Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 5157–5170.
DOI:10.3934/math.2020331
Received: 03 December 2019
Accepted: 25 May 2020
Published: 15 June 2020

Research article

Majorization problem for two subclasses of meromorphic functions associated with a convolution operator

Akhter Rasheed[1], **Saqib Hussain**[1,*], **Syed Ghoos Ali Shah**[1], **Maslina Darus**[2] **and Saeed Lodhi**[3]

[1] Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Pakistan
[2] Department of Mathematical Sciences, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia
[3] Department of Management Sciences, COMSATS University Islamabad, Abbottabad Campus, Pakistan

* **Correspondence:** Email: saqib_math@yahoo.com.

Abstract: In the present paper, we investigate a majorization problem for the class $M_{\alpha,\beta}^{\nu,j}(\eta,\varkappa;A,B)$ of meromorphic functions and the class $N_{\alpha,\beta}^{\nu,j}(\theta,b;A,B)$ of meromorphic spirllike functions related with a convolution operator. We extend the results existing in literature for higher order derivative. Several consequences of the main results in the form of corollaries are also pointed out.

Keywords: meromorphic function; subordination and majorization
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

Let $\sum$ denote the class of meromorphic function of the form:

$$\lambda(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} a_t \omega^t, \tag{1.1}$$

which are analytic in the punctured open unit disc $U^* = \{\omega : \omega \in \mathbb{C} \text{ and } 0 < |\omega| < 1\} = U - \{0\}$, where $U = U^* \cup \{0\}$. Let $\delta(\omega) \in \sum$, be given by

$$\delta(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} b_t \omega^t, \tag{1.2}$$

then the Convolution (Hadamard product) of $\lambda(\omega)$ and $\delta(\omega)$ is denoted and defined as:

$$(\lambda * \delta)(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} a_t b_t \omega^t.$$

In 1967, MacGregor [17] introduced the concept of majorization as follows.

Definition 1. *Let λ and δ be analytic in U^*. We say that λ is majorized by δ in U^* and written as $\lambda(\omega) \ll \delta(\omega)$ $\omega \in U^*$, if there exists a function $\varphi(\omega)$, analytic in U^*, satisfying*

$$|\varphi(\omega)| \leq 1, \quad and \quad \lambda(\omega) = \varphi(\omega)\delta(\omega), \quad \omega \in U^*. \tag{1.3}$$

In 1970, Robertson [19] gave the idea of quasi-subordination as:

Definition 2. *A function $\lambda(\omega)$ is subordinate to $\delta(\omega)$ in U and written as: $\lambda(\omega) \prec \delta(\omega)$, if there exists a Schwarz function $k(\omega)$, which is holomorphic in U^* with $|k(\omega)| < 1$, such that $\lambda(\omega) = \delta(k(\omega))$. Furthermore, if the function $\delta(\omega)$ is univalent in U^*, then we have the following equivalence (see [16]):*

$$\lambda(\omega) \prec \delta(\omega) \text{ and } \lambda(U) \subset \delta(U). \tag{1.4}$$

Further, $\lambda(\omega)$ is quasi-subordinate to $\delta(\omega)$ in U^* and written is

$$\lambda(\omega) \prec_q \delta(\omega) \quad (\omega \in U^*),$$

if there exist two analytic functions $\varphi(\omega)$ and $k(\omega)$ in U^* such that $\frac{\lambda(\omega)}{\varphi(\omega)}$ is analytic in U^* and

$$|\varphi(\omega)| \leq 1 \text{ and } k(\omega) \leq |\omega| < 1 \quad \omega \in U^*,$$

satisfying

$$\lambda(\omega) = \varphi(\omega)\delta(k(\omega)) \quad \omega \in U^*. \tag{1.5}$$

(i) For $\varphi(\omega) = 1$ in (1.5), we have

$$\lambda(\omega) = \delta(k(\omega)) \quad \omega \in U^*,$$

and we say that the λ function is subordinate to δ in U^*, denoted by (see [20])

$$\lambda(\omega) \prec \delta(\omega) \quad (\omega \in U^*).$$

(ii) If $k(\omega) = \omega$, the quasi-subordination (1.5) becomes the majorization given in (1.3). For related work on majorization see [1,4,9,21].
Let us consider the second order linear homogenous differential equation (see, Baricz [6]):

$$\omega^2 k''(\omega) + \alpha\omega k'(\omega) + \left[\beta\omega^2 - \nu^2 + (1-\alpha)\right] k(\omega) = 0. \tag{1.6}$$

The function $k_{\nu,\alpha,\beta}(\omega)$, is known as generalized Bessel's function of first kind and is the solution of differential equation given in (1.6)

$$k_{\nu,\alpha,\beta}(\omega) = \sum_{t=0}^{\infty} \frac{(-\beta)^t}{\Gamma(t+1)\Gamma(t+\nu+1+\frac{\alpha+1}{2})} \left(\frac{\omega}{2}\right)^{2t+\nu}. \tag{1.7}$$

Let us denote

$$\begin{aligned}\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega) &= \frac{2^{\nu}\Gamma(\nu+\frac{\alpha+1}{2})}{\omega^{\frac{\nu}{2}+1}}k_{\nu,\alpha,\beta}(\omega^{\frac{1}{2}}),\\ &= \frac{1}{\omega}+\sum_{t=0}^{\infty}\frac{(-\beta)^{t+1}\Gamma(\nu+\frac{\alpha+1}{2})}{4^{t+1}\Gamma(t+2)\Gamma(t+\nu+1+\frac{\alpha+1}{2})}(\omega)^{t},\end{aligned}$$

where ν, α and β are positive real numbers. The operator $\mathcal{L}_{\nu,\alpha,\beta}$ is a variation of the operator introduced by Deniz [7] (see also Baricz et al. [5]) for analytic functions. By using the convolution, we define the operator $\mathcal{L}_{\nu,\alpha,\beta}$ as follows:

$$\begin{aligned}(\mathcal{L}_{\nu,\alpha,\beta}\lambda)(\omega) &= \mathcal{L}_{\nu,\alpha,\beta}(\omega)*\lambda(\omega),\\ &= \frac{1}{\omega}+\sum_{t=0}^{\infty}\frac{(-\beta)^{t+1}\Gamma(\nu+\frac{\alpha+1}{2})}{4^{t+1}\Gamma(t+2)\Gamma(t+\nu+1+\frac{\alpha+1}{2})}a_{t}(\omega)^{t}. \qquad (1.8)\end{aligned}$$

The operator $\mathcal{L}_{\nu,\alpha,\beta}$ was introduced and studied by Mostafa et al. [15] (see also [2]). From (1.8), we have

$$\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}=\left(\nu-1+\frac{\alpha+1}{2}\right)\left(\mathcal{L}_{\nu-1,\alpha,\beta}\lambda(\omega)\right)^{j}-\left(\nu+\frac{\alpha+1}{2}\right)\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}. \qquad (1.9)$$

By taking $\alpha=\beta=1$, the above operator reduces to $(\mathcal{L}_{\nu}\lambda)(\omega)$ studied by Aouf et al. [2].

Definition 3. *Let $-1\leq B<A\leq 1$, $\eta\in\mathbb{C}-\{0\}$, $j\in W$ and $\nu,\alpha,\beta>0$. A function $\lambda\in\sum$ is said to be in the class $M^{\nu,j}_{\alpha,\beta}(\eta,\varkappa;A,B)$ of meromorphic functions of complex order $\eta\neq 0$ in U^{*} if and only if*

$$1-\frac{1}{\eta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}}+\nu+j\right)-\varkappa\left|-\frac{1}{\eta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}}+\nu+j\right)\right|\prec\frac{1+A\omega}{1+B\omega}. \qquad (1.10)$$

Remark 1.

(i). For $A=1$, $B=-1$ and $\varkappa=0$, we denote the class

$$M^{\nu,j}_{\alpha,\beta}(\eta,0;1,-1)=M^{\nu,j}_{\alpha,\beta}(\eta).$$

So, $\lambda\in M^{\nu,j}_{\alpha,\beta}(\eta,\varkappa;A,B)$ if and only if

$$\Re\left[1-\frac{1}{\eta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}}+\nu+j\right)\right]>0.$$

(ii). For $\alpha=1,\beta=1$, $M^{\nu,j}_{1,1}(\eta,0;1,-1)$ reduces to the class $M^{\nu,j}(\eta)$.

$$\Re\left[1-\frac{1}{\eta}\left(\frac{\omega\left(\mathcal{L}_{\nu}\lambda(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu}\lambda(\omega)\right)^{j}}+\nu+j\right)\right]>0.$$

Definition 4. *A function $\lambda \in \sum$ is said to be in the class $N^{\nu,j}_{\alpha,\beta}(\theta, b; A, B)$ of meromorphic spirllike functions of complex order $b \neq 0$ in U^*, if and only if*

$$1 - \frac{e^{i\theta}}{b\cos\theta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}} + j + 1\right) \prec \frac{1 + A\omega}{1 + B\omega}, \tag{1.11}$$

where,

$$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \ -1 \leq \beta < A \leq 1, \eta \in \mathbb{C} - \{0\}, \ j \in W, \ \nu, \alpha, \beta > 0 \text{ and } \omega \in U^*\right).$$

(i). For $A = 1$ and $B = -1$, we set

$$N^{\nu,j}_{\alpha,\beta}(\theta, b; 1, -1) = N^{\nu,j}_{\alpha,\beta}(\theta, b),$$

where $N^{\nu,j}_{\alpha,\beta}(\theta, b)$ denote the class of functions $\lambda \in \sum$ satisfying the following inequality:

$$\Re\left[\frac{e^{i\theta}}{b\cos\theta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}} + j + 1\right)\right] < 1.$$

(ii). For $\theta = 0$ and $\alpha = \beta = 1$ we write

$$N^{\nu,j}_{1,1}(0, b; 1, -1) = N^{\nu,j}(b),$$

where $N^{\nu,j}(b)$ denote the class of functions $\lambda \in \sum$ satisfying the following inequality:

$$\Re\left[\frac{1}{b}\left(\frac{\omega\left(\mathcal{L}_{\nu}\lambda(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu}\lambda(\omega)\right)^{j}} + j + 1\right)\right] < 1.$$

A majorization problem for the normalized class of starlike functions has been examined by MacGregor [17] and Altintas et al. [3,4]. Recently, Eljamal et al. [8], Goyal et al. [12,13], Goswami et al. [10,11], Li et al. [14], Tang et al. [21,22] and Prajapat and Aouf [18] generalized these results for different classes of analytic functions.

The objective of this paper is to examined the problems of majorization for the classes $M^{\nu,j}_{\alpha,\beta}(\eta, \varkappa; A, B)$ and $N^{\nu,j}_{\alpha,\beta}(\theta, b; A, B)$.

2. Majorization problem for the class $M^{\nu,j}_{\alpha,\beta}(\eta, \varkappa; A, B)$

In Theorem 1, we prove majorization property for the class $M^{\nu,j}_{\alpha,\beta}(\eta, \varkappa; A, B)$.

Theorem 1. *Let the function $\lambda \in \sum$ and suppose that $\delta \in M^{\nu,j}_{\alpha,\beta}(\eta, \varkappa; A, B)$. If $\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}$ is majorized by $\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}$ in U^*, then*

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}\right| \leq \left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}\right|, \qquad (|\omega| < r_0), \tag{2.1}$$

where $r_0 = r_0(\eta, \varkappa, \nu, \alpha, \beta, A, B)$ *is the smallest positive roots of the equation*

$$\begin{aligned} & -\rho\left(\nu-1+\frac{\alpha+1}{2}\right)\left[\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)|B|\right]r^3-\left(\nu-1+\frac{\alpha+1}{2}\right)\left[\rho\left(\frac{\alpha+1}{2}\right)+\rho^2|B|-|B|\right]r^2 \\ & -\left(\nu-1+\frac{\alpha+1}{2}\right)\left[\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)|B|+\rho^2|B|-1\right]r+ \\ & \rho\left(\nu-1+\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}\right) \\ = & \ 0. \end{aligned} \tag{2.2}$$

Proof. Since $\delta \in M_{\alpha,\beta}^{\nu,j}(\eta, \varkappa; A, B)$, we have

$$1-\frac{1}{\eta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}}+\nu+j\right)-\varkappa\left|-\frac{1}{\eta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}}+\nu+j\right)\right|=\frac{1+Ak(\omega)}{1+Bk(\omega)}, \tag{2.3}$$

where $k(\omega) = c_1\omega + c_2\omega^2 + ...$, is analytic and bounded functions in U^* with

$$|k(\omega)| \le |\omega| \quad (\omega \in U^*). \tag{2.4}$$

Taking

$$\S(\omega)=1-\frac{1}{\eta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}}+\nu+j\right), \tag{2.5}$$

In (2.3), we have

$$\S(\omega)-\varkappa|\S(\omega)-1|=\frac{1+Ak(\omega)}{1+Bk(\omega)},$$

which implies

$$\S(\omega)=\frac{1+\left(\frac{A-B\varkappa e^{-i\theta}}{1-\varkappa e^{-i\theta}}\right)k(\omega)}{1+Bk(\omega)}. \tag{2.6}$$

Using (2.6) in (2.5), we get

$$\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}}=-\frac{\nu+j+\left[\frac{(A-B)\eta}{1-\varkappa e^{-i\theta}}+(\nu+j)B\right]k(\omega)}{1+Bk(\omega)}. \tag{2.7}$$

Application of Leibnitz's Theorem on (1.9) gives

$$\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}=\left(\nu-1+\frac{\alpha+1}{2}\right)\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}-\left(\nu+j+\frac{\alpha+1}{2}\right)\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}. \tag{2.8}$$

By using (2.8) in (2.7) and making simple calculations, we have

$$\frac{\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}}=\frac{\frac{\alpha+1}{2}-\left[\frac{(A-B)\eta}{1-\varkappa e^{-i\theta}}-\left(\frac{\alpha+1}{2}\right)B\right]k(\omega)}{(1+Bk(\omega))\left(\nu-1+\frac{\alpha+1}{2}\right)}. \tag{2.9}$$

Or, equivalently

$$\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}=\frac{(1+Bk(\omega))\left(\nu-1+\frac{\alpha+1}{2}\right)}{\frac{\alpha+1}{2}-\left[\frac{(A-B)\eta}{1-\varkappa e^{-i\theta}}-\left(\frac{\alpha+1}{2}\right)B\right]k(\omega)}\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}. \tag{2.10}$$

Since $|k(\omega)|\leq|\omega|$, (2.10) gives us

$$\begin{aligned}\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}\right| &\leq \frac{[1+|B|\,|\omega|]\left(\nu-1+\frac{\alpha+1}{2}\right)}{\frac{\alpha+1}{2}-\left|\frac{(A-B)\eta}{1-\varkappa e^{-i\theta}}-\left(\frac{\alpha+1}{2}\right)B\right||\omega|}\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}\right| \\ &\leq \frac{[1+|B|\,|\omega|]\left(\nu-1+\frac{\alpha+1}{2}\right)}{\frac{\alpha+1}{2}-\left[\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)|B|\right]|\omega|}\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}\right|. \end{aligned} \tag{2.11}$$

Since $\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}$ is majorized by $\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}$ in U^{*}. So from (1.3), we have

$$\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}=\varphi(\omega)\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}. \tag{2.12}$$

Differentiating (2.12) with respect to ω then multiplying with ω, we get

$$\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}=\omega\varphi^{'}(\omega)\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}+\omega\varphi(\omega)\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}. \tag{2.13}$$

By using (2.8), (2.12) and (2.13), we have

$$\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}=\frac{1}{\left(\nu-1+\frac{\alpha+1}{2}\right)}\omega\varphi^{'}(\omega)\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}+\varphi(\omega)\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j+1}. \tag{2.14}$$

On the other hand, noticing that the Schwarz function φ satisfies the inequality

$$\left|\varphi^{'}(\omega)\right|\leq\frac{1-|\varphi(\omega)|^{2}}{1-|\omega|^{2}}\qquad(\omega\in U^{*}). \tag{2.15}$$

Using (2.8) and (2.15) in (2.14), we get

$$\begin{aligned}\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}\right| &\leq \left[|\varphi(\omega)|+\frac{\omega\left(1-|\varphi(\omega)|^{2}\right)[1+|B|\,|\omega|]\left(\nu-1+\frac{\alpha+1}{2}\right)}{\left(\nu-1+\frac{\alpha+1}{2}\right)\left(1-|\omega|^{2}\right)\left(\frac{\alpha+1}{2}-\left[\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)B\right]|\omega|\right)}\right] \\ &\quad\times\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}\right|,\end{aligned}$$

By taking

$$|\omega|=r,\quad|\varphi(\omega)|=\rho\qquad(0\leq\rho\leq1),$$

reduces to the inequality

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}\right|\leq\frac{\Phi_{1}(\rho)}{\left(\nu-1+\frac{\alpha+1}{2}\right)(1-r^{2})\left(\frac{\alpha+1}{2}-\left[\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)B\right]r\right)}\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}\right|,$$

where

$$\Phi_1(\rho) = \begin{bmatrix} \rho\left(\nu-1+\frac{\alpha+1}{2}\right)\left(1-r^2\right)\left(\frac{\alpha+1}{2}-\left[\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)B\right]r\right) \\ +r\left(1-\rho^2\right)[1+|B|r]\left(\nu-1+\frac{\alpha+1}{2}\right) \end{bmatrix}$$

$$= -r[1+|B|r]\left(\nu-1+\frac{\alpha+1}{2}\right)\rho^2+\rho\left(\nu-1+\frac{\alpha+1}{2}\right)\left(1-r^2\right)$$
$$\left(\frac{\alpha+1}{2}-\left[\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)B\right]r\right)+r[1+|B|r]\left(\nu-1+\frac{\alpha+1}{2}\right), \tag{2.16}$$

takes in maximum value at $\rho = 1$ with $r_0 = r_0(\eta, \varkappa, \nu, \alpha, \beta, A, B)$ where r_0 is the least positive root of the (2.2). Furthermore, if $0 \leq \xi_0 \leq r_0(\eta, \varkappa, \nu, \alpha, \beta, A, B)$, then the function $\psi_1(\rho)$ defined by

$$\psi_1(\rho) = -\xi_0[1+|B|\xi_0]\left(\nu-1+\frac{\alpha+1}{2}\right)\rho^2+\rho\left(\nu-1+\frac{\alpha+1}{2}\right)\left(1-\xi_0^2\right)$$
$$\left(\frac{\alpha+1}{2}-\left[\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)B\right]\xi_0\right)+\xi_0[1+|B|\xi_0]\left(\nu-1+\frac{\alpha+1}{2}\right), \tag{2.17}$$

is an increasing function on the interval $(0 \leq \rho \leq 1)$, so that

$$\psi_1(\rho) \leq \psi_1(1) = \left(\nu-1+\frac{\alpha+1}{2}\right)\left(1-\xi_0^2\right)\left[\frac{\alpha+1}{2}-\left(\frac{(A-B)|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)B\right)\xi_0\right]$$
$$(0 \leq \rho \leq 1,\ 0 \leq \xi_0 \leq r_0(\eta, \varkappa, A, B)).$$

Hence, upon setting $\rho = 1$ in (2.17), we achieve (2.1). □

Special Cases: Let $A = 1$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let the function $\lambda \in \sum$ and suppose that $\delta \in M^{\nu,j}_{\alpha,\beta}(\eta, \varkappa; A, B)$. If $\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^j$ is majorized by $\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^j$ in U^*, then*

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}\right| \leq \left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}\right|, \quad (|\omega| < r_1),$$

where $r_1 = r_1(\eta, \varkappa, \nu, \alpha, \beta)$ is the least positive roots of the equation

$$\rho\left(\nu-1+\frac{\alpha+1}{2}\right)\left[\frac{2|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)\right]r^3-\left(\nu-1+\frac{\alpha+1}{2}\right)\left[\rho\left(\frac{\alpha+1}{2}\right)+\rho^2-1\right]r^2-$$
$$\left(\nu-1+\frac{\alpha+1}{2}\right)\left[\rho\left\{\frac{2|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)\right\}+\rho^2-1\right]r+\rho\left(\nu-1+\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}\right)$$
$$= 0. \tag{2.18}$$

Here, $r = -1$ is one of the roots (2.18) and the other roots are given by

$$r_1 = \frac{k_0-\sqrt{k_0^2-4\rho^2\left(\nu-1+\frac{\alpha+1}{2}\right)\left[\frac{2|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)\right]\left(\nu-1+\frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}\right)}}{2\rho\left(\nu-1+\frac{\alpha+1}{2}\right)\left[\frac{2|\eta|}{1-\varkappa}-\left(\frac{\alpha+1}{2}\right)\right]},$$

where

$$k_0 = \left(\nu-1+\frac{\alpha+1}{2}\right)\left[\rho\left\{\frac{2|\eta|}{1-\varkappa}-2\left(\frac{\alpha+1}{2}\right)\right\}+\rho^2-1\right].$$

Taking $\varkappa = 0$ in corollary 1, we state the following:

Corollary 2. *Let the function* $\lambda \in \sum$ *and suppose that* $\delta \in M^{\nu,j}_{\alpha,\beta}(\eta, \varkappa; A, B)$. *If* $\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^j$ *is majorized by* $\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^j$ *in* U^*, *then*

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}\right| \leq \left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}\right|, \qquad (|\omega| < r_2),$$

where $r_2 = r_2(\eta, \nu, \alpha, \beta)$ *is the lowest positive roots of the equation*

$$\begin{aligned} &\rho\left(\nu - 1 + \frac{\alpha+1}{2}\right)\left[2|\eta| - \left(\frac{\alpha+1}{2}\right)\right]r^3 - \left(\nu - 1 + \frac{\alpha+1}{2}\right)\left[\rho\left(\frac{\alpha+1}{2}\right) + \rho^2 - 1\right]r^2 - \\ &\left(\nu - 1 + \frac{\alpha+1}{2}\right)\left[\rho\left\{2|\eta| - \left(\frac{\alpha+1}{2}\right)\right\} + \rho^2 - 1\right]r + \rho\left(\nu - 1 + \frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}\right) \\ = \;& 0, \end{aligned} \tag{2.19}$$

given by

$$r_2 = \frac{k_1 - \sqrt{k_1^2 - 4\rho^2\left(\nu - 1 + \frac{\alpha+1}{2}\right)\left[2|\eta| - \left(\frac{\alpha+1}{2}\right)\right]\left(\nu - 1 + \frac{\alpha+1}{2}\right)\left(\frac{\alpha+1}{2}\right)}}{2\rho\left(\nu - 1 + \frac{\alpha+1}{2}\right)\left[2|\eta| - \left(\frac{\alpha+1}{2}\right)\right]},$$

where

$$k_1 = \left(\nu - 1 + \frac{\alpha+1}{2}\right)\left[\rho\left\{2|\eta| - 2\left(\frac{\alpha+1}{2}\right)\right\} + \rho^2 - 1\right].$$

Taking $\alpha = \beta = 1$ in corollary 2, we get the following:

Corollary 3. *Let the function* $\lambda \in \sum$ *and suppose that* $\delta \in M^{\nu,j}_{\alpha,\beta}(\eta, \varkappa; A, B)$. *If* $\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^j$ *is majorized by* $\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^j$ *in* U^*, *then*

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}\right| \leq \left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}\right|, \qquad (|\omega| < r_3),$$

where $r_3 = r_3(\eta, \nu)$ *is the lowest positive roots of the equation*

$$\begin{aligned} &\rho\nu\left[2|\eta| - 1\right]r^3 - \nu\left[\rho + \rho^2 - 1\right]r^2 - \nu\left[\rho\left(2|\eta| - 1\right) + \rho^2 - 1\right]r + \rho\nu \\ = \;& 0, \end{aligned} \tag{2.20}$$

given by

$$r_3 = \frac{k_2 - \sqrt{k_2^2 - 4\rho^2\nu\left[2|\eta| - 1\right]\nu}}{2\rho\nu\left[2|\eta| - 1\right]},$$

where

$$k_2 = \nu\left[\rho\left\{2|\eta| - 2\right\} + \rho^2 - 1\right].$$

3. Majorization problem for the class $N_{\alpha,\beta}^{\nu,j}(\theta, b; A, B)$

Secondly, we exam majorization property for the class $N_{\alpha,\beta}^{\nu,j}(\theta, b; A, B)$.

Theorem 2. *Let the function $\lambda \in \sum$ and suppose that $\delta \in N_{\alpha,\beta}^{\nu,j}(\theta, b; A, B)$. If*

$$\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j} \ll \left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}, \qquad (j \in 0, 1, 2...),$$

then

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}\right| \leq \left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}\right|, \qquad (|\omega| < r_4), \tag{3.1}$$

where $r_4 = r_4(\theta, b, \nu, \alpha, \beta, A, B)$ is the smallest positive roots of the equation

$$\begin{aligned} &-\rho\left[\left|(B-A)\, b\cos\theta + \left(\nu + \frac{\alpha+1}{2} - 1\right)|B|\right|\right] r^3 - \\ &\left[\rho\left\{\nu + \frac{\alpha+1}{2} - 1\right\} - |B|\left(1-\rho^2\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right)\right] r^2 \\ &+\left[\rho\left\{\left|(B-A)\, b\cos\theta + \left(\nu + \frac{\alpha+1}{2} - 1\right)|B|\right|\right\} + \left(1-\rho^2\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right)\right] r \\ &+\rho\left[\nu + \frac{\alpha+1}{2} - 1\right] = 0, \\ &\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2},\ -1 \leq \beta < A \leq 1, \eta \in \mathbb{C} - \{0\},\ \nu, \alpha, \beta > 0,\ \text{and } \omega \in U^{*}\right). \end{aligned} \tag{3.2}$$

Proof. Since $\delta \in N_{\alpha,\beta}^{\nu,j}(\theta, b; A, B)$, so

$$1 - \frac{e^{i\theta}}{b\cos\theta}\left(\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j}} + j + 1\right) = \frac{1 + A\omega}{1 + B\omega}, \tag{3.3}$$

where, $k(\omega)$ is defined as (2.4).
From (3.3), we have

$$\frac{\omega\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}} = \frac{\left[(B-A)\, b\cos\theta - (j+1)\, Be^{i\theta}\right] k(\omega) - (j+1)\, e^{i\theta}}{e^{i\theta}\left(1 + Bk(\omega)\right)}. \tag{3.4}$$

Now, using (2.8) in (3.4) and making simple calculations, we obtain

$$\frac{\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}}{\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}} = \frac{\begin{array}{c}\left[(B-A)\, b\cos\theta + \left(\nu + \frac{\alpha+1}{2} - 1\right) Be^{i\theta}\right] k(\omega) \\ + \left[\left(\nu + j + \frac{\alpha+1}{2}\right) - 1\right] e^{i\theta}\end{array}}{e^{i\theta}\left(1 + Bk(\omega)\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right)}, \tag{3.5}$$

which, in view of $|k(\omega)| \leq |\omega|$ $(\omega \in U^{*})$, immediately yields the following inequality

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}\right| \leq \frac{\left|e^{i\theta}\right|\left(1 + |B|\,|k(\omega)|\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right)}{\begin{array}{c}\left[\left|(B-A)\, b\cos\theta + \left(\nu + \frac{\alpha+1}{2} - 1\right) Be^{i\theta}\right|\right] |k(\omega)| \\ + \left[\left(\nu + \frac{\alpha+1}{2}\right) - 1\right]\left|e^{i\theta}\right|\end{array}}$$

$$\times\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}\right|. \tag{3.6}$$

Now, using (2.15) and (3.6) in (2.14) and working on the similar lines as in Theorem 1, we have

$$\begin{aligned}\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\lambda(\omega)\right)^{j}\right| \leq & \left[|\varphi(\omega)| + \frac{|\omega|\left(1-|\varphi(\omega)|^{2}\right)(1+|B||\omega|)\left(\nu-1+\frac{\alpha+1}{2}\right)}{\left(1-|\omega|^{2}\right)\left[\begin{array}{c}\left\{\left|(B-A)\,b\cos\theta+\left(\nu+\frac{\alpha+1}{2}-1\right)B\right|\right\}|\omega| \\ +\left[\left(\nu+\frac{\alpha+1}{2}\right)-1\right]\end{array}\right]}\right] \\ & \times\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}\right|.\end{aligned}$$

By setting $|\omega| = r, \quad |\varphi(\omega)| = \rho \quad (0 \leq \rho \leq 1)$, leads us to the inequality

$$\begin{aligned}\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\lambda(\omega)\right)^{j}\right| \leq & \left[\frac{\Phi_2(\rho)}{(1-r^{2})\left[\begin{array}{c}\left\{\left|(B-A)\,b\cos\theta+\left(\nu+\frac{\alpha+1}{2}-1\right)B\right|\right\}r \\ +\left(\nu+\frac{\alpha+1}{2}\right)-1\end{array}\right]}\right] \\ & \times\left|\left(\mathcal{L}_{\nu-1,\alpha,\beta}\delta(\omega)\right)^{j}\right|, \end{aligned} \tag{3.7}$$

where the function $\Phi_2(\rho)$ is given by

$$\begin{aligned}\Phi_2(\rho) = & \rho\left(1-r^{2}\right)\left[\begin{array}{c}\left\{\left|(B-A)\,b\cos\theta+\left(\nu+\frac{\alpha+1}{2}-1\right)B\right|\right\}r \\ +\left(\nu+\frac{\alpha+1}{2}\right)-1\end{array}\right] \\ & +r\left(1-\rho^{2}\right)(1+Br)\left(\nu-1+\frac{\alpha+1}{2}\right).\end{aligned}$$

$\Phi_2(\rho)$ its maximum value at $\rho = 1$ with $r_4 = r_4(\theta, b, \nu, \alpha, \beta, A, B)$ given in (3.2). Moreover if $0 \leq \xi_1 \leq r_4(\theta, b, \nu, \alpha, \beta, A, B)$, then the function.

$$\begin{aligned}\psi_2(\rho) = & \rho\left(1-\xi_1^{2}\right)\left[\begin{array}{c}\left\{\left|(B-A)\,b\cos\theta+\left(\nu+\frac{\alpha+1}{2}-1\right)B\right|\right\}\xi_1 \\ +\left(\nu+\frac{\alpha+1}{2}\right)-1\end{array}\right] \\ & +\xi_1\left(1-\rho^{2}\right)(1+B\xi_1)\left(\nu-1+\frac{\alpha+1}{2}\right),\end{aligned}$$

increasing on the interval $0 \leq \rho \leq 1$, so that $\psi_2(\rho)$ does not exceed

$$\psi_2(1) = \left(1-\xi_1^{2}\right)\left[\begin{array}{c}\left\{\left|(B-A)\,b\cos\theta+\left(\nu+\frac{\alpha+1}{2}-1\right)B\right|\right\}\xi_1 \\ +\left(\nu+\frac{\alpha+1}{2}\right)-1\end{array}\right].$$

Therefore, from this fact (3.7) gives the inequality (3.1). We complete the proof. □

Special Cases: Let $A = 1$ and $B = -1$ in Theorem 2, we obtain the following corollary.

Corollary 4. *Let the function* $\lambda \in \sum$ *and suppose that* $\delta \in N^{\nu,j}_{\alpha,\beta}(\theta, b; A, B)$*. If*

$$\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j} \ll \left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}, \qquad (j \in 0, 1, 2, ...),$$

then

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}\right| \le \left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}\right|, \qquad (|\omega| < r_5),$$

where $r_5 = r_5(\theta, b, \nu, \alpha, \beta)$ *is the lowest positive roots of the equation*

$$\begin{aligned}
&-\rho\left[\left|-2b\cos\theta + \left(\nu + \frac{\alpha+1}{2} - 1\right)\right|\right] r^3 - \\
&\left[\rho\left\{\nu + \frac{\alpha+1}{2} - 1\right\} - \left(1-\rho^2\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right)\right] r^2 + \\
&\left[\rho\left\{\left|-2b\cos\theta + \left(\nu + \frac{\alpha+1}{2} - 1\right)\right|\right\} + \left(1-\rho^2\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right)\right] r + \\
&\rho\left[\nu + \frac{\alpha+1}{2} - 1\right] = 0.
\end{aligned} \tag{3.8}$$

Where $r = -1$ *is first roots and the other two roots are given by*

$$r_5 = \frac{\kappa_0 - \sqrt{\kappa_0^2 + 4\rho^2\left[\left|-2b\cos\theta + \left(\nu + \frac{\alpha+1}{2} - 1\right)\right|\right]\left[\nu + \frac{\alpha+1}{2} - 1\right]}}{-2\rho\left[\left|-2b\cos\theta + \left(\nu + \frac{\alpha+1}{2} - 1\right)\right|\right]},$$

and

$$\kappa_0 = \left[\left(1-\rho^2\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right) - \rho\left\{\left|-2b\cos\theta + 2\left(\nu + \frac{\alpha+1}{2} - 1\right)\right|\right\}\right].$$

Which reduces to Corollary 4 for $\theta = 0$.

Corollary 5. *Let the function* $\lambda \in \sum$ *and suppose that* $\delta \in N^{\nu,j}_{\alpha,\beta}(\theta, b; A, B)$*. If*

$$\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j} \ll \left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j}, \qquad (j \in 0, 1, 2, ...),$$

then

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}\right| \le \left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}\right|, \qquad (|\omega| < r_6),$$

where $r_6 = r_6(b, \nu, \alpha, \beta)$ *is the least positive roots of the equation*

$$\begin{aligned}
&-\rho\left[\left|-2b + \left(\nu + \frac{\alpha+1}{2} - 1\right)\right|\right] r^3 - \\
&\left[\rho\left\{\nu + \frac{\alpha+1}{2} - 1\right\} - \left(1-\rho^2\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right)\right] r^2 + \\
&\left[\rho\left\{\left|-2b + \left(\nu + \frac{\alpha+1}{2} - 1\right)\right|\right\} + \left(1-\rho^2\right)\left(\nu - 1 + \frac{\alpha+1}{2}\right)\right] r +
\end{aligned}$$

$$\rho\left[\nu+\frac{\alpha+1}{2}-1\right]=0, \tag{3.9}$$

given by

$$r_6=\frac{\kappa_1-\sqrt{\kappa_1^2+4\rho^2\left[\left|-2b+\left(\nu+\frac{\alpha+1}{2}-1\right)\right|\right]\left[\nu+\frac{\alpha+1}{2}-1\right]}}{-2\rho\left[\left|-2b+\left(\nu+\frac{\alpha+1}{2}-1\right)\right|\right]},$$

and

$$\kappa_1=\left[\left(1-\rho^2\right)\left(\nu-1+\frac{\alpha+1}{2}\right)-\rho\left\{\left|-2b+2\left(\nu+\frac{\alpha+1}{2}-1\right)\right|\right\}\right].$$

Taking $\alpha=\beta=1$ in corollary 5, we get.

Corollary 6. *Let the function $\lambda\in\sum$ and suppose that $\delta\in N_{\alpha,\beta}^{\nu,j}(\theta,b;A,B)$. If*

$$\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^j \ll \left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^j, \qquad (j\in 0,1,2,...),$$

then

$$\left|\left(\mathcal{L}_{\nu,\alpha,\beta}\lambda(\omega)\right)^{j+1}\right| \leq \left|\left(\mathcal{L}_{\nu,\alpha,\beta}\delta(\omega)\right)^{j+1}\right|, \qquad (|\omega|<r_7),$$

where $r_7=r_7(b,\nu)$ is the lowest positive roots of the equation

$$-\rho\left|-2b+\nu\right|r^3-\left[\rho\nu-\left(1-\rho^2\right)\nu\right]r^2+$$
$$\left[\rho\left|-2b+\nu\right|+\left(1-\rho^2\right)\nu\right]r+\rho\left[\nu\right]=0, \tag{3.10}$$

given by

$$r_7=\frac{\kappa_2-\sqrt{\kappa_2^2+4\rho^2\left[\left|-2b+\nu\right|\right]\left[\nu\right]}}{-2\rho\left[\left|-2b+\nu\right|\right]},$$

and

$$\kappa_2=\left[\left(1-\rho^2\right)\nu-\rho\left\{\left|-2b+2\nu\right|\right\}\right].$$

4. Conclusion

In this paper, we explore the problems of majorization for the classes $M_{\alpha,\beta}^{\nu,j}(\eta,\varkappa;A,B)$ and $N_{\alpha,\beta}^{\nu,j}(\theta,b;A,B)$ by using a convolution operator $\mathcal{L}_{\nu,\alpha,\beta}$. These results generalizes and unify the theory of majorization which is an active part of current ongoing research in Geometric Function Theory. By specializing different parameters like ν, η, $\varkappa$, θ and b, we obtain a number of important corollaries in Geometric Function Theory.

Acknowledgments

The work here is supported by GUP-2019-032.

Conflict of interest

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

References

1. A. A. A. Abubaker, M. Darus, D. Breaz, *Majorization for a subclass of β-spiral functions of order α involving a generalized linear operator*, Adv. Decis. Sci., **2011** (2011), 1–9.
2. M. K. Aouf, A. O. Mostafa, H. M. Zayed, *Convolution properties for some subclasses of meromorphic functions of complex order*, Abstr. Appl. Anal., **2015** (2015), 1–6.
3. O. Altintas, H. M. Srivastava, *Some majorization problems associated with p-valently starlike and convex functions of complex order*, East Asian Math. J., **17** (2001), 175–183.
4. O. Altintas, O. Ozkan, H. M. Srivastava, *Majorization by starlike functions of complex order*, Complex Var., **46** (2001), 207–218.
5. A. Baricz, E. Deniz, M. Caglar, et al. *Differential subordinations involving the generalized Bessel functions*, Bull. Malays. Math. Sci. Soc., **38** (2015), 1255–1280.
6. A. Baricz, *Generalized Bessel Functions of the First Kind*, Springer-Verlag, Berlin, 2010.
7. E. Deniz, *Differential subordination and superordination results for an operator associated with the generalized Bessel function*, Preprint, 2012.
8. E. A. Eljamal, M. Darus, *Majorization for certain classes of analytic functions defined by a new operator*, CUBO, **14** (2012), 119–125.
9. S. P. Goyal, P. Goswami, *Majorization for certain classes of meromorphic functions defined by integral operator*, Ann. Univ. Mariae Curie Sklodowska Lublin-Polonia., **66** (2016), 57–62.
10. P. Goswami, M. K. Aouf, *Majorization properties for certain classes of analytic functions using the Salagean operator*, Appl. Math. Lett., **23** (2010), 1351–1354.
11. P. Goswami, Z. G. Wang, *Majorization for certain classes of analytic functions*, Acta Univ. Apulensis Math. Inform., **21** (2009), 97–104.
12. S. P. Goyal, S. K. Bansal, P. Goswami, *Majorization for certain classes of analytic functions defined by linear operator using differential subordination*, J. Appl. Math. Stat. Inform., **6** (2010), 45–50.
13. S. P. Goyal, P. Goswami, *Majorization for certain classes of analytic functions defined by fractional derivatives*, Appl. Math. Lett., **22** (2009), 1855–1858.
14. S. H. Li, H. Tang, E. Ao, *Majorization properties for certain new classes of analytic functions using the Salagean operator*, J. Inequal. Appl., **2013** (2013), 1–8.
15. A. O. Mostafa, M. K. Aouf, H. M. Zayed, *Convolution properties for some subclasses of meromorphic bounded functions of complex order*, Int. J. Open Probl. Complex Anal., **236** (2016), 1–8.
16. S. S. Miller, P. T. Mocanu, *Differential Subordinations Theory and Applications*, Marcel Dekker, New York, 2000.

17. T. H. MacGregor, *Majorization by univalent functions*, Duke Math. J., **34** (1967), 95–102.

18. J. K. Prajapat, M. K. Aouf, *Majorization problem for certain class of p-valently analytic function defined by generalized fractional differintegral operator*, Comput. Math. Appl., **63** (2012), 42–47.

19. M. S. Robertson, *Quasi-subordination and coefficient conjectures*, Bull. Am. Math. Soc., **76** (1970), 1–9.

20. H. M. Srivastava, S. Owa, *Current Topics in Analytic Function Theory*, World Scientific, Singapore, 1992.

21. H. Tang, G. T. Deng, S. H. Li, *Majorization properties for certain classes of analytic functions involving a generalized differential operator*, J. Math. Res. Appl., **33** (2013), 578–586.

22. H. Tang, S. H. Li, G. T. Deng, *Majorization properties for a new subclass of θ- spiral functions of order γ*, Math. Slovaca, **64** (2014), 1–12.