha + k)}{2(y_2 - y_1)^{\frac{\alpha}{k}}} \left[{}_k I^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-} \left((\Upsilon \circ \Psi)(\Psi^{-1}(\ell_1 + \ell_2 - y_2)) \right) \right]$$
$$= \frac{\alpha}{2k(y_2 - y_1)^{\frac{\alpha}{k}}} \int_{\Psi^{-1}(\ell_1+\ell_2-y_2)}^{\Psi^{-1}(\ell_1+\ell_2-y_1)} \Psi'(\gamma) \left(-(\ell_1 + \ell_2 - y_2)) + \Psi(\gamma)\right)^{\frac{\alpha}{k}-1} (\Upsilon \circ \Psi)(\gamma) d\gamma$$
$$= \frac{1}{2(y_2 - y_1)^{\frac{\alpha}{k}}} \left[\Upsilon(\ell_1 + \ell_2 - y_1)(y_2 - y_1)^{\frac{\alpha}{k}} \right]$$
$$- \int_{\Psi^{-1}(\ell_1+\ell_2-y_2)}^{\Psi^{-1}(\ell_1+\ell_2-y_1)} \Psi'(\gamma) \left(-(\ell_1 + \ell_2 - y_2) + \Psi(\gamma)\right)^{\frac{\alpha}{k}} (\Upsilon' \circ \Psi)(\gamma) d\gamma. \tag{3.4}$$

Substituting (3.3) and (3.4) in (3.2), we get (3.1). □

Corollary 4. *Choosing $\Psi(\gamma) = \gamma$ in Lemma 3.1, we get the following equality*

$$\frac{\Upsilon(\ell_1 + \ell_2 - y_1) + \Upsilon(\ell_1 + \ell_2 - y_2)}{2}$$
$$- \frac{\Gamma_k(\alpha + k)}{2(y_2 - y_1)^{\frac{\alpha}{k}}} \left[{}_k J^{\alpha}_{(\ell_1+\ell_2-y_2)^+} \Upsilon(\ell_1 + \ell_2 - y_1) + {}_k J^{\alpha}_{(\ell_1+\ell_2-y_1)^-} \Upsilon(\ell_1 + \ell_2 - y_2) \right]$$
$$= \frac{1}{2(y_2 - y_1)^{\frac{\alpha}{k}}} \int_{(\ell_1+\ell_2-y_2)}^{(\ell_1+\ell_2-y_1)} \left[(\gamma - (\ell_1 + \ell_2 - y_2))^{\frac{\alpha}{k}} - ((\ell_1 + \ell_2 - y_1) - \gamma)^{\frac{\alpha}{k}} \right] \Upsilon'(\gamma) d\gamma.$$

Remark 6. For $k = 1$ and taking $y_1 = \ell_1$ and $y_2 = \ell_2$, we get the following Lemma by O'Regan et al. in [17]:

$$\frac{\Upsilon(\ell_1) + \Upsilon(\ell_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\ell_2 - \ell_1)^{\alpha}} \left\{ \left(I^{\alpha:\Psi}_{\Psi^{-1}(\ell_1)^+} \right) \left(\Upsilon \circ \Psi)(\Psi^{-1}(\ell_2) \right) + \left(I^{\alpha:\Psi}_{\Psi^{-1}(\ell_2)^-} \right) \left(\Upsilon \circ \Psi)(\Psi^{-1}(\ell_1) \right) \right\}$$
$$= \frac{1}{2(\ell_2 - \ell_1)^{\alpha}} \int_{\Psi^{-1}(\ell_1)}^{\Psi^{-1}(\ell_2)} \left((\Psi(\gamma) - (\ell_1))^{\alpha} - ((\ell_2) - \Psi(\gamma))^{\alpha} \right) (\Upsilon' \circ \Psi)(\gamma) \Psi'(\gamma) d\gamma.$$

Theorem 3.2. *Let (A_1) holds. Also suppose that $|\Upsilon'|$ is a convex function on $[\ell_1, \ell_2]$, then the following fractional integral inequality holds:*

$$\left| \frac{\Upsilon(\ell_1 + \ell_2 - y_1) + \Upsilon(\ell_1 + \ell_2 - y_2)}{2} - \frac{\Gamma_k(\alpha + k)}{2(y_2 - y_1)^{\frac{\alpha}{k}}} \left[({}_k I^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+})\Upsilon(\ell_1 + \ell_2 - y_1) \right. \right.$$
$$\left. \left. + ({}_k I^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-})\Upsilon(\ell_1 + \ell_2 - y_2) \right] \right|$$
$$\leq \left(\frac{y_2 - y_1}{\frac{\alpha}{k} + 1} \right) \left(1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \left\{ \left| \Upsilon'(\ell_1) \right| + \left| \Upsilon'(\ell_2) \right| - \left(\frac{\left| \Upsilon'(y_1) \right| + \left| \Upsilon'(y_2) \right|}{2} \right) \right\} \tag{3.5}$$

for all $y_1, y_2 \in [\ell_1, \ell_2]$.

Proof. By using Lemma 3.1, Jensen–Mercer inequality and properties of modulus, we have

$$
\begin{aligned}
&\Bigg|\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2}-\frac{\Gamma_k(\alpha+k)}{2(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kI^{\alpha:\Psi}_{\left(\ell_1+\ell_2-y_2\right)^+}\right)\Upsilon(\ell_1+\ell_2-y_1)\\
&+\left({}_kI^{\alpha:\Psi}_{\left(\ell_1+\ell_2-y_1\right)^-}\right)\Upsilon(\ell_1+\ell_2-y_2)\Big]\Bigg|\\
&\le\int_{\Psi^{-1}(\ell_1+\ell_2-y_2)}^{\Psi^{-1}(\ell_1+\ell_2-y_1)}\left|\left((\Psi(\gamma)-(\ell_1+\ell_2-y_2))^{\frac{\alpha}{k}}-((\ell_1+\ell_2-y_1)-\Psi(\gamma))^{\frac{\alpha}{k}}\right)\right|\\
&\times\frac{1}{2(y_2-y_1)^{\frac{\alpha}{k}}}\left|(\Upsilon'\circ\Psi)(\gamma)\right|\Psi'(\gamma)d\gamma\\
&=\frac{(y_2-y_1)}{2}\int_0^1\left|\lambda^{\frac{\alpha}{k}}-(1-\lambda)^{\frac{\alpha}{k}}\right|\left|\Upsilon'\left((\ell_1+\ell_2-(\lambda y_1+(1-\lambda)\,y_2)\right)\right|d\lambda\\
&\le\frac{(y_2-y_1)}{2}\int_0^1\left|\lambda^{\frac{\alpha}{k}}-(1-\lambda)^{\frac{\alpha}{k}}\right|\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\lambda\left|\Upsilon'(y_1)\right|+(1-\lambda)\left|\Upsilon'(y_2)\right|\right)\right\}d\lambda\\
&=\frac{(y_2-y_1)}{2}\left[I_1+I_2\right],
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&=\int_0^{\frac{1}{2}}\left((1-\lambda)^{\frac{\alpha}{k}}-\lambda^{\frac{\alpha}{k}}\right)\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\lambda\left|\Upsilon'(y_1)\right|+(1-\lambda)\left|\Upsilon'(y_2)\right|\right)\right\}\\
&=\left(\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|\right)\left(\frac{1}{(\frac{\alpha}{k}+1)}-\frac{2^{-\frac{\alpha}{k}}}{(\frac{\alpha}{k}+1)}\right)\\
&-\left\{\left|\Upsilon'(y_1)\right|\left(\frac{1}{\left(\frac{\alpha}{k}+1\right)(\alpha+2)}-\frac{2^{-\frac{\alpha}{k}-1}}{(\frac{\alpha}{k}+1)}\right)+\left|\Upsilon'(y_2)\right|\left(\frac{1}{\left(\frac{\alpha}{k}+2\right)}-\frac{2^{-\frac{\alpha}{k}-1}}{(\frac{\alpha}{k}+1)}\right)\right\}
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
I_2&=\int_{\frac{1}{2}}^1\left(\lambda^{\frac{\alpha}{k}}-(1-\lambda)^{\frac{\alpha}{k}}\right)\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\lambda\left|\Upsilon'(y_1)\right|+(1-\lambda)\left|\Upsilon'(y_2)\right|\right)\right\}\\
&=\left(\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|\right)\left(\frac{1}{\left(\frac{\alpha}{k}+1\right)}-\frac{2^{-\frac{\alpha}{k}}}{(\frac{\alpha}{k}+1)}\right)\\
&-\left\{\left|\Upsilon'(y_1)\right|\left(\frac{1}{\left(\frac{\alpha}{k}+2\right)}-\frac{2^{-\frac{\alpha}{k}-1}}{(\frac{\alpha}{k}+1)}\right)+\left|\Upsilon'(y_2)\right|\left(\frac{1}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)}-\frac{2^{-\frac{\alpha}{k}-1}}{(\frac{\alpha}{k}+1)}\right)\right\}.
\end{aligned}
\tag{3.7}
$$

Combining inequalities (3.6) and (3.7), we get the desired inequality (3.5). □

Corollary 5. *Choosing $\Psi(\gamma)=\gamma$ in Theorem 3.2, we get the following inequality*

$$
\Bigg|\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2}-\frac{\Gamma_k(\alpha+k)}{2(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_2)^+}\right)\Upsilon(\ell_1+\ell_2-y_1)
$$

$$+ \left({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_1)^-}\right)\Upsilon(\ell_1+\ell_2-y_2)\Big]\Big|$$
$$\leq \left(\frac{y_2-y_1}{\frac{\alpha}{k}+1}\right)\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\frac{\left|\Upsilon'(y_1)\right|+\left|\Upsilon'(y_2)\right|}{2}\right)\right\}.$$

Remark 7. For $y_1 = \ell_1$ and $y_2 = \ell_2$ and $k = 1$ in Theorem 3.2, we will get inequality proved in [17]:

$$\left|\frac{\Upsilon(\ell_1)+\Upsilon(\ell_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\ell_2-\ell_1)^{\alpha}}\Big[(I^{\alpha:\Psi}_{\Psi^{-1}(\ell_1)^+})\Upsilon(\ell_2) + (I^{\alpha:\Psi}_{\Psi^{-1}(\ell_2)^-})\Upsilon(\ell_1)\Big]\right|$$
$$\leq \frac{(\ell_2-\ell_1)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\Big[\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|\Big].$$

Remark 8. For $k = 1$ and taking $\Psi(\gamma) = \gamma$, we obtain Theorem 4 in [23].

Lemma 3.3. *Let $0 \leq \ell_1 < \ell_2$, $\Upsilon : [\ell_1, \ell_2] \to \Re$ be a positive function and $\Upsilon \in L_1[\ell_1, \ell_2]$. Also suppose that Υ is a convex function on $[\ell_1, \ell_2]$, $\Psi(\gamma)$ is an increasing and positive monotone function on $(\ell_1, \ell_2]$, having a continuous derivative Ψ' on (ℓ_1, ℓ_2) and $\alpha, k > 0$. Then the following identity holds:*

$$\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$
$$\times\Big[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)$$
$$+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)\Big]$$
$$=\frac{(y_2-y_1)}{4}\Big[\int_0^1 \lambda^{\frac{\alpha}{k}}\,\Upsilon'\left(\ell_1+\ell_2-\left(\frac{1+\lambda}{2}y_1+\frac{1-\lambda}{2}y_2\right)\right)d\lambda$$
$$-\int_0^1 \lambda^{\frac{\alpha}{k}}\,\Upsilon'\left(\ell_1+\ell_2-\left(\frac{1-\lambda}{2}y_1+\frac{1+\lambda}{2}y_2\right)\right)d\lambda\Big]. \tag{3.8}$$

Proof. It suffices to note that

$$I = \frac{(y_2-y_1)}{4}\{I_1 - I_2\}, \tag{3.9}$$

where

$$I_1 = \int_0^1 \lambda^{\frac{\alpha}{k}}\,\Upsilon'\left(\ell_1+\ell_2-\left(\frac{1+\lambda}{2}y_1+\frac{1-\lambda}{2}y_2\right)\right)d\lambda$$
$$= \frac{2}{(y_2-y_1)}\,\Upsilon(\ell_1+\ell_2-y_1) - \frac{2\left(\frac{\alpha}{k}\right)}{y_2-y_1}\int_0^1 \lambda^{\frac{\alpha}{k}-1}\,\Upsilon\left(\ell_1+\ell_2-\left(\frac{1+\lambda}{2}y_1+\frac{1-\lambda}{2}y_2\right)\right)d\lambda$$
$$= \frac{2}{(y_2-y_1)}\,\Upsilon(\ell_1+\ell_2-y_1)$$
$$-\frac{2^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}+1}}\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right) \tag{3.10}$$

and

$$I_2 = \int_0^1 \lambda^{\frac{\alpha}{k}}\Upsilon'\left(\ell_1+\ell_2-\left(\frac{1-\lambda}{2}y_1+\frac{1+\lambda}{2}y_2\right)\right)d\lambda$$

$$= -\frac{2}{(y_2-y_1)}\Upsilon(\ell_1+\ell_2-y_2) - \frac{2\left(\frac{\alpha}{k}\right)}{y_2-y_1}\int_0^1 \lambda^{\frac{\alpha}{k}-1}\,\Upsilon\left(\ell_1+\ell_2-\left(\frac{1-\lambda}{2}y_1+\frac{1+\lambda}{2}y_2\right)\right)d\lambda$$

$$= -\frac{2}{(y_2-y_1)}\Upsilon(\ell_1+\ell_2-y_2)$$

$$-\frac{2^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}+1}}\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right) \tag{3.11}$$

Substituting (3.10) and (3.11) in (3.9), we get (3.8). □

Corollary 6. *Choosing $\Psi(\gamma)=\gamma$ in Lemma 3.3, we get the following equality*

$$\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$
$$\times\left\{\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_2)^+}\right)\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)+\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_1)^-}\right)\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\right\}$$
$$= \frac{(y_2-y_1)}{4}\int_0^1 \lambda^{\frac{\alpha}{k}}\left[\Upsilon'\left(\ell_1+\ell_2-\left(\frac{1+\lambda}{2}y_1+\frac{1-\lambda}{2}y_2\right)\right)-\right.$$
$$\left.\Upsilon'\left(\ell_1+\ell_2-\left(\frac{1-\lambda}{2}y_1+\frac{1+\lambda}{2}y_2\right)\right)\right]d\lambda.$$

Theorem 3.4. *Let (A_1) holds. Also suppose that Υ' is a differentiable and Υ'' is bounded function on $[\ell_1,\ell_2]$, then the following fractional integral inequality holds:*

$$\left|\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\right.$$
$$\times\left[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)\right.$$
$$\left.\left.+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)\right]\right|$$
$$\leq \frac{(y_2-y_1)^2}{4(\frac{\alpha}{k}+2)}\sup_{\xi\in[\ell_1,\ell_2]}\left|\Upsilon''(\xi)\right| \tag{3.12}$$

for all $y_1,y_2\in[\ell_1,\ell_2]$.

Proof. By using Lemma 3.3 and applying mean value theorem for the function Υ', we have

$$\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$
$$\times\left[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)\right.$$
$$\left.+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)\right]$$
$$= \frac{(y_2-y_1)^2}{4}\int_0^1 (\lambda)^{\frac{\alpha}{k}}\,\lambda\Upsilon''(\xi)\,d\lambda, \tag{3.13}$$

where $\xi(\lambda) \in [\ell_1, \ell_2]$. This leads us to

$$\Bigg|\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}} \\ \times\Bigg[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_2)^+}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right) \\ +\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-y_1)^-}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)\right)\Bigg]\Bigg| \\ \leq\frac{(y_2-y_1)^2}{4}\int_0^1(\lambda)^{\frac{\alpha}{k}}\lambda\left|\Upsilon''(\xi)\right|d\lambda \\ \leq\frac{(y_2-y_1)^2}{4}\sup_{\xi\in[\ell_1,\ell_2]}\left|\Upsilon''(\xi)\right|\times\int_0^1(\lambda)^{\frac{\alpha}{k}+1}d\lambda \\ =\frac{(y_2-y_1)^2}{4(\frac{\alpha}{k}+2)}\sup_{\xi\in[\ell_1,\ell_2]}\left|\Upsilon''(\xi)\right|,$$

which completes the proof. □

Corollary 7. *Choosing $\Psi(\gamma)=\gamma$ in Theorem 3.4, we get the following inequality*

$$\Bigg|\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}} \\ \times\left\{\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_2)^+}\right)\left(\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)+\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_1)^-}\right)\left(\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\right)\right\}\Bigg| \\ \leq\frac{(y_2-y_1)^2}{4(\frac{\alpha}{k}+2)}\sup_{\xi\in[\ell_1,\ell_2]}\left|\Upsilon''(\xi)\right|.$$

Theorem 3.5. *Let (A_1) holds. Also suppose that $|\Upsilon'|$ is a convex function on $[\ell_1,\ell_2]$, then the following fractional integral inequality holds:*

$$\Bigg|\frac{\Upsilon(\ell_1+\ell_2-y_2)+\Upsilon(\ell_1+\ell_2-y_1)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}} \\ \times\Bigg[\left({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-y_2)\right)^+}\right)\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right) \\ +\left({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-y_1)\right)^-}\right)\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)\Bigg]\Bigg| \\ \leq\frac{(y_2-y_1)}{2(\frac{\alpha}{k}+1)}\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\frac{\left|\Upsilon'(y_1)\right|+\left|\Upsilon'(y_2)\right|}{2}\right)\right\} \tag{3.14}$$

for all $y_1, y_2 \in [\ell_1,\ell_2]$.

Proof. By using Lemma 3.3, Jensen–Mercer inequality and properties of modulus, we have

$$\Bigg|\frac{\Upsilon(\ell_1+\ell_2-y_2)+\Upsilon(\ell_1+\ell_2-y_1)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$

$$\times\Big[\Big({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-y_2)\right)^+}\Big)\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)$$
$$+\Big({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-y_1)\right)^-}\Big)\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)\Big]\Big|$$
$$\leq\frac{(y_2-y_1)}{4}\Big[\int_0^1\lambda^{\frac{\alpha}{k}}\left|\Upsilon'\Big((\ell_1+\ell_2-\Big(\frac{1+\lambda}{2}y_1+\frac{1-\lambda}{2}y_2\Big)\Big)\right|d\lambda$$
$$+\int_0^1\lambda^{\frac{\alpha}{k}}\left|\Upsilon'\Big((\ell_1+\ell_2-\Big(\frac{1-\lambda}{2}y_1+\frac{1+\lambda}{2}y_2\Big)\Big)\right|d\lambda\Big]$$
$$\leq\frac{(y_2-y_1)}{4}\Big[\int_0^1\lambda^{\frac{\alpha}{k}}\Big\{|\Upsilon'(\ell_1)|+|\Upsilon'(\ell_2)|-\Big(\frac{1+\lambda}{2}|\Upsilon'(y_1)|+\frac{(1-\lambda)}{2}|\Upsilon'(y_2)|\Big)\Big\}d\lambda$$
$$+\int_0^1\lambda^{\frac{\alpha}{k}}\Big\{|\Upsilon'(\ell_1)|+|\Upsilon'(\ell_2)|-\Big(\frac{(1-\lambda)}{2}|\Upsilon'(y_1)|+\frac{1+\lambda}{2}|\Upsilon'(y_2)|\Big)\Big\}d\lambda\Big].$$

After integration, we obtain the required result. □

Corollary 8. *Choosing $\Psi(\gamma)=\gamma$ in Theorem 3.5, we get the following inequality*

$$\Big|\frac{\Upsilon(\ell_1+\ell_2-y_1)+\Upsilon(\ell_1+\ell_2-y_2)}{2}-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$
$$\times\Big\{\Big({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_2)^+}\Big)\Big(\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)\Big)+\Big({}_kJ^{\alpha}_{(\ell_1+\ell_2-y_1)^-}\Big)\Big(\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)\Big)\Big\}\Big|$$
$$\leq\frac{(y_2-y_1)}{2(\frac{\alpha}{k}+1)}\Big\{|\Upsilon'(\ell_1)|+|\Upsilon'(\ell_2)|-\Big(\frac{|\Upsilon'(y_1)|+|\Upsilon'(y_2)|}{2}\Big)\Big\}.$$

Lemma 3.6. *Let $0\leq\ell_1<\ell_2$, $\Upsilon:[\ell_1,\ell_2]\to\Re$ be a positive function and $\Upsilon\in L_1[\ell_1,\ell_2]$. Also suppose that Υ is a convex function on $[\ell_1,\ell_2]$, $\Psi(\gamma)$ is an increasing and positive monotone function on $(\ell_1,\ell_2]$, having a continuous derivative Ψ' on (ℓ_1,ℓ_2) and $\alpha,k>0$. Then the following identity holds:*

$$\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$
$$\Big[\Big({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\Big)\Big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_1)\Big)$$
$$+\Big({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\Big)\Big(\Upsilon\circ\Psi)(\Psi^{-1}(\ell_1+\ell_2-y_2)\Big)\Big]$$
$$=\frac{(y_2-y_1)}{4}\Big[\int_0^1\lambda^{\frac{\alpha}{k}}\,\Upsilon'\Big(\ell_1+\ell_2-\Big(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\Big)\Big)d\lambda$$
$$-\int_0^1\lambda^{\frac{\alpha}{k}}\,\Upsilon'\Big(\ell_1+\ell_2-\Big(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\Big)\Big)d\lambda\Big].$$

Proof. The proof of this Lemma is similar to the proof of Lemma 3.3. □

Corollary 9. *Choosing $\Psi(\gamma)=\gamma$ in Lemma 3.6, we get the following identity*

$$\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$

$$\times\left[\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\Upsilon(\ell_1+\ell_2-y_1)+\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\Upsilon(\ell_1+\ell_2-y_2)\right]$$
$$=\frac{(y_2-y_1)}{4}\left[\int_0^1\lambda^{\frac{\alpha}{k}}\,\Upsilon'\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)d\lambda\right.$$
$$\left.-\int_0^1\lambda^{\frac{\alpha}{k}}\,\Upsilon'\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)d\lambda\right].$$

Remark 9. For $k=1$ and taking $\Psi(\gamma)=\gamma$ in Lemma 3.6, we obtain Lemma 2 in [23].

Remark 10. For $\Psi(\gamma)=\gamma$, $y_1=\ell_1$ and $y_2=\ell_2$, we have Lemma 3 in [25].

Theorem 3.7. *Let (A_1) holds. Also suppose that $|\Upsilon'|$ is a convex function on $[\ell_1,\ell_2]$, then the following fractional integral inequality holds:*

$$\left|\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\right.$$
$$\left.\times\left[({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)^+})\Upsilon(\ell_1+\ell_2-y_1)+({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)^-})\Upsilon(\ell_1+\ell_2-y_2)\right]\right|$$
$$\leq\frac{(y_2-y_1)}{2(\frac{\alpha}{k}+1)}\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\frac{\left|\Upsilon'(y_1)\right|+\left|\Upsilon'(y_2)\right|}{2}\right)\right\}\tag{3.15}$$

for all $y_1,y_2\in[\ell_1,\ell_2]$.

Proof. By using Lemma 3.6, Jensen–Mercer inequality and properties of modulus, we have

$$\left|\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\right.$$
$$\left.\times\left[({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)^+})\Upsilon(\ell_1+\ell_2-y_1)+({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)^-})\Upsilon(\ell_1+\ell_2-y_2)\right]\right|$$
$$\leq\frac{(y_2-y_1)}{4}\left[\int_0^1\lambda^{\frac{\alpha}{k}}\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)\right|d\lambda\right.$$
$$\left.+\int_0^1\lambda^{\frac{\alpha}{k}}\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\right|d\lambda\right]$$
$$\leq\frac{(y_2-y_1)}{4}\left[\int_0^1\lambda^{\frac{\alpha}{k}}\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\frac{\lambda}{2}\left|\Upsilon'(y_1)\right|+\frac{(2-\lambda)}{2}\left|\Upsilon'(y_2)\right|\right)\right\}d\lambda\right.$$
$$\left.+\int_0^1\lambda^{\frac{\alpha}{k}}\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\frac{(2-\lambda)}{2}\left|\Upsilon'(y_1)\right|+\frac{\lambda}{2}\left|\Upsilon'(y_2)\right|\right)\right\}d\lambda\right]$$
$$\leq\frac{(y_2-y_1)}{2(\frac{\alpha}{k}+1)}\left\{\left|\Upsilon'(\ell_1)\right|+\left|\Upsilon'(\ell_2)\right|-\left(\frac{\left|\Upsilon'(y_1)\right|+\left|\Upsilon'(y_2)\right|}{2}\right)\right\},$$

which completes the proof. □

Corollary 10. *Choosing $\Psi(\gamma)=\gamma$ in Theorem 3.7, we get the following inequality*

$$\left|\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\right.$$

$$\times\Big[\Big({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\Big)\Upsilon(\ell_1+\ell_2-y_1)+\Big({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\Big)\Upsilon(\ell_1+\ell_2-y_2)\Big]\Big|$$
$$\leq\frac{(y_2-y_1)}{2(\frac{\alpha}{k}+1)}\Big\{\big|\Upsilon'(\ell_1)\big|+\big|\Upsilon'(\ell_2)\big|-\Big(\frac{\big|\Upsilon'(y_1)\big|+\big|\Upsilon'(y_2)\big|}{2}\Big)\Big\}.$$

Remark 11. For $k=1$ and taking $\Psi(\gamma)=\gamma$ in Theorem 3.7, we obtain Theorem 5 in [23].

Theorem 3.8. *Let (A_1) holds. If $|\Upsilon'|^q$ is convex function, then for $q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$, the following fractional integral inequality holds:*

$$\Big|\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\Big({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)^+}\Big)\Upsilon(\ell_1+\ell_2-y_1)$$
$$+\Big({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)^-}\Big)\Upsilon(\ell_1+\ell_2-y_2)\Big]\Big|$$
$$\leq\frac{(y_2-y_1)}{4}\Big(\frac{k}{p\alpha+k}\Big)^{\frac{1}{p}}\Big[\Big(\Big(\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q-\Big(\frac{1}{4}\big|\Upsilon'(y_1)\big|^q+\frac{3}{4}\big|\Upsilon'(y_2)\big|^q\Big)\Big)^{\frac{1}{q}}$$
$$+\Big(\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q-\Big(\frac{3}{4}\big|\Upsilon'(y_1)\big|^q+\frac{1}{4}\big|\Upsilon'(y_2)\big|^q\Big)\Big)^{\frac{1}{q}}\Big]\tag{3.16}$$

for all $y_1,y_2\in[\ell_1,\ell_2]$.

Proof. By using Lemma 3.6, Holder's inequality, Jensen-Mercer inequality, the fact that $|\Upsilon'|^q$ is convex function and properties of modulus, we have

$$\Big|\Upsilon(\ell_1+\ell_2-\frac{y_1+y_2}{2})-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\Big[\Big({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)^+}\Big)\Upsilon(\ell_1+\ell_2-y_1)$$
$$+\Big({}_kI^{\alpha:\Psi}_{\left(\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})\right)^-}\Big)\Upsilon(\ell_1+\ell_2-y_2)\Big]\Big|$$
$$\leq\frac{(y_2-y_1)}{4}\Big[\int_0^1\lambda^{\frac{\alpha}{k}}\Big|\Upsilon'\Big(\ell_1+\ell_2-\Big(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\Big)\Big)\Big|d\lambda$$
$$+\int_0^1\lambda^{\frac{\alpha}{k}}\Big|\Upsilon'\Big(\ell_1+\ell_2-\Big(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\Big)\Big)\Big|d\lambda\Big]$$
$$\leq\Big(\frac{y_2-y_1}{4}\Big)\int_0^1\lambda^{\frac{\alpha}{k}}\Big\{\Big|\Upsilon'(\ell_1)+\Upsilon'(\ell_2)-\Big(\frac{\lambda}{2}\Upsilon'(y_1)+\frac{(2-\lambda)}{2}\Upsilon'(y_2)\Big)\Big|\Big\}d\lambda$$
$$+\Big(\frac{(y_2-y_1)}{4}\Big)\int_0^1\lambda^{\frac{\alpha}{k}}\Big\{\Big|\Upsilon'(\ell_1)+\Upsilon'(\ell_2)-\Big(\frac{(2-\lambda)}{2}\Upsilon'(y_1)+\frac{\lambda}{2}\Upsilon'(y_2)\Big)\Big|\Big\}d\lambda$$
$$\leq\frac{(y_2-y_1)}{4}\Big(\int_0^1\lambda^{p(\frac{\alpha}{k})}d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|\Upsilon'(\ell_1)+\Upsilon'(\ell_1)-\Big(\frac{\lambda}{2}\Upsilon'(y_1)+\frac{(2-\lambda)}{2}\Upsilon'(y_2)\Big)\Big|^q d\lambda\Big)^{\frac{1}{q}}$$
$$+\frac{(y_2-y_1)}{4}\Big(\int_0^1\lambda^{p(\frac{\alpha}{k})}d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|\Upsilon'(\ell_1)+\Upsilon'(\ell_2)-\Big(\frac{(2-\lambda)}{2}\Upsilon'(y_1)+\frac{\lambda}{2}\Upsilon'(y_2)\Big)\Big|^q d\lambda\Big)^{\frac{1}{q}}$$
$$\leq\frac{(y_2-y_1)}{4}\Big(\frac{k}{p\alpha+k}\Big)^{\frac{1}{p}}\Big(\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q-\Big(\frac{1}{4}\big|\Upsilon'(y_1)\big|^q+\frac{3}{4}\big|\Upsilon'(y_2)\big|^q\Big)\Big)^{\frac{1}{q}}$$

$$+\frac{(y_2-y_1)}{4}\Big(\frac{k}{p\alpha+k}\Big)^{\frac{1}{p}}\Big(\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q-\Big(\frac{3}{4}\big|\Upsilon'(y_2)\big|^q+\frac{1}{4}\big|\Upsilon'(y_1)\big|^q\Big)\Big)^{\frac{1}{q}}$$

$$=\frac{(y_2-y_1)}{4}\Big(\frac{k}{p\alpha+k}\Big)^{\frac{1}{p}}\Big[\Big(\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q-\Big(\frac{1}{4}\big|\Upsilon'(y_1))\big|^q+\frac{3}{4}\big|\Upsilon'(y_2)\big|^q\Big)\Big)^{\frac{1}{q}}$$

$$+\Big(\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q-\Big(\frac{3}{4}\big|\Upsilon'(y_1)\big|^q+\frac{1}{4}\big|\Upsilon'(y_2)\big|^q\Big)\Big)^{\frac{1}{q}}\Big],$$

which completes the proof. □

Corollary 11. *Choosing $\Psi(\gamma)=\gamma$ in Theorem 3.8, we get the following inequality*

$$\Big|\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$

$$\times\Big[\Big({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\Big)\Upsilon(\ell_1+\ell_2-y_1)+\Big({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\Big)\Upsilon(\ell_1+\ell_2-y_2)\Big]\Big|$$

$$\leq\frac{(y_2-y_1)}{4}\Big(\frac{k}{p(\alpha)+k}\Big)^{\frac{1}{p}}\Big[\Big(\Big(\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q-\Big(\frac{1}{4}\big|\Upsilon'(y_1))\big|^q+\frac{3}{4}\big|\Upsilon'(y_2)\big|^q\Big)\Big)^{\frac{1}{q}}$$

$$+\Big(\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q-\Big(\frac{3}{4}\big|\Upsilon'(y_2)\big|^q+\frac{1}{4}\big|\Upsilon'(y_1)\big|^q\Big)\Big)^{\frac{1}{q}}\Big].$$

Remark 12. For $k=1$ and taking $\Psi(\gamma)=\gamma$ in Theorem 3.8, we have Theorem 6 in [23].

4. New inequalities via improved Hölder inequality and related results

Theorem 4.1. *Let (A_1) holds. Also suppose that $|\Upsilon'|^q$ is a convex function on $[\ell_1,\ell_2]$, then for $q\geq 1$, the following fractional integral inequality holds:*

$$\Big|\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$

$$\times\Big[\Big({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\Big)\Big((\Upsilon\circ\Psi)\Big(\Psi^{-1}(\ell_1+\ell_2-y_1)\Big)\Big)$$

$$+\Big({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\Big)\Big((\Upsilon\circ\Psi)\Big(\Psi^{-1}(\ell_1+\ell_2-y_2)\Big)\Big)\Big]\Big|$$

$$\leq\frac{(y_2-y_1)}{4}\Big[\Big\{\Big(\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\Big(\frac{\big|\Upsilon'(y_1)\big|^q}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}$$

$$+\frac{(\frac{\alpha}{k}+5)\big|\Upsilon'(y_2)\big|^q}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}\Big)\Big)^{\frac{1}{q}}$$

$$+\Big(\frac{1}{(\frac{\alpha}{k}+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q}{(\frac{\alpha}{k}+2)}-\Big(\frac{\big|\Upsilon'(y_1)\big|^q}{2(\frac{\alpha}{k}+3)}+\frac{\big|\Upsilon'(y_2)\big|^q}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}\Big)\Big)^{\frac{1}{q}}\Big\}$$

$$+\Big\{\Big(\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{\big|\Upsilon'(\ell_1)\big|^q+\big|\Upsilon'(\ell_2)\big|^q}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}-\Big(\frac{(\frac{\alpha}{k}+5)\big|\Upsilon'(y_1)\big|^q}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}$$

$$+\frac{\left|\Upsilon'(y_2)\right|^q}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}\Big)\Big)^{\frac{1}{q}}+\Big(\frac{1}{(\frac{\alpha}{k}+2)}\Big)^{1-\frac{1}{q}}$$
$$\times\Big(\frac{\left|\Upsilon'(\ell_1)\right|^q+\left|\Upsilon'(\ell_2)\right|^q}{(\frac{\alpha}{k}+2)}-\Big(\frac{(\frac{\alpha}{k}+4)\left|\Upsilon'(y_1)\right|^q}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}+\frac{\left|\Upsilon'(y_2)\right|^q}{2(\frac{\alpha}{k}+3)}\Big)\Big)^{\frac{1}{q}}\Big\}\Big] \tag{4.1}$$

for all $y_1, y_2 \in [\ell_1, \ell_2]$.

Proof. By using Lemma 3.6, Jensen–Mercer inequality for $|\Upsilon'|^q$, applying the Improved power-mean integral inequality (see [11]) and properties of modulus, we have

$$\Bigg|\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$
$$\times\Bigg[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-y_1)\right)\right)$$
$$+\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\left((\Upsilon\circ\Psi)\left(\Psi^{-1}(\ell_1+\ell_2-y_2)\right)\right)\Bigg]\Bigg|$$

$$\leq\frac{(y_2-y_1)}{4}\Big\{\Big(\int_0^1(1-\lambda)\lambda^{\frac{\alpha}{k}}\,d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-\lambda)\lambda^{\frac{\alpha}{k}}\Big|\Upsilon'(\ell_1+\ell_2-(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2))\Big|^q\,d\lambda\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1\lambda^{\frac{\alpha}{k}+1}\,d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^1\lambda^{\frac{\alpha}{k}+1}\Big|\Upsilon'(\ell_1+\ell_2-(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2))\Big|^q\,d\lambda\Big)^{\frac{1}{q}}\Big\}$$
$$+\Big\{\Big(\int_0^1(1-\lambda)\lambda^{\frac{\alpha}{k}}\,d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-\lambda)\lambda^{\frac{\alpha}{k}}\Big|\Upsilon'(\ell_1+\ell_2-(\frac{\lambda}{2}y_2+\frac{2-\lambda}{2}y_1))\Big|^q\,d\lambda\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1\lambda^{\frac{\alpha}{k}+1}\,d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^1\lambda^{\frac{\alpha}{k}+1}\Big|\Upsilon'(\ell_1+\ell_2-(\frac{\lambda}{2}y_2+\frac{2-\lambda}{2}y_1))\Big|^q\,d\lambda\Big)^{\frac{1}{q}}\Big\}. \tag{4.2}$$

It is easy to see that

$$\int_0^1(1-\lambda)\;\lambda^{\frac{\alpha}{k}}\,d\lambda=\frac{1}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} \tag{4.3}$$

$$\int_0^1(1-\lambda)\,\lambda^{\frac{\alpha}{k}}\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)\right|^q\,d\lambda$$
$$\leq\frac{(|\Upsilon'(\ell_1)|^q+|\Upsilon'(\ell_2)|^q)}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)}-\left(\frac{|\Upsilon'(y_1)|^q}{2\left(\frac{\alpha}{k}+2\right)\left(\frac{\alpha}{k}+3\right)}+\frac{|\Upsilon'(y_2)|^q}{2\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)\left(\frac{\alpha}{k}+3\right)}\right) \tag{4.4}$$

$$\int_0^1\lambda^{\frac{\alpha}{k}+1}\,d\lambda=\frac{1}{\left(\frac{\alpha}{k}+2\right)} \tag{4.5}$$

$$\int_0^1\lambda^{\frac{\alpha}{k}+1}\;\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)\right|^q\,d\lambda$$

$$\leq \frac{|\Upsilon'(\ell_1)|^q + |\Upsilon'(\ell_2)|^q}{\left(\frac{\alpha}{k}+2\right)} - \left(\frac{|\Upsilon'(y_1)|^q}{2\left(\frac{\alpha}{k}+3\right)} + \frac{|\Upsilon'(y_2)|^q}{2\left(\frac{\alpha}{k}+2\right)\left(\frac{\alpha}{k}+3\right)}\right) \tag{4.6}$$

$$\int_0^1 (1-\lambda)\lambda^{\frac{\alpha}{k}} \left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\right|^q d\lambda$$
$$\leq \frac{|\Upsilon'(\ell_1)|^q + |\Upsilon'(\ell_2)|^q}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} - \left(\frac{\left(\frac{\alpha}{k}+5\right)|\Upsilon'(y_1)|^q}{2\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)\left(\frac{\alpha}{k}+3\right)} + \frac{|\Upsilon'(y_2)|^q}{2\left(\frac{\alpha}{k}+2\right)\left(\frac{\alpha}{k}+3\right)}\right) \tag{4.7}$$

$$\int_0^1 \lambda^{\frac{\alpha}{k}+1} \left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\right|^q d\lambda$$
$$\leq \frac{|\Upsilon'(\ell_1)|^q + |\Upsilon'(\ell_2)|^q}{\left(\frac{\alpha}{k}+2\right)} - \left(\frac{\left(\frac{\alpha}{k}+4\right)|\Upsilon'(y_1)|^q}{2\left(\frac{\alpha}{k}+2\right)\left(\frac{\alpha}{k}+3\right)} + \frac{|\Upsilon'(y_2)|^q}{2\left(\frac{\alpha}{k}+3\right)}\right). \tag{4.8}$$

By substituting (4.3)–(4.8) in (4.2), we obtain the desired inequality (4.1). □

Corollary 12. *Choosing $\Psi(\gamma) = \gamma$ in Theorem 4.1, we get the following inequality*

$$\Bigg|\Upsilon\left(\ell_1+\ell_2-\frac{y_1+y_2}{2}\right) - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}$$
$$\times\left[\left({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\Upsilon(\ell_1+\ell_2-y_1) + \left({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\Upsilon(\ell_1+\ell_2-y_2)\right]\Bigg|$$

$$\leq \frac{(y_2-y_1)}{4}\Bigg[\Bigg\{\Big(\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{\left|\Upsilon'(\ell_1)\right|^q + \left|\Upsilon'(\ell_2)\right|^q}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)} - \Big(\frac{\left|\Upsilon'(y_1)\right|^q}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}$$
$$+\frac{(\frac{\alpha}{k}+5)\left|\Upsilon'(y_2)\right|^q}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}\Big)\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{1}{(\frac{\alpha}{k}+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{\left|\Upsilon'(\ell_1)\right|^q + \left|\Upsilon'(\ell_2)\right|^q}{(\frac{\alpha}{k}+2)} - \Big(\frac{\left|\Upsilon'(y_1)\right|^q}{2(\frac{\alpha}{k}+3)} + \frac{\left|\Upsilon'(y_2)\right|^q}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}\Big)\Big)^{\frac{1}{q}}\Bigg\}$$
$$+\Bigg\{\Big(\frac{1}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{\left|\Upsilon'(\ell_1)\right|^q + \left|\Upsilon'(\ell_2)\right|^q}{(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)} - \Big(\frac{(\frac{\alpha}{k}+5)\left|\Upsilon'(y_1)\right|^q}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}$$
$$+\frac{\left|\Upsilon'(y_2)\right|^q}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)}\Big)\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{1}{(\frac{\alpha}{k}+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{\left|\Upsilon'(\ell_1)\right|^q + \left|\Upsilon'(\ell_2)\right|^q}{(\frac{\alpha}{k}+2)} - \Big(\frac{(\frac{\alpha}{k}+4)\left|\Upsilon'(y_1)\right|^q}{2(\frac{\alpha}{k}+2)(\frac{\alpha}{k}+3)} + \frac{\left|\Upsilon'(y_2)\right|^q}{2(\frac{\alpha}{k}+3)}\Big)\Big)^{\frac{1}{q}}\Bigg\}\Bigg].$$

Theorem 4.2. *Let* (A_1) *holds. Also suppose that* $|\Upsilon'|^q$ *is a convex function on* $[\ell_1, \ell_2]$*, then for* $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *the following fractional integral inequalities holds:*

$$\Bigg|\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right) - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha + k)}{(y_2 - y_1)^{\frac{\alpha}{k}}}\Bigg[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\left(\Upsilon \circ \Psi)(\Psi^{-1}(\ell_1 + \ell_2 - y_1)\right)$$
$$+ \left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\left(\Upsilon \circ \Psi)(\Psi^{-1}(\ell_1 + \ell_2 - y_2)\right)\Bigg]\Bigg|$$

$$\le \frac{(y_2 - y_1)}{4}\Bigg[\Bigg\{\Big(\frac{1}{(\frac{\alpha p}{k} + 1)(\frac{\alpha p}{k} + 2)}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}(\left|\Upsilon'(\ell_1)\right|^q + \left|\Upsilon'(\ell_2)\right|^q)$$
$$-\Big(\frac{1}{12}\left|\Upsilon'(y_1)\right|^q + \frac{5}{12}\left|\Upsilon'(y_2)\right|^q\Big)\Big)^{\frac{1}{q}} + \Big(\frac{1}{(\frac{\alpha p}{k} + 2)}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}(\left|\Upsilon'(\ell_1)\right|^q + \left|\Upsilon'(\ell_2)\right|^q)$$
$$-(\frac{1}{6}\left|\Upsilon'(y_1)\right|^q + \frac{1}{3}\left|\Upsilon'(y_2)\right|^q)\Big)^{\frac{1}{q}}\Bigg\} + \Bigg\{\Big(\frac{1}{(\frac{\alpha p}{k} + 1)(\frac{\alpha p}{k} + 2)}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}(\left|\Upsilon'(\ell_1)\right|^q$$
$$+\left|\Upsilon'(\ell_2)\right|^q) - \Big(\frac{5}{12}\left|\Upsilon'(y_1)\right|^q + \frac{1}{12}\left|\Upsilon'(y_2)\right|^q\Big)\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{1}{\frac{\alpha p}{k} + 2}\Big)^{\frac{1}{p}}$$
$$\times\Big(\frac{1}{2}(\left|\Upsilon'(\ell_1)\right|^q + \left|\Upsilon'(\ell_2)\right|^q) - \Big(\frac{1}{3}\left|\Upsilon'(y_1)\right|^q + \frac{1}{6}\left|\Upsilon'(y_2)\right|^q\Big)\Big)^{\frac{1}{q}}\Bigg\}\Bigg] \tag{4.9}$$

for all $y_1, y_2 \in [\ell_1, \ell_2]$.

Proof. By using Lemma 3.6, Jensen–Mercer inequality, applying the Hölder- İşcan integral inequality (see [10]) and properties of modulus, we have

$$\Bigg|\Upsilon\left(\ell_1 + \ell_2 - \frac{y_1 + y_2}{2}\right) - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha + k)}{(y_2 - y_1)^{\frac{\alpha}{k}}}$$
$$\times\Bigg[\left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\right)\left(\Upsilon \circ \Psi)(\Psi^{-1}(\ell_1 + \ell_2 - y_1)\right)$$
$$+ \left({}_kI^{\alpha:\Psi}_{\Psi^{-1}(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\right)\left(\Upsilon \circ \Psi)(\Psi^{-1}(\ell_1 + \ell_2 - y_2)\right)\Bigg]\Bigg|$$

$$\le \frac{(y_2 - y_1)}{4}\Bigg\{\Big(\int_0^1 (1 - \lambda)\lambda^{\frac{\alpha}{k}p}\, d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1 (1 - \lambda)\left|\Upsilon'(\ell_1 + \ell_2 - (\frac{\lambda}{2}y_1 + \frac{2 - \lambda}{2}y_2))\right|^q d\lambda\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 \lambda^{\frac{\alpha}{k}p+1}\, d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1 \lambda\left|\Upsilon'(\ell_1 + \ell_2 - (\frac{\lambda}{2}y_1 + \frac{2 - \lambda}{2}y_2))\right|^q d\lambda\Big)^{\frac{1}{q}}\Bigg\}$$
$$+ \Bigg\{\Big(\int_0^1 (1 - \lambda)\lambda^{\frac{\alpha p}{k}}\, d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1 (1 - \lambda)\left|\Upsilon'(\ell_1 + \ell_2 - (\frac{2 - \lambda}{2}y_1 + \frac{t}{2}y_2))\right|^q d\lambda\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 \lambda^{\frac{\alpha p}{k}+1}\, d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1 \lambda\left|\Upsilon'(\ell_1 + \ell_2 - (\frac{2 - \lambda}{2}y_1 + \frac{\lambda}{2}y_2))\right|^q d\lambda\Big)^{\frac{1}{q}}\Bigg\}. \tag{4.10}$$

By the convexity of $|\Upsilon'|^q$, we get

$$\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)\right|^q$$

$$\leq \left|\Upsilon'(\ell_1)\right|^q+\left|\Upsilon'(\ell_2)\right|^q-\left(\frac{\lambda}{2}\left|\Upsilon'(y_1)\right|^q+\frac{2-\lambda}{2}\left|\Upsilon'(y_2)\right|^q\right). \tag{4.11}$$

It is easy to see that

$$\int_0^1 (1-\lambda)\ \lambda^{\frac{\alpha p}{k}}\ d\lambda = \frac{1}{\left(\frac{\alpha p}{k}+1\right)\left(\frac{\alpha p}{k}+2\right)}; \tag{4.12}$$

$$\int_0^1 (1-\lambda)\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)\right|^q d\lambda$$
$$\leq \frac{1}{2}\left(\left|\Upsilon'(\ell_1)\right|^q+\left|\Upsilon'(\ell_2)\right|^q\right)-\left(\frac{1}{12}\left|\Upsilon'(y_1)\right|^q+\frac{5}{12}\left|\Upsilon'(y_2)\right|^q\right); \tag{4.13}$$

$$\int_0^1 \lambda^{\frac{\alpha p}{k}+1}\ d\lambda = \frac{1}{\left(\frac{\alpha p}{k}+2\right)}; \tag{4.14}$$

$$\int_0^1 \lambda\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{\lambda}{2}y_1+\frac{2-\lambda}{2}y_2\right)\right)\right|^q d\lambda$$
$$\leq \frac{1}{2}\left(\left|\Upsilon'(\ell_1)\right|^q+\left|\Upsilon'(\ell_2)\right|^q\right)-\left(\frac{1}{6}\left|\Upsilon'(y_1)\right|^q+\frac{1}{3}\left|\Upsilon'(y_2)\right|^q\right); \tag{4.15}$$

$$\int_0^1 (1-\lambda)\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\right|^q d\lambda$$
$$\leq \frac{1}{2}\left(\left|\Upsilon'(\ell_1)\right|^q+\left|\Upsilon'(\ell_2)\right|^q\right)-\left(\frac{5}{12}\left|\Upsilon'(y_1)\right|^q+\frac{1}{12}\left|\Upsilon'(y_2)\right|^q\right) \tag{4.16}$$

and

$$\int_0^1 \lambda\left|\Upsilon'\left(\ell_1+\ell_2-\left(\frac{2-\lambda}{2}y_1+\frac{\lambda}{2}y_2\right)\right)\right|^q d\lambda$$
$$\leq \frac{1}{2}\left(\left|\Upsilon'(\ell_1)\right|^q+\left|\Upsilon'(\ell_2)\right|^q\right)-\left(\frac{1}{3}\left|\Upsilon'(y_1)\right|^q+\frac{1}{6}\left|\Upsilon'(y_2)\right|^q\right). \tag{4.17}$$

By substituting (4.11)–(4.17) in (4.10), we obtain the required inequality (4.9). □

Corollary 13. *Choosing $\Psi(\gamma)=\gamma$ in Theorem 4.2, we obtain the following inequality*

$$\begin{aligned}&\Big|\Upsilon\Big(\ell_1+\ell_2-\frac{y_1+y_2}{2}\Big)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y_2-y_1)^{\frac{\alpha}{k}}}\\&\times\Big[\Big({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^+}\Big)\Upsilon(\ell_1+\ell_2-y_1)+\Big({}_kJ^{\alpha}_{(\ell_1+\ell_2-\frac{y_1+y_2}{2})^-}\Big)\Upsilon(\ell_1+\ell_2-y_2)\Big]\Big|\end{aligned}$$

$$\begin{aligned}&\leq\frac{(y_2-y_1)}{4}\Big[\Big\{\Big(\frac{1}{(\frac{\alpha p}{k}+1)(\frac{\alpha p}{k}+2)}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}(|\Upsilon'(\ell_1)|^q+|\Upsilon'(\ell_2)|^q)\\&-\Big(\frac{1}{12}|\Upsilon'(y_1)|^q+\frac{5}{12}|\Upsilon'(y_2)|^q\Big)\Big)^{\frac{1}{q}}+\Big(\frac{1}{(\frac{\alpha p}{k}+2)}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}(|\Upsilon'(\ell_1)|^q+|\Upsilon'(\ell_2)|^q)\\&-(\frac{1}{6}|\Upsilon'(y_1)|^q+\frac{1}{3}|\Upsilon'(y_2)|^q)\Big)^{\frac{1}{q}}\Big\}+\Big\{\Big(\frac{1}{(\frac{\alpha p}{k}+1)(\frac{\alpha p}{k}+2)}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}(|\Upsilon'(\ell_1)|^q\\&+|\Upsilon'(\ell_2)|^q)-\Big(\frac{5}{12}|\Upsilon'(y_1)|^q+\frac{1}{12}|\Upsilon'(y_2)|^q\Big)\Big)^{\frac{1}{q}}\\&+\Big(\frac{1}{\frac{\alpha p}{k}+2}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}(|\Upsilon'(\ell_1)|^q+|\Upsilon'(\ell_2)|^q)-\Big(\frac{1}{3}|\Upsilon'(y_1)|^q+\frac{1}{6}|\Upsilon'(y_2)|^q\Big)\Big)^{\frac{1}{q}}\Big\}\Big].\end{aligned}$$

5. Conclusion

In this article, authors obtained some Hermite–Jensen–Mercer type inequalities using ψ–Riemann–Liouville k–Fractional integrals and several ψ–Riemann–Liouville k–Fractional integral inequalities are provided as well. Some known results are recaptured as special cases of our results. We hope that our new idea and technique may inspired many researcher in this fascinating field.

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Conflict of interest

The authors declare no conflict of interest.

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