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Research article

Entropy dimension of shifts of finite type on free groups

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Abstract: This paper considers the topological degree of G-shifts of finite type for the case where G is a finitely generated free group. Topological degree is the logarithm of entropy dimension; that is, topological degree is a characterization for zero entropy systems. Following the conjugacy-invariance of topological degree, we show that it is equivalent to solving a system of nonlinear recurrence equations. More explicitly, the topological degree of G-shift of finite type is achieved as the maximal spectral radius of a collection of matrices corresponding to the shift itself.

Keywords: topological degree; entropy dimension; free group; finitely generated group; Cayley graph; conjugacy-invariant
Mathematics Subject Classification: 37A35, 37B10

1. Introduction

A discrete dynamical system consists of a pair (X, T) in which X is a set, and $T : \mathbb{Z}^+ \times X \to X$ is a function which describes the evolution of elements of X in discrete time steps. It is known that, if the evolution of the system is reversible in time, then the evolution function can be treated as a group acting over X. For example, consider the joint action of two homeomorphisms T_1 and T_2 of the same space X with $T_1 \circ T_2 = T_2 \circ T_1$, then the joint action of T_1 and T_2 can be viewed as a $\mathbb{Z}^2$-action over X. A natural question is to study a countable or finitely generated group acting over the space.

Generally, these systems could be difficult to elucidate. One of the most important tools is to partition the set X into finitely many parts and subsequently code each element of the set as the sequence of partitions visited by its orbit. This technique is nowadays known as "symbolic dynamics," and the first study dedicated to this technique is referred to Hedlund and Morse [1]. In other words, symbolic dynamics is the discipline which studies dynamical systems obtained through codings. These objects are called *shift spaces* or *shifts*. For instance, studying symbolic dynamical systems helps for the investigation of hyperbolic topological dynamical systems. The interested reader can consult standard literature such as [2–6].

Let $\mathcal{A}$ be a finite alphabet and G a group. A *configuration* is a function from G to $\mathcal{A}$, and a *pattern* is a function from a finite subset of G to $\mathcal{A}$. A *shift space* is a subset $X \subseteq \mathcal{A}^G$ consists of configurations which avoid patterns from some set $\mathcal{F}$. We denote such a space by $X = \mathsf{X}_{\mathcal{F}}$. A shift space is called a *shift of finite type* (SFT) if $\mathcal{F}$ is a finite set. Shifts of finite type have been thoroughly studied in the case of $\mathbb{Z}$-actions, most of the core results can be found in the celebrated book by Lind and Marcus [7]. In particular, SFTs are characterized by graphs (hence they are nonempty except their graphs containing no bi-infinite walks), and they always contain periodic configurations. What is more, their topological entropies correspond to nonnegative rational multiples of logarithms of Perron numbers [8]. However, the investigation of SFTs is rife for the case of $\mathbb{Z}^d$-actions when $d \geq 2$ [9–13].

Several properties which are seen in one-dimensional SFTs no longer hold in higher dimensions. For instance, it is even undecidable if an SFT is nonempty [9]; there exist two-dimensional SFTs which contain no periodic cinfigurations [9, 14, 15]; different kinds of mixing properties are introduced for examining the existence and denseness of periodic configurations [10]. For the set of numbers achieved as topological entropies of two-dimensional shifts of finite type, the groundbreaking theorem of Hochman and Meyerovitch [16] indicates that they are right recursively enumerable.

In the recent years, shift spaces defined on monoids or groups have also gained attention, see [17–25] for instance. Whenever G is a free monoid, many properties hold again. For example, the conjugacy between two irreducible G-SFTs is decidable [17]; the nonemptiness, extensibility, and the existence and denseness of periodic configurations are decidable for G-SFTs [19, 21]. Aside from the qualitative behavior of G-SFTs, the phenomena from the computational perspective are also fruitful. Petersen and Salama [26] reveal an algorithm to estimate the entropy of a *hom-shift* [27]. (A hom-shift, roughly speaking, is a G-SFT which is isotropic and symmetric; alternatively, each direction of a hom-shift shares the same rule.) What is more, there are fruitful results about computational aspects of shifts on finitely generated groups [23, 28–32].

This paper studies the properties of G-SFTs for the case where G is a finitely generated free group via *topological degree*. The topological degree of a G-shift reflects the idea of *entropy dimension*, which has been extensively investigated for zero entropy systems over $\mathbb{Z}^d$-actions [33–37]. The importance of zero entropy systems has been revealed recently since they exhibit diverse complexities. For example, Katok and Thouvenot revealed a $\mathbb{Z}^2$-action in [38] which has zero directional entropy in each direction but

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} H\Big(\bigvee_{(i,j)\in C_n} \sigma^{-(i,j)} P\Big) > 0$$

for any $0 < \alpha < 2$, where C_n denotes the square of size $n \times n$ and P is some finite measurable partition. Another motivation of elucidating the topological degrees of G-shifts is that they are conjugacy-invariant while the topological entropies* (defined in (3.1)) are not when G is non-amenable [39].

After demonstrating the conjugacy-invariance of topological degree (Proposition 3.1) and the fact

*When G is a countable amenable group and $X \subseteq \mathcal{A}^G$ is a subshift. The topological entropy of X is defined as the growth rate of $B_{f_n}(X)$ with respect to f_n, where $(f_n)_{n\in\mathbb{N}}$ is any Følner sequence. When $G = \mathbb{Z}^d$, one can pick $f_n = \{0, 1, \ldots, n-1\}^d$ the n-ball for $n \in \mathbb{N}$. The same idea extends to finitely generated free groups.

that every G-SFT is topologically conjugated to a vertex shift (Proposition 2.5), where a vertex shift is a G-SFT characterized by a set of matrices, we reveal that the calculation of topological degree (of an SFT) is equivalent to solving a system of nonlinear recurrence equations (Proposition 4.1 and Theorem 5.1). Finally, the algorithm is derived by decomposing the system into several subsystems (Theorem 4.7). Section 6 extends the study to a class of finitely generated groups.

2. Shift spaces and higher block shifts

Let F_d be the free group on d generators with basis $\Sigma_d = \{s_1, \ldots, s_d\}$ for some $d \in \mathbb{N}$. A *word* in F_d is a product of the form

$$w = s_{i_1}^{o_1} s_{i_2}^{o_2} \cdots s_{i_n}^{o_n}, \quad \text{where} \quad 1 \leq i_j \leq d, o_j \in \{-1, 1\}, n \geq 0. \tag{2.1}$$

Herein, the *empty word* (or the empty product), such as $n = 0$, refers to the identity element $e_{F_d} \in F_d$. Each element of F_d has the unique expression of the form (2.1) in the sense that $s_{i_j}^{o_j} s_{i_{j+1}}^{o_{j+1}} \neq e_{F_d}$. A product of this form is called the *minimal presentation*. Suppose g is an element of F_d. The *length* of g is defined as the number of elements of its minimal presentation; that is,

$$|g| = \min\{n : g = s_{i_1}^{o_1} s_{i_2}^{o_2} \cdots s_{i_n}^{o_n}, 1 \leq i_j \leq d, o_j \in \{-1, 1\}, n \geq 0\}.$$

It follows that e_{F_d} is the unique word of length 0. For the simplification, we denote by e the identity element of F_d for the rest of this paper unless otherwise stated.

Let $\mathcal{A} = \{1, 2, \ldots, k\}$ be a finite alphabet. A *configuration* (or a *coloring*) is a function $x : F_d \to \mathcal{A}$ and a *pattern* is a function from a finite subset of F_d to $\mathcal{A}$. For each $n \in \mathbb{N}$, denote by $E_n = \{g \in F_d : |g| \leq n\}$ the *n-ball* in G centered at e. A pattern is called an *n-block* if its domain (or support) is E_n for some $n \geq 0$. For each $g \in F_d$, $x_g := x(g)$ denotes the label attached to the Cayley graph of F_d at the vertex g. The *full shift* $\mathcal{A}^{F_d}$ consists of all configurations from F_d to $\mathcal{A}$, and the (*right*-)*shift action* $\sigma : F_d \times \mathcal{A}^{F_d} \to \mathcal{A}^{F_d}$ is defined as $(\sigma_g x)_{g'} := \sigma(g, x)_{g'} = x_{gg'}$. For $H \subset F_d$ and $x \in \mathcal{A}^{F_d}$, we denote by $x|_H$ the restriction of x on H given by $(x|_H)_g = x_g$ for $g \in H$.

Remark 2.1. Notably, there are four possibilities to define σ as a group action; one could either define $(\sigma_g x)_h$ as $x_{g^{-1}h}, x_{gh}, x_{hg^{-1}}$, or x_{hg}. The first and the last define left group actions while the other two are right group actions. From a theoretical point of view, the choice is arbitrary. We choose the second one so that σ corresponds to the left shift in $\mathbb{Z}$.

Suppose $x \in \mathcal{A}^{F_d}$ is a configuration and $p : H \to \mathcal{A}$ is a pattern. We say *x accepts p* (at g) or *p appears in x* (at g) if there exists $g \in F_d$ such that $\sigma_g x|_H = p$, i.e., $x_{gh} = p_h$ for all $h \in H$. Let $\mathcal{P}$ be the set of all possible patterns and $\mathcal{F} \subseteq \mathcal{P}$. A shift space over F_d with $\mathcal{A}$ is defined as follows.

Definition 2.2. Let F_d be a free group generated by $\Sigma_d = \{s_1, \ldots, s_d\}$ and $\mathcal{A}$ a finite alphabet. Suppose $\mathcal{P}$ is the set of all possible configurations. A set of configurations $X \subseteq \mathcal{A}^{F_d}$ is called a *shift space* if there exists $\mathcal{F} \subseteq \mathcal{P}$ such that

$$X = \mathsf{X}_{\mathcal{F}} = \{x \in \mathcal{A}^{F_d} : \text{ no } p \in \mathcal{P} \text{ appears in } x\}.$$

For any two configurations $x, y \in \mathcal{A}^{F_d}$, define $\mathrm{d} : \mathcal{A}^{F_d} \times \mathcal{A}^{F_d} \to \mathbb{R}$ as

$$\mathrm{d}(x, y) = \begin{cases} k^{-n}, & n = \min\{|g| : x_g \neq y_g\} < \infty; \\ 0, & \text{otherwise}. \end{cases}$$

Then d is a metric on $\mathcal{A}^{F_d}$ and is similar to the one defined in one-dimensional shift spaces. A straightforward elucidation demonstrates that every shift space X is topologically closed and shift invariant; that is, $\sigma_g X \subseteq X$ for all $g \in F_d$. A shift space X is called a *shift of finite type* (SFT) if $X = \mathsf{X}_{\mathcal{F}}$ for some finite set $\mathcal{F} \subseteq \mathcal{P}$.

Suppose $X \subseteq \mathcal{A}^{F_d}$ and $Y \subseteq \mathcal{B}^{F_d}$ are two shift spaces with finite alphabets $\mathcal{A}$ and $\mathcal{B}$, respectively. A transformation $\phi : X \to Y$ is called a *sliding block code* if there exists a *block map* $\Phi : \Gamma_m(X) \to \mathcal{B}$ for some $m \geq 0$ such that $(\phi x)_g = \Phi(x|_{gE_m})$ for all $g \in F_d$, where $\Gamma_m(X) = \{x|_{E_m} : x \in X\}$ and $gE_m = \{gg' : |g'| \leq m\}$. For the case where $\mathcal{B} = \Gamma_m(X)$, the sliding block code ϕ is called the *mth higher block code*. It is not difficult to see that the well-known Curtis-Lyndon-Hedlund theorem in classical symbolic dynamical systems remains true for shift spaces over F_d. More explicitly, a transformation $\phi : X \to Y$ is a sliding block code if and only if ϕ is continuous and $\phi \circ \sigma_g = \sigma_g \circ \phi$ for all $g \in F_d$ [40].

A sliding block code $\phi : X \to Y$ is called an *embedding code* (resp. a *factor code*) if it is one to one (resp. onto). Furthermore, we say that ϕ is a *conjugacy* if there exists a sliding block code $\psi : Y \to X$ such that $(\psi \circ \phi)(x) = x$ and $(\phi \circ \psi)(y) = y$ for all $x \in X$ and $y \in Y$; two shifts X and Y are *topologically conjugated*, denoted by $X \cong Y$, if there is a conjugacy from X to Y.

Definition 2.3. Suppose $X \subseteq \mathcal{A}^{F_d}$ is a shift space and m is a nonnegative integer. We define the *mth higher block presentation* of X, denoted by $X^{[m]}$, as the image of the mth higher block code ϕ, i.e., $X^{[m]} = \phi(X) \subseteq \Gamma_m(X)^{F_d}$.

It is known that $X^{[m]} \cong X$ for $m \geq 0$ when $d = 1$ [7]. (Note that F_1 is isomorphic with $\mathbb{Z}$.) Theorem 2.4 indicates that, for $d \in \mathbb{N}$, every shift space over F_d is topologically conjugated to its mth higher block presentation.

Theorem 2.4. *Suppose X is a shift space and $m \geq 0$. Then $X \cong X^{[m]}$.*

Proof. Let $\psi : X^{[m]} \to X$ be the sliding block code derived from the block map $\Psi : \Gamma_m(X) \to \mathcal{A}$ defined as $\Psi(u) = u_e$. It can be verified without difficulty that ϕ is the inverse of ψ, and vice versa. □

Suppose X is an SFT over F_d with the alphabet $\mathcal{A}$. Then there exists $m \geq 0$ such that $X = \mathsf{X}_{\mathcal{F}}$ for some $\mathcal{F} \subseteq \mathcal{A}^{E_m}$. We say that X is an SFT with the *nearest neighborhood* (or X is a *Markov shift*) provided $m = 1$. Theorem 2.4 infers that we may assume that X is an SFT with the nearest neighborhood without loss of generality.

A classical result for $\mathbb{Z}$-shifts is that every $\mathbb{Z}$-SFT is topologically conjugated to an SFT

$$\mathsf{X}_A = \{x \in \mathcal{A}^{\mathbb{Z}} : A(x_i, x_{i+1}) = 1 \text{ for } i \in \mathbb{Z}\}$$

for some binary matrix A indexed by the alphabet $\mathcal{A}$. Such a result extends to SFTs over F_d with the alphabet $\mathcal{A}$. Let $\mathbf{A} = \{A_1, A_2, \ldots, A_d\}$ be a collection of binary matrices indexed by $\mathcal{A}$. We define the *vertex shift* $\mathsf{X}_{\mathbf{A}}$ as

$$\mathsf{X}_{\mathbf{A}} = \{x \in \mathcal{A}^{F_d} : A_i(x_g, x_{gs_i}) = 1 \text{ for } 1 \leq i \leq d, g \in F_d\}. \tag{2.2}$$

It follows immediately that $\mathsf{X}_{\mathbf{A}}$ is a Markov shift. Proposition 2.5, which extends the above classical result to F_d-SFTs, infers that it suffices to investigate vertex shifts instead of general SFTs.

Proposition 2.5. *Every shift of finite type is topologically conjugated to a vertex shift.*

Proof. The demonstration is achieved yielding similar discussion in classical symbolic dynamical systems, thus it is omitted. □

3. Topological degree of shift spaces

Suppose X is a shift space over F_d with the alphabet $\mathcal{A}$. Let $\Gamma_n(X)$ be the collection of allowable n-blocks of X; that is, $u \in \Gamma_n(X)$ if and only if x accepts u for some $x \in X$. The *topological entropy* and the *topological degree* of X are defined as

$$h(X) = \limsup_{n\to\infty} \frac{\ln \gamma_n(X)}{|E_n|} \tag{3.1}$$

and

$$\deg(X) = \limsup_{n\to\infty} \frac{\ln\ln \gamma_n(X)}{n}, \tag{3.2}$$

respectively. When $d = 1$, both the limits of (3.1) and (3.2) exists since $\ln \gamma_n(X)$ is subadditive [7]; the subadditivity of $\ln \gamma_n(X)$ infers zero degree of X.

While the topological entropy is conjugacy invariant for F_1-shifts (i.e., $\mathbb{Z}$-shifts), Ornstein and Weiss exhibited an example in which the factor admitted larger topological entropy (see [39,41] for more details). This makes topological entropy no longer classifies F_d-SFTs since it is not conjugacy invariant. Hence, a general entropy theory (now known as the *topological sofic entropy*, see [39, 42]) has to abandon the property that factor maps cannot increase the entropy. It is of interest if topological degree remains to be conjugacy invariant; Proposition 3.1 demonstrates that the topological degree is conjugacy invariant.

Proposition 3.1. *The topological degree of shift spaces over F_d is conjugacy invariant.*

Proof. Obviously, the topological degree is conjugacy invariant for $d = 1$ since $\deg(X) = 0$ for all X.

For the case where $d \geq 2$, it suffices to show that $\deg(X) \geq \deg(Y)$ for any two shifts X and Y such that Y is a factor of X. Let $\phi : X \to Y$ be a factor code comes from the block map $\Phi : \Gamma_m(X) \to \mathcal{A}_Y$ for some $m \geq 0$, where $\mathcal{A}_Y$ is the alphabet of Y. Observe that

$$\gamma_n(Y) \leq \gamma_{m+n}(X) \leq \gamma_m(X)\gamma_n(X)^{2d(2d-1)^{m-1}} \quad \text{for} \quad n \in \mathbb{N}.$$

Therefore,

$$\ln\ln \gamma_n(Y) \leq \ln(\ln \gamma_m(X) + 2d(2d-1)^{m-1} \ln \gamma_n(X)) \leq \ln K + \ln(1 + \ln \gamma_n(X))$$

for some constant $K > 0$. This concludes that $\deg(Y) \leq \deg(X)$. □

For the rest of this paper, we consider the case where $d = 2$ unless otherwise stated. Although the limit (3.1) may not exist in general, Proposition 3.2 indicates that the limit (3.2) exists provided the specific shift has positive entropy.

Proposition 3.2. *Suppose X is a shift space over F_d with the alphabet $\mathcal{A}$ and $\liminf_{n\to\infty} \frac{\ln \gamma_n(X)}{|E_n|} > 0$. Then the limit* (3.2) *exists; that is,*

$$\deg(X) = \lim_{n\to\infty} \frac{\ln\ln \gamma_n(X)}{n}.$$

Proof. We denote by $\gamma_n := \gamma_n(X)$ for the simplification of the notation. For $n \in \mathbb{N}$, observe that

$$|E_n| = \frac{c+1}{c-1}(c^n - 1) \quad \text{and} \quad |\widehat{E_n}| = (c+1)c^{n-1},$$

where $c = 2d - 1$ and $\widehat{E_n} = \{g \in F_d : |g| = n\}$. Let

$$\alpha = \liminf_{n\to\infty} \frac{\ln \gamma_n}{|E_n|} = \liminf_{n\to\infty} \frac{\ln \gamma_n}{\frac{c+1}{c-1}c^n} > 0.$$

For each $\epsilon > 0$ such that $\epsilon < \alpha/2$, there exists $m \in \mathbb{N}$ such that

$$\alpha - \epsilon < \frac{\ln \gamma_m}{\frac{c+1}{c-1}c^m} < \alpha + \epsilon \quad \text{and} \quad \frac{\left|\ln \frac{c-1}{c+1}\frac{2}{\alpha}\right|}{m} < \epsilon. \tag{3.3}$$

For $n \in \mathbb{N}$, write $n = \ell m + r$ with $0 \le r \le m - 1$. Notably,

$$\begin{aligned}\gamma_n = \gamma_{r+\ell m} &\le \gamma_r \cdot \gamma_{\ell m}^{(c+1)c^{r-1}} \le \gamma_r \left(\gamma_m \gamma_{(\ell-1)m}^{c^m}\right)^{(c+1)c^{r-1}} \\ &\le \gamma_r \left(\gamma_m \left(\gamma_m \gamma_{(\ell-2)m}^{c^m}\right)^{c^m}\right)^{(c+1)c^{r-1}} \le \gamma_r \gamma_m^{\frac{c^{\ell m}-1}{c^m-1}(c+1)c^{r-1}}.\end{aligned}$$

Let $\beta = \liminf_{n\to\infty} \frac{\ln\ln\gamma_n}{n}$. Then

$$\begin{aligned}\beta - \epsilon < \frac{\ln\ln\gamma_n}{n} &\le \frac{\ln\left(\ln\gamma_r + \frac{c^{\ell m}-1}{c^m-1}(c+1)c^{r-1}\ln\gamma_m\right)}{n} \\ &\le \frac{\ln\gamma_r + \ln\left(\frac{c^{\ell m}-1}{c^m-1}(c+1)c^{r-1}\ln\gamma_m\right)}{n} \\ &\le \frac{\ln\gamma_r}{n} + \frac{\ln\left(\left(\frac{c-1}{c+1}\frac{1}{\alpha-\epsilon}\ln\gamma_m\right)^{\ell}\cdot\ln\gamma_m\right)}{n} + \frac{\ln\frac{(c+1)c^{r-1}}{c^m-1}}{n} \\ &< 2\epsilon + \frac{\ell\ln\frac{c-1}{c+1}\frac{2}{\alpha}}{\ell m} + \frac{(\ell+1)\ln\ln\gamma_m}{\ell m} < \beta + 3\epsilon\end{aligned}$$

whenever n is large enough, where the third inequality is driven by the mean value theorem, and the fourth and the last inequalities come from (3.3). The desired result then follows. □

Suppose X is a $\mathbb{Z}^d$-shift space with zero topological entropy

$$h(X) = \lim_{n\to\infty} \frac{\ln|C_n(X)|}{n^d} = 0,$$

where $C_n(X)$ is the set of all allowable patterns over hypercube of length n.

Definition 3.3. Let X be a $\mathbb{Z}^d$-shift space. The *entropy dimension* of X is then defined as

$$D(X) = \inf\left\{s > 0 : \limsup_{n\to\infty} \frac{\ln|C_n(X)|}{n^s} = 0\right\} = \sup\left\{s > 0 : \limsup_{n\to\infty} \frac{\ln|C_n(X)|}{n^s} = \infty\right\}.$$

The idea of entropy dimension extends to F_d-shift space as follows. Let

$$K = \inf\left\{s > 0 : \limsup_{n\to\infty} \frac{\ln \gamma_n(X)}{s^n} = 0\right\} = \sup\left\{s > 0 : \limsup_{n\to\infty} \frac{\ln \gamma_n(X)}{s^n} = \infty\right\}. \tag{3.4}$$

Then $K \leq 2d-1$ and $K = 2d-1$ if $h(X) > 0$. Proposition 3.2 demonstrates that $h(X) > 0$ is a sufficient condition for the existence of the limit (3.2). For the case where $h(X) = 0$ (thus the limit (3.1) exists), the following proposition reveals another sufficient condition for the existence of the limit (3.2). The proof is delivered via similar discussion in the proof of Proposition 3.2, thus it is omitted.

Proposition 3.4. *Suppose X is a shift space over F_d with the alphabet $\mathcal{A}$ and $h(X) = 0$. Let K be defined as* (3.4). *If* $\liminf_{n\to\infty} \frac{\ln \gamma_n(X)}{K^n} > 0$, *then the limit* (3.2) *exists.*

Remark 3.5. Following Proposition 3.2, it is seen that $h(X) > 0$ implies $\deg(X) = \ln(2d-1)$. Furthermore, it can be verified without difficulty from the definition of entropy dimension and Proposition 3.4 that $\deg(X) = \ln K$ whenever $h(X) = 0$.

4. Algorithm for degree of SFTs

For each F_d-shift space X, $K < 2d-1$ infers that X is a zero entropy system. It is of interest how to calculate the topological degree of X. This section is devoted to developing an algorithm for the computation of topological degrees of F_d-SFTs.

4.1. SFTs over monoid

Suppose X is a shift of finite type over F_d with the alphabet $\mathcal{A} = \{1, 2, \ldots, k\}$. Propositions 2.5 and 3.1 indicate that, without loss of generality, we may assume $X = \mathsf{X}_{\mathbf{A}}$ is a vertex shift for some $k \times k$ binary matrices $\mathbf{A} = \{A_1, \ldots, A_d\}$.

Notably, the free group F_d can be treated as a monoid $G = \langle S|R\rangle$ with $S = \{t_1, t_2, \ldots, t_{2d}\}$ and $R = \{t_i t_{i+d} = t_{i+d} t_i = e \text{ for } 1 \leq i \leq d\}$. Indeed, G is obtained from F_d by renaming the elements of F_d. What is more, write $w \in F_d$ as its unique expression

$$w = s_{i_1}^{o_1} s_{i_2}^{o_2} \cdots s_{i_n}^{o_n}, \quad \text{where} \quad 1 \leq i_j \leq d, o_j \in \{-1, 1\}, n \geq 0.$$

Then

$$g = t_{j_1} t_{j_2} \cdots t_{j_n}, \quad \text{where} \quad j_\ell = i_\ell + \frac{1 - o_\ell}{2} d, 1 \leq \ell \leq n,$$

is the rename of w in G, and vice versa. The Cayley graph of G is then a rooted tree such that every node has $(2d-1)$-children except the root, which has $(2d)$-children. We call G a *monoid representation* of F_d. Let Ω be the set of vertex shifts over F_d and

$$\Xi = \{X \subseteq \mathcal{A}^G : X = \mathsf{X}_{\mathbf{B}} : \mathbf{B} = \{B_i\}_{i=1}^{2d}, B_{i+d} = B_i' \text{ for } 1 \leq i \leq d, B_i \in \{0, 1\}^{k \times k}\}$$

a proper subset of vertex shifts over G, where M' denotes the transpose of M. It is seen that there is a one-to-one correspondence between Ω and Ξ. Indeed, suppose $X = \mathsf{X}_{\mathbf{A}}$ is a vertex shift over F_d determined by some $k \times k$ binary matrices $\mathbf{A} = \{A_1, \ldots, A_d\}$ and $Y = \mathsf{X}_{\mathbf{B}}$ is a vertex shift over G with

$\mathbf{B} = \{A_1, \ldots, A_{2d}\}, A_{i+d} = A_i'$ for $1 \le i \le d$. Since $x \in X$ if and only if $A_i(x_w, x_{ws_i}) = 1$ for $1 \le i \le d$ and $w \in F_d$, it follows immediately that

$$A_{i+d}(x_w, x_{ws_i^{-1}}) = A_i'(x_w, x_{ws_i^{-1}}) = 1 \quad \text{for} \quad 1 \le i \le d.$$

Let $g \in G$ be the (unique) representation of w, and let y be defined as $y_g = x_w$. Then y is well-defined and $y \in Y$. Similarly, for each $y \in Y$, we can construct the unique $x \in X$ that is the representation of y in X. Note that $X \cong Y$ in this case. Conversely, for each $Y \in \Xi$, we can construct the unique $X \in \Omega$ such that $X \cong Y$. The above discussion yields the following proposition.

Proposition 4.1. *Let $G = \langle t_1, \ldots, t_{2d} | t_1 t_{d+1}, \ldots, t_d t_{2d} \rangle$ be the monoid representation of F_d. For each vertex shift X over F_d, there exists a vertex shift Y over G such that $X \cong Y$. More explicitly, the vertex shift $\mathsf{X}_{\mathbf{A}}$ over F_d with $\mathbf{A} = \{A_1, \ldots, A_d\}$ is topologically conjugated to the vertex shift $\mathsf{X}_{\mathbf{B}}$ over G with $\mathbf{B} = \{A_1, \ldots, A_{2d}\}, A_{i+d} = A_i'$ for $1 \le i \le d$.*

Proposition 4.1 reveals that, to calculate the topological degree of vertex shifts over F_d, it suffices to investigate vertex shifts over G. Roughly speaking, there is no difference between the structure of F_d and G since the Cayley graph of G is obtained by "twisting" the Cayley graph of F_d appropriately (cf. Figure 1). For the rest of this section, X is a vertex shift $\mathsf{X}_{\mathbf{A}}$, where $\mathbf{A} = \{A_1, \ldots, A_{2d}\}$, over G with the alphabet $\mathcal{A}$ unless otherwise stated.

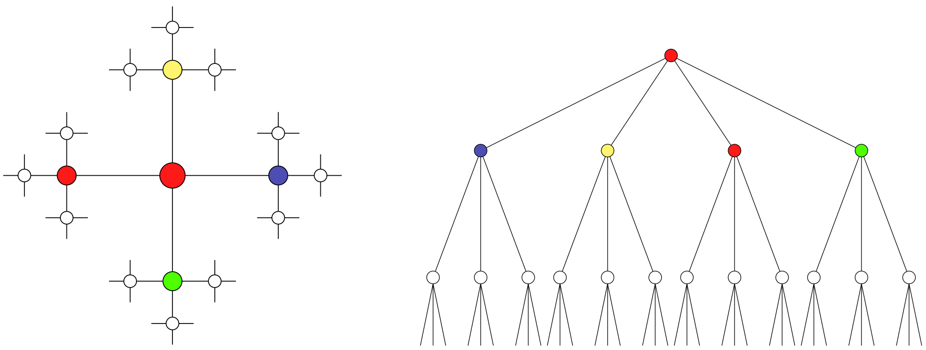

Figure 1. A finitely generated free group can be treated as a finitely generated monoid whose generators satisfy some relations. For example, the free group F_2 can be seen as the monoid $G = \langle t_1, t_2, t_3, t_4 | t_1 t_3, t_2 t_4 \rangle$. More explicitly, F_2 is isomorphic with G.

Let $E_n = \{g \in G : |g| \le n\}$ and $\Gamma_{n;a} = \{x|_{E_n} : x \in X, x_e = a\}$, where $a \in \mathcal{A}, n \in \mathbb{N}$. Write $\mathcal{A} = \mathcal{A}_E \bigcup \mathcal{A}_I$, herein $a \in \mathcal{A}_E$ if and only if $\gamma_{n;a} \ge 2$ for some $n \in \mathbb{N}$. (Without loss of generality, we assume that $\gamma_{n;a} \ge 1$ for all $a \in \mathcal{A}, n \in \mathbb{N}$.) Proposition 4.2 then follows.

Proposition 4.2. *Suppose X is a vertex shift over G with the alphabet $\mathcal{A}$. Then*

$$\deg(X) = \limsup_{n\to\infty} \frac{\ln \sum_{a\in\mathcal{A}} \ln \gamma_{n;a}}{n} = \limsup_{n\to\infty} \frac{\ln \sum_{a\in\mathcal{A}_E} \ln \gamma_{n;a}}{n}. \tag{4.1}$$

Proof. Recall that $\mathcal{A} = \{1, 2, \ldots, k\}$. Since

$$\gamma_{n;1} \cdot \gamma_{n;2} \cdots \gamma_{n;k} \le \left(\frac{\sum_{i=1}^k \gamma_{n;i}}{k} \right)^k,$$

We derive that

$$\limsup_{n\to\infty} \frac{\ln \sum_{i=1}^{k} \ln \gamma_{n;i}}{n} \le \limsup_{n\to\infty} \frac{\ln\ln \sum_{i=1}^{k} \gamma_{n;i}}{n} = \deg(X).$$

On the other hand, denote by $a_n = \max_{1\le i\le k} \gamma_{n;i}$. The inequality

$$a_n \le \sum_{i=1}^{k} \gamma_{n;i} \le k a_n$$

yields that

$$\limsup_{n\to\infty} \frac{\ln\ln \sum_{i=1}^{k} \gamma_{n;i}}{n} = \limsup_{n\to\infty} \frac{\ln\ln a_n}{n}. \tag{4.2}$$

Observe that

$$\sum_{i=1}^{k} \ln \gamma_{n;i} = \ln \prod_{i=1}^{k} \gamma_{n;i} \ge \ln a_n. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\limsup_{n\to\infty} \frac{\ln \sum_{i=1}^{k} \ln \gamma_{n;i}}{n} \ge \limsup_{n\to\infty} \frac{\ln\ln a_n}{n} = \limsup_{n\to\infty} \frac{\ln\ln \sum_{i=1}^{k} \gamma_{n;i}}{n} = \deg(X).$$

This completes the proof. □

For $i \in \mathcal{A}$, $g \in G$, and $n \in \mathbb{N}$, denote by

$$\Gamma_{n;i}^{[g]} = \{x|_{gE_n} : x \in X\}$$

the set of all n-blocks centered at g and labeled i at g. Notably, G is a monoid whose Cayley graph satisfies that only the root (which represents the identity element $e \in G$) has $(2d)$-children and g has $(2d-1)$-children for $g \ne e$. Since $X = \mathsf{X}_{\mathbf{A}}$ with $\mathbf{A} = \{A_1, \ldots, A_{2d}\}$ is a vertex shift, it is seen that

$$\gamma_n = \gamma_{n;1} + \gamma_{n;2} + \cdots + \gamma_{n;k}$$

and

$$\gamma_{n;i} = \gamma_{n;i}^{[e]} = \prod_{\ell=1}^{2d} \sum_{j=1}^{k} A_\ell(i,j) \gamma_{n-1;j}^{[t_\ell]}, \tag{4.4}$$

$$\gamma_{n-1;i}^{[t_l]} = \prod_{1\le \ell\le 2d, |l-\ell|\ne d} \sum_{j=1}^{k} A_\ell(i,j) \gamma_{n-2;j}^{[t_l t_\ell]} \tag{4.5}$$

for $1 \le i \le k$ and $1 \le l \le 2d$. For $n \in \mathbb{N}$ and $g \in G$ such that $g \ne e$, define

$$E_n(g) = \{g' \in G : 1 \le |g'| \le n \text{ and } |gg'| = |g| + |g'|\}.$$

It is easily seen that $E_n(g) = E_n(t_l)$ if and only if $g = g' t_l$ with $|g| = |g'| + 1$. Hence, we can rewrite (4.5) as

$$\gamma_{n-1;i}^{[t_l]} = \prod_{1\le \ell\le 2d, |l-\ell|\ne d} \sum_{j=1}^{k} A_\ell(i,j) \gamma_{n-2;j}^{[t_\ell]}. \tag{4.6}$$

Conclusively, computing the topological degree of $\mathsf{X}_{\mathbf{A}}$, where $\mathbf{A} = \{A_1, \ldots, A_{2d}\}$, is equivalent to study the system of nonlinear recurrence equations defined as

$$\begin{aligned} \gamma_{n;i} &= \prod_{\ell=1}^{2d} \sum_{j=1}^{k} A_\ell(i,j)\gamma_{n-1;j}^{[t_\ell]}, \\ \gamma_{n;i}^{[t_l]} &= \prod_{1\le\ell\le 2d, |l-\ell|\ne d} \sum_{j=1}^{k} A_\ell(i,j)\gamma_{n-1;j}^{[t_\ell]}, \\ \gamma_{0;i}^{[t_l]} &= 1; \end{aligned} \tag{4.7}$$

herein, $n \in \mathbb{N}$ and $1 \le l \le 2d$.

Example 4.3. Suppose $d = 2$. A monoid representation of F_2 is

$$G = \langle t_1, t_2, t_3, t_4 | t_1t_3, t_3t_1, t_2t_4, t_4t_2 \rangle.$$

Each $w \in F_2$ has the unique representation $g \in G$, and vice versa. For example, the representation of $s_1^2 s_2 s_1^{-1} s_2 \in F_2$ in G is $t_1^2 t_2 t_3 t_2$. Furthermore, for $n \in \mathbb{N}$ and $g \in G$, $E_n(g)$ is uniquely determined by the last digit of the minimal presentation of g; an example is

$$E_2(t_1t_2) = \{t_1, t_2, t_3, t_1^2, t_1t_2, t_1t_4, t_2t_1, t_2^2, t_2t_3, t_3t_2, t_3^2, t_3t_4, \ldots\} = E_2(t_2).$$

Suppose $X = \mathsf{X}_{\mathbf{A}}$ is a vertex shift over G with the alphabet $\mathcal{A} = \{1, 2\}$ and $\mathbf{A} = \{A_1, A_2, A_3, A_4\}$, where

$$A_1 = A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, A_2 = A_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Evaluating the topological degree of X is equivalent to investigating the system of nonlinear recurrence equations

$$\begin{aligned} \gamma_{n;1} &= \left(\gamma_{n-1;1}^{[t_1]} + \gamma_{n-1;2}^{[t_1]}\right)\left(\gamma_{n-1;1}^{[t_3]} + \gamma_{n-1;2}^{[t_3]}\right)\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;2} &= \left(\gamma_{n-1;1}^{[t_2]} + \gamma_{n-1;2}^{[t_2]}\right)\left(\gamma_{n-1;1}^{[t_4]} + \gamma_{n-1;2}^{[t_4]}\right)\gamma_{n-1;1}^{[t_1]}\gamma_{n-1;1}^{[t_3]}, \\ \gamma_{n;1}^{[t_1]} &= \left(\gamma_{n-1;1}^{[t_1]} + \gamma_{n-1;2}^{[t_1]}\right)\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;1}^{[t_2]} &= \left(\gamma_{n-1;1}^{[t_1]} + \gamma_{n-1;2}^{[t_1]}\right)\left(\gamma_{n-1;1}^{[t_3]} + \gamma_{n-1;2}^{[t_3]}\right)\gamma_{n-1;2}^{[t_2]}, \\ \gamma_{n;1}^{[t_3]} &= \left(\gamma_{n-1;1}^{[t_3]} + \gamma_{n-1;2}^{[t_3]}\right)\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;1}^{[t_4]} &= \left(\gamma_{n-1;1}^{[t_1]} + \gamma_{n-1;2}^{[t_1]}\right)\left(\gamma_{n-1;1}^{[t_3]} + \gamma_{n-1;2}^{[t_3]}\right)\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;2}^{[t_1]} &= \left(\gamma_{n-1;1}^{[t_2]} + \gamma_{n-1;2}^{[t_2]}\right)\left(\gamma_{n-1;1}^{[t_4]} + \gamma_{n-1;2}^{[t_4]}\right)\gamma_{n-1;1}^{[t_1]}, \\ \gamma_{n;2}^{[t_2]} &= \left(\gamma_{n-1;1}^{[t_2]} + \gamma_{n-1;2}^{[t_2]}\right)\gamma_{n-1;1}^{[t_1]}\gamma_{n-1;1}^{[t_3]}, \\ \gamma_{n;2}^{[t_3]} &= \left(\gamma_{n-1;1}^{[t_2]} + \gamma_{n-1;2}^{[t_2]}\right)\left(\gamma_{n-1;1}^{[t_4]} + \gamma_{n-1;2}^{[t_4]}\right)\gamma_{n-1;1}^{[t_3]}, \\ \gamma_{n;2}^{[t_4]} &= \left(\gamma_{n-1;1}^{[t_4]} + \gamma_{n-1;2}^{[t_4]}\right)\gamma_{n-1;1}^{[t_1]}\gamma_{n-1;1}^{[t_3]}, \quad n \in \mathbb{N}, \\ \gamma_{0;j}^{[t_i]} &= 1, \quad 1 \le i \le 4, 1 \le j \le 2. \end{aligned}$$

4.2. System of nonlinear recurrence equations

In this subsection, we investigate the growth rate of sequences described by systems of nonlinear recurrence equations.

Suppose $\{a_{n;1}, a_{n;2}, \dots, a_{n;p}\}_{n\in\mathbb{N}}$ is determined by

$$\begin{cases} a_{n;i} = f_i(a_{n-1;1}, a_{n-1;2}, \dots, a_{n-1;p}), & n \geq 2, \\ a_{1;i} = c_i, \end{cases} \tag{4.8}$$

for some polynomials $f_i : \mathbb{R}^p \to \mathbb{R}$ with zero constant term and nonnegative coefficients, and $c_i \in \mathbb{R}^+$, $1 \leq i \leq p$. Let $\mathsf{F} = (\{a_{n;1}, a_{n;2}, \dots, a_{n;p}\}_{n\in\mathbb{N}}, \{f_i\}_{i=1}^p)$ denote the system (4.8). With the abuse of terminology, we define the *degree* of F as

$$\deg(\mathsf{F}) = \limsup_{n\to\infty} \frac{\ln \sum_{i=1}^p \ln a_{n;i}}{n}.$$

A system of nonlinear recurrence equations F is *simple* if f_i contains only one term for each i. For the simplicity, we focus on the case where $c_i \geq 1$ for all i.

Definition 4.4. Suppose $\mathsf{F} = (\{a_{n;1}, a_{n;2}, \dots, a_{n;p}\}_{n\in\mathbb{N}}, \{f_i\}_{i=1}^p)$ is a simple system of nonlinear recurrence equations. The *weighted adjacency matrix* of F is a $p \times p$ integral matrix defined as

$$M(i, j) = \max_{m\geq 0} \left\{ m \geq 0 : \alpha_j^m | f_i(\alpha_1, \dots, \alpha_p) \right\}. \tag{4.9}$$

For the case where F is a simple system, let $b_n = (\ln a_{n;1}, \ln a_{n;2}, \dots, \ln a_{n;p})' \in \mathbb{R}^p$. The definition of weighted adjacency matrix indicates that $b_n = Mb_{n-1}$ for $n \geq 2$. Suppose there exists $N \in \mathbb{N}$ such that $a_{n;i} > 1$ for $1 \leq i \leq p$. The Perron-Frobenius theorem asserts that

$$\deg(\mathsf{F}) = \limsup_{n\to\infty} \frac{\ln \sum_{i=1}^p \ln a_{n;i}}{n} = \limsup_{n\to\infty} \frac{\ln \sum_{i,j=1}^p M^{n-1}(i, j)}{n} = \ln \rho_M,$$

where ρ_M is the spectral radius of M. This derives Proposition 4.5, which is also demonstrated in [43].

Proposition 4.5. *Suppose $\mathsf{F} = (\{a_{n;1}, a_{n;2}, \dots, a_{n;p}\}_{n\in\mathbb{N}}, \{f_i\}_{i=1}^p)$ is a simple system of nonlinear recurrence equations and M is the weighted adjacency matrix of F. If $a_{n;i} > 1$ for $1 \leq i \leq p$ and n large enough, then*

$$\deg(\mathsf{F}) = \ln \rho_M,$$

where ρ_M is the spectral radius of M.

Remark 4.6. Suppose that $a_{n;i} = 1$ for some $1 \leq i \leq p$ and $n \in \mathbb{N}$. Then $\ln a_{n;i} = 0$ makes no contribution to the degree of F. If this is the case, let $\overline{M}$ be the matrix obtained by deleting the ith row and the ith column of M. It is seen that $\deg(\mathsf{F}) = \ln \rho_{\overline{M}}$.

Proposition 4.5 is analogous to the classical result of the topological entropy of one-dimensional shifts of finite type; that is, the topological entropy of a shift of finite type is the logarithm of the spectral radius of some matrix. Theorem 4.7, which is an extension of Proposition 4.5, elucidates that the degree of a system of nonlinear recurrence equations is the degree of its maximal simple subsystem.

Theorem 4.7. *Suppose* $\mathsf{F} = (\{a_{n;1}, a_{n;2}, \ldots, a_{n;p}\}_{n\in\mathbb{N}}, \{f_i\}_{i=1}^{p})$ *is a system of nonlinear recurrence equations. If, for* $1 \le i \le p$, $a_{n;i} > 1$ *for n large enough, then*

$$\deg(\mathsf{F}) = \max\{\ln \rho_{M_{\mathsf{E}}} : \mathsf{E} \text{ is a simple subsystem of } \mathsf{F}\}. \tag{4.10}$$

Proof. Let E be a simple subsystem of F such that $\deg(\mathsf{F}) = \ln \rho_{M_{\mathsf{E}}}$. Obviously, $\deg(\mathsf{E}) \le \deg(\mathsf{F})$. It remains to show that $\deg(\mathsf{F}) \le \deg(\mathsf{E})$.

Without loss of generality, we may assume that $a_{1;i} > 1$ for $1 \le i \le p$ and

$$a_{n;1} \ge a_{n;2} \ge \cdots \ge a_{n;p} \quad \text{for} \quad n \in \mathbb{N}.$$

For each i, write

$$f_i(a_{n;1}, \ldots, a_{n;p}) = a_{n;1}^{m_{i,1}} a_{n;2}^{m_{i,2}} \cdots a_{n;p}^{m_{i,p}} \cdot \overline{f_i}(a_{n;1}, \ldots, a_{n;p}),$$

where $m_{i,j} = M_{\mathsf{E}}(i, j)$ for $1 \le j \le p$. Observe that $\rho_{M_{\mathsf{E}}} \ge \rho_{M_{\mathsf{E}'}}$ for any simple subsystem E' implies $M_{\mathsf{E}}(i, j) \ge M_{\mathsf{E}'}(i, j)$ for $1 \le i, j \le p$. Therefore, there exists $C > 0$ such that

$$1 < \overline{f_i}(a_{n;1}, \ldots, a_{n;p}) < C \quad \text{for} \quad 1 \le i \le p, n \in \mathbb{N}.$$

Let $\alpha_n = (\ln a_{n;1}, \ln a_{n;2}, \ldots, \ln a_{n;p})'$. Then

$$\alpha_n = M_{\mathsf{E}}\alpha_{n-1} + \beta_{n-1}, \quad \text{where} \quad \beta_{n-1} = (\ln \overline{f_1}, \ldots, \ln \overline{f_p})', n \ge 2.$$

It follows from

$$\alpha_n = M_{\mathsf{E}}^{n-1}\alpha_1 + M_{\mathsf{E}}^{n-2}\beta_1 + \cdots + \beta_{n-1}$$

that

$$\|\alpha_n\| = \sum_{i=1}^{p} \alpha_n^{(i)} \le d_0 \|\sum_{i=1}^{n-1} M_{\mathsf{E}}^{i}\| \le d_1 \sum_{i=1}^{n-1} \rho_{M_{\mathsf{E}}}^{i} \le d_2 \rho_{M_{\mathsf{E}}}^{n}$$

for some constants d_0, d_1, and d_2 depending on M_{E}. Thus, we have derive

$$\deg(\mathsf{F}) = \limsup_{n\to\infty} \frac{\ln \sum_{i=1}^{p} \alpha_n^{(i)}}{n} \le \ln \rho_{M_{\mathsf{E}}} = \deg(\mathsf{E}).$$

This completes the proof. □

Example 4.8. Given a system of nonlinear recurrence equations F as

$$\begin{aligned} a_{n;1} &= (a_{n-1;1} + a_{n-1;2})a_{n-1;2}, \\ a_{n;2} &= (a_{n-1;1} + a_{n-1;2})a_{n-1;1}, \quad n \ge 2, \\ a_{1;1} &= a_{1;2} = 1. \end{aligned}$$

Then $a_{n;1} \ge 2$ and $a_{n;2} \ge 2$ for $n \ge 2$. Consider the following simple subsystem

$$\begin{aligned} a_{n;1} &= a_{n-1;1}a_{n-1;2}, \\ a_{n;2} &= a_{n-1;1}^2. \end{aligned}$$

The weighted adjacency matrix is

$$M = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix},$$

and Proposition 4.5 demonstrates that the degree of this subsystem is $\ln 2$. Observe that $\deg(\mathsf{E}) = \ln 2$ for each simple subsystem E. Theorem 4.7 indicates that $\deg(\mathsf{F}) = \ln 2$.

5. Topological degree of SFTs over free groups

Suppose $X = \mathsf{X}_{\mathbf{A}}$ is a vertex shift over F_d with the alphabet $\mathcal{A} = \{1, 2, \ldots, k\}$ and $\mathbf{A} = \{A_1, A_2, \ldots, A_d\}$ for some $k \times k$ binary matrices. Section 4 reveals that the cardinality of set of n-blocks in X forms a system of nonlinear recurrence equations F_X defined as

$$\begin{cases} \gamma_{n;i} = \prod_{\ell=1}^{2d} \sum_{j=1}^{k} A_\ell(i,j)\gamma_{n-1;j}^{[t_\ell]}, \\ \gamma_{n;i}^{[t_l]} = \prod_{1\le \ell \le 2d, |l-\ell|\neq d} \sum_{j=1}^{k} A_\ell(i,j)\gamma_{n-1;j}^{[t_\ell]}, \\ \gamma_{0;i}^{[t_l]} = 1, \end{cases} \tag{5.1}$$

where $1 \le i \le k, 1 \le l \le 2d, n \in \mathbb{N}$, and $A_{r+d} = A'_r$ for $1 \le r \le d$. Recall that $\{t_1, t_2, \ldots, t_{2d}\}$ represents the set of generators $\{s_1, \ldots, s_d, s_1^{-1}, \ldots, s_d^{-1}\}$ of F_d. Since $\gamma_{0;i}^{[t_l]} = 1$ for $1 \le i \le k$ and $1 \le l \le 2d$, we conclude that $\gamma_{n;i} \ge \gamma_{n;i}^{[t_l]}$ for all i, l. Therefore,

$$\sum_{i=1}^{k} \gamma_{n;i} \le \sum_{i=1}^{k} (\gamma_{n;i} + \sum_{l=1}^{2d} \gamma_{n;i}^{[t_l]}) \le (2d+1) \sum_{i=1}^{k} \gamma_{n;i}.$$

Since $\deg(X)$ and $\deg(\mathsf{F}_X)$ measure the growth rate of $\sum_{i=1}^{k} \ln \gamma_{n;i}$ and $\sum_{i=1}^{k} (\ln \gamma_{n;i} + \sum_{l=1}^{2d} \ln \gamma_{n;i}^{[t_l]})$, respectively, the inequality above derives the following theorem that demonstrates the coincidence of degrees of X and F_X. In addition, $\deg(\mathsf{F}_X)$ is obtained by Theorem 4.7.

Theorem 5.1. *Given a set of binary matrices $\mathbf{A} = \{A_1, A_2, \ldots, A_d\}$. Suppose $X = \mathsf{X}_{\mathbf{A}}$ is a vertex shift over F_d and F_X is the corresponding system of nonlinear recurrence equations. Then* $\deg(X) = \deg(\mathsf{F}_X)$.

Recall that the alphabet $\mathcal{A} = \mathcal{A}_E \bigcup \mathcal{A}_I$ is decomposed as the union of two disjoint subsets, where $i \in \mathcal{A}_E$ if and only if $\gamma_{n;i} \ge 2$ for some $n \in \mathbb{N}$. Observe that, for each simple subsystem of F_X, the weighted adjacency matrix is an upper triangular block matrix

$$M = \begin{pmatrix} N_{11} & N_{12} \\ O & N_{22} \end{pmatrix},$$

where N_{11} is a $k \times k$ matrix indexed by $\{\gamma_{n;i}\}_{i=1}^{k}$ and N_{22} is a $2dk \times 2dk$ matrix indexed by $\{\gamma_{n;i}^{[t_l]}\}_{1\le i\le k, 1\le l\le 2d}$. It is seen from (5.1) that N_{11} is the zero matrix and

$$\sum_{j=1}^{2dk} N_{22}(i,j) = 2d - 1 \quad \text{for} \quad 1 \le i \le 2dk.$$

Theorem 4.7 explains that the degree of F_X is $\ln \rho_{N_{22}}$ if and only if $\mathcal{A} = \mathcal{A}_E$. For the case where $\mathcal{A}_I \neq \emptyset$, we denote by N the matrix obtained from N_{22} by eliminating those rows and columns indexed by $\mathcal{A}_I$. Then $\deg(\mathsf{F}_X) = \ln \rho_N$. This concludes Proposition 5.2.

Proposition 5.2. *Suppose X is a vertex shift over F_d with the alphabet $\mathcal{A}$. If $\mathcal{A} = \mathcal{A}_E$, then* $\deg(\mathsf{F}_X) = \ln(2d - 1)$.

Example 5.3. Suppose X is the vertex shift studies in Example 4.3. It is easily seen that $\mathcal{A} = \mathcal{A}_E$. Hence, we can conclude that $\deg(X) = \ln 3$.

Example 5.4. Let $X = \mathsf{X}_{\mathbf{A}}$ be the vertex shift over F_2 with the alphabet $\mathcal{A} = \{1, 2, 3\}$ and

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The system of nonlinear recurrence equations F_X is

$$\begin{cases} \gamma_{n;1} = \left(\gamma_{n-1;1}^{[t_1]} + \gamma_{n-1;2}^{[t_1]}\right)\left(\gamma_{n-1;1}^{[t_3]} + \gamma_{n-1;2}^{[t_3]}\right)\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;2} = \gamma_{n-1;1}^{[t_1]}\gamma_{n-1;1}^{[t_2]}\gamma_{n-1;1}^{[t_3]}\gamma_{n-1;1}^{[t_4]}, \\ \gamma_{n;3} = \gamma_{n-1;3}^{[t_1]}\gamma_{n-1;3}^{[t_2]}\gamma_{n-1;3}^{[t_3]}\gamma_{n-1;3}^{[t_4]}, \\ \gamma_{n;1}^{[t_1]} = \left(\gamma_{n-1;1}^{[t_1]} + \gamma_{n-1;2}^{[t_1]}\right)\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;1}^{[t_2]} = \left(\gamma_{n-1;1}^{[t_1]} + \gamma_{n-1;2}^{[t_1]}\right)\left(\gamma_{n-1;1}^{[t_3]} + \gamma_{n-1;2}^{[t_3]}\right)\gamma_{n-1;2}^{[t_2]}, \\ \gamma_{n;1}^{[t_3]} = \left(\gamma_{n-1;1}^{[t_3]} + \gamma_{n-1;2}^{[t_3]}\right)\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;1}^{[t_4]} = \left(\gamma_{n-1;1}^{[t_1]} + \gamma_{n-1;2}^{[t_1]}\right)\left(\gamma_{n-1;1}^{[t_3]} + \gamma_{n-1;2}^{[t_3]}\right)\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;2}^{[t_1]} = \gamma_{n-1;1}^{[t_1]}\gamma_{n-1;1}^{[t_2]}\gamma_{n-1;1}^{[t_4]}, \\ \gamma_{n;2}^{[t_2]} = \gamma_{n-1;1}^{[t_1]}\gamma_{n-1;1}^{[t_2]}\gamma_{n-1;1}^{[t_3]}, \\ \gamma_{n;2}^{[t_3]} = \gamma_{n-1;1}^{[t_2]}\gamma_{n-1;1}^{[t_3]}\gamma_{n-1;1}^{[t_4]}, \\ \gamma_{n;2}^{[t_4]} = \gamma_{n-1;1}^{[t_1]}\gamma_{n-1;1}^{[t_3]}\gamma_{n-1;1}^{[t_4]}, \\ \gamma_{n;3}^{[t_1]} = \gamma_{n-1;3}^{[t_1]}\gamma_{n-1;3}^{[t_2]}\gamma_{n-1;3}^{[t_4]}, \\ \gamma_{n;3}^{[t_2]} = \gamma_{n-1;3}^{[t_1]}\gamma_{n-1;3}^{[t_2]}\gamma_{n-1;3}^{[t_3]}, \\ \gamma_{n;3}^{[t_3]} = \gamma_{n-1;3}^{[t_2]}\gamma_{n-1;3}^{[t_3]}\gamma_{n-1;3}^{[t_4]}, \\ \gamma_{n;3}^{[t_4]} = \gamma_{n-1;3}^{[t_1]}\gamma_{n-1;3}^{[t_3]}\gamma_{n-1;3}^{[t_4]}, \quad n \in \mathbb{N}, \\ \gamma_{0;j}^{[t_i]} = 1, \quad 1 \le i \le 4, 1 \le j \le 3. \end{cases}$$

It is seen that $\mathcal{A}_E = \{1, 2\}$ and $\mathcal{A}_I = \{3\}$. Consider the following simple subsystem

$$\begin{cases} \gamma_{n;1} = \gamma_{n-1;1}^{[t_1]}\gamma_{n-1;2}^{[t_3]}\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;1}^{[t_1]} = \gamma_{n-1;1}^{[t_1]}\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;1}^{[t_2]} = \gamma_{n-1;2}^{[t_1]}\gamma_{n-1;1}^{[t_3]}\gamma_{n-1;2}^{[t_2]}, \\ \gamma_{n;1}^{[t_3]} = \gamma_{n-1;1}^{[t_3]}\gamma_{n-1;2}^{[t_2]}\gamma_{n-1;2}^{[t_4]}, \\ \gamma_{n;1}^{[t_4]} = \gamma_{n-1;1}^{[t_1]}\gamma_{n-1;1}^{[t_3]}\gamma_{n-1;2}^{[t_4]}. \end{cases}$$

(Herein, we only state those equations with multiple terms.) Then the weighted adjacency matrix is

$$M = \begin{pmatrix} N_{11} & N_{12} \\ O & N_{22} \end{pmatrix} \quad \text{with} \quad N_{22} = \begin{pmatrix} 1&0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&1&0&1&1&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0&1&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0&0&0&0&0 \\ 1&0&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&0&1 \\ 0&0&0&0&0&0&0&0&1&1&1&0 \\ 0&0&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&0&1&0&1&1 \end{pmatrix}.$$

Since $\mathcal{A}_I = \{3\}$, by eliminating the last 4 rows and columns of N_{22} we derive

$$N = \begin{pmatrix} 1&0&0&0&0&1&0&1 \\ 0&0&1&0&1&1&0&0 \\ 0&0&1&0&0&1&0&1 \\ 1&0&1&0&0&0&0&1 \\ 1&1&0&1&0&0&0&0 \\ 1&1&1&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0 \\ 1&0&1&1&0&0&0&0 \end{pmatrix}.$$

Theorems 4.7 and 5.1 demonstrate that

$$\deg(X) \geq \ln \rho_N = \ln 3.$$

Since $\deg(X) \leq \ln 3$, we have concluded that $\deg(X) = \ln 3$.

6. Finitely generated group action over SFTs

The discussion in Sections 3 and 4 can easily extend to a class of finitely generated groups. For the completeness of this paper, this section illustrates the methodology of the computation of topological degree via an example. The detailed investigation will be studied in the future work.

Let $G = \langle \alpha_1, \ldots, \alpha_d | R \rangle$ be a finitely generated group such that $S_d = \{\alpha_1, \ldots, \alpha_d\}$ is a minimal generating set, and

$$R = \{\alpha_i^2 = e \text{ for } 1 \leq i \leq d\}.$$

Then there is only one homomorphism $\pi : F_d \to G$ satisfying $\pi(s_i) = \alpha_i$ for $1 \leq i \leq d$. For each $g \in G$, define the length of g as

$$|g|_\pi = \min\{|w| : w \in \pi^{-1}(g)\},$$

recall that $|w|$ is the length of $w \in F_d$. This definition is equivalent to

$$|g|_{S_d} = \min\{n : g = \alpha_{i_1} \cdots \alpha_{i_n}, \alpha_{i_j} \in S_d \bigcup S_d^{-1}\}.$$

Suppose $X = \mathsf{X}_{\mathbf{A}}$ is a vertex shift over G with the alphabet $\mathcal{A}$, where $\mathbf{A} = \{A_1, \ldots, A_d\}$ is a collection of binary matrices indexed by $\mathcal{A}$. To compute the topological degree of X, following the discussion in Section 4 we derive a system of nonlinear recurrence equations since the relation

$$E_n(g) = E_n(\alpha) \quad \text{if and only if} \quad g = g'\alpha \text{ with } |g|_\pi = |g'|_\pi + 1$$

still holds in G, where $\alpha \in S_d \bigcup S_d^{-1}$ and

$$E_n(g) = \{g' \in G : 1 \leq |g'|_\pi \leq n \text{ and } |gg'|_\pi = |g|_\pi + |g'|_\pi\}.$$

We use an example to show how the discussion for free groups extends to finitely generated groups satisfying the property mentioned above.

Example 6.1. Suppose $G = \langle \alpha_1, \alpha_2 | \alpha_2^2 \rangle$ is a finitely generated group. Let $X = \mathsf{X}_{\mathbf{A}}$ is a vertex shift over G with the alphabet $\mathcal{A} = \{1, 2, 3\}$ and $\mathbf{A} = \{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\alpha_2^2 = e$ is of order 2, it follows that the monoid representation of G is $G' = \langle t_1, t_2, t_3 | t_1 t_3, t_3 t_1 \rangle$. Observe from the structure of Cayley graph of G' that the cardinality of n-blocks of X, γ_n, satisfies

$$\begin{cases}
\gamma_{n;1} = \left(\gamma_{n-1;1}^{[\alpha_1]} + \gamma_{n-1;2}^{[\alpha_1]}\right) \gamma_{n-1;2}^{[\alpha_2]} \gamma_{n-1;1}^{[\alpha_3]}, \\
\gamma_{n;2} = \left(\gamma_{n-1;1}^{[\alpha_3]} + \gamma_{n-1;2}^{[\alpha_3]}\right) \gamma_{n-1;2}^{[\alpha_1]} \gamma_{n-1;3}^{[\alpha_2]}, \\
\gamma_{n;3} = \gamma_{n-1;3}^{[\alpha_1]} \gamma_{n-1;3}^{[\alpha_2]} \gamma_{n-1;3}^{[\alpha_3]}, \\
\gamma_{n;1}^{[\alpha_1]} = \left(\gamma_{n-1;1}^{[\alpha_1\alpha_1]} + \gamma_{n-1;2}^{[\alpha_1\alpha_1]}\right) \gamma_{n-1;2}^{[\alpha_1\alpha_2]}, \\
\gamma_{n;1}^{[\alpha_2]} = \left(\gamma_{n-1;1}^{[\alpha_2\alpha_1]} + \gamma_{n-1;2}^{[\alpha_2\alpha_1]}\right) \gamma_{n-1;1}^{[\alpha_2\alpha_3]}, \\
\gamma_{n;1}^{[\alpha_3]} = \gamma_{n-1;2}^{[\alpha_3\alpha_2]} \gamma_{n-1;1}^{[\alpha_3\alpha_3]}, \\
\gamma_{n;2}^{[\alpha_1]} = \gamma_{n-1;2}^{[\alpha_1\alpha_1]} \gamma_{n-1;3}^{[\alpha_1\alpha_2]}, \\
\gamma_{n;2}^{[\alpha_2]} = \left(\gamma_{n-1;1}^{[\alpha_2\alpha_3]} + \gamma_{n-1;2}^{[\alpha_2\alpha_3]}\right) \gamma_{n-1;2}^{[\alpha_2\alpha_1]}, \\
\gamma_{n;2}^{[\alpha_3]} = \left(\gamma_{n-1;1}^{[\alpha_3\alpha_3]} + \gamma_{n-1;2}^{[\alpha_3\alpha_3]}\right) \gamma_{n-1;3}^{[\alpha_3\alpha_2]}, \\
\gamma_{n;3}^{[\alpha_1]} = \gamma_{n-1;3}^{[\alpha_1\alpha_1]} \gamma_{n-1;3}^{[\alpha_1\alpha_2]}, \\
\gamma_{n;3}^{[\alpha_2]} = \gamma_{n-1;3}^{[\alpha_2\alpha_1]} \gamma_{n-1;3}^{[\alpha_2\alpha_3]}, \\
\gamma_{n;3}^{[\alpha_3]} = \gamma_{n-1;3}^{[\alpha_3\alpha_2]} \gamma_{n-1;3}^{[\alpha_3\alpha_3]}, \\
\gamma_{0;i}^{[\alpha_j]} = 1, \quad 1 \leq i, j \leq 3,
\end{cases}$$

where $\alpha_3 = \alpha_1^{-1}$. Since $E_n(\alpha_i\alpha_j) = E_n(\alpha_j)$ for $1 \leq i, j \leq 3$, we can rewrite the system as

$$\begin{cases} \gamma_{n;1} = \left(\gamma_{n-1;1}^{[\alpha_1]} + \gamma_{n-1;2}^{[\alpha_1]}\right)\gamma_{n-1;2}^{[\alpha_2]}\gamma_{n-1;1}^{[\alpha_3]}, \\ \gamma_{n;2} = \left(\gamma_{n-1;1}^{[\alpha_3]} + \gamma_{n-1;2}^{[\alpha_3]}\right)\gamma_{n-1;2}^{[\alpha_1]}\gamma_{n-1;3}^{[\alpha_2]}, \\ \gamma_{n;3} = \gamma_{n-1;3}^{[\alpha_1]}\gamma_{n-1;3}^{[\alpha_2]}\gamma_{n-1;3}^{[\alpha_3]}, \\ \gamma_{n;1}^{[\alpha_1]} = \left(\gamma_{n-1;1}^{[\alpha_1]} + \gamma_{n-1;2}^{[\alpha_1]}\right)\gamma_{n-1;2}^{[\alpha_2]}, \\ \gamma_{n;1}^{[\alpha_2]} = \left(\gamma_{n-1;1}^{[\alpha_1]} + \gamma_{n-1;2}^{[\alpha_1]}\right)\gamma_{n-1;1}^{[\alpha_3]}, \\ \gamma_{n;1}^{[\alpha_3]} = \gamma_{n-1;2}^{[\alpha_2]}\gamma_{n-1;1}^{[\alpha_3]}, \\ \gamma_{n;2}^{[\alpha_1]} = \gamma_{n-1;2}^{[\alpha_1]}\gamma_{n-1;3}^{[\alpha_2]}, \\ \gamma_{n;2}^{[\alpha_2]} = \left(\gamma_{n-1;1}^{[\alpha_3]} + \gamma_{n-1;2}^{[\alpha_3]}\right)\gamma_{n-1;2}^{[\alpha_1]}, \\ \gamma_{n;2}^{[\alpha_3]} = \left(\gamma_{n-1;1}^{[\alpha_3]} + \gamma_{n-1;2}^{[\alpha_3]}\right)\gamma_{n-1;3}^{[\alpha_2]}, \\ \gamma_{n;3}^{[\alpha_1]} = \gamma_{n-1;3}^{[\alpha_1]}\gamma_{n-1;3}^{[\alpha_2]}, \\ \gamma_{n;3}^{[\alpha_2]} = \gamma_{n-1;3}^{[\alpha_1]}\gamma_{n-1;3}^{[\alpha_3]}, \\ \gamma_{n;3}^{[\alpha_3]} = \gamma_{n-1;3}^{[\alpha_2]}\gamma_{n-1;3}^{[\alpha_3]}. \end{cases}$$

Observe that $\mathcal{A}_E = \{1, 2\}$ and $\mathcal{A}_I = \{3\}$. Theorems 4.7 and 5.1 demonstrate that

$$\deg(X) = \ln \frac{1 + \sqrt{5}}{2}.$$

7. Conclusions

In this paper, we consider the topological degree of G-shifts of finite type for the case where G is a finitely generated free group. Topological degree, which is the logarithm of entropy dimension, characterizes zero entropy systems in more details. Since the topological entropy is no longer conjugacy-invariant for shifts over free groups, the conjugacy-invariance of topological degree may be treated as a criterion for determining whether two shift spaces over free group are topological conjugate. After showing that finding topological degree is equivalent to solving a system of nonlinear recurrence equations, we reveal that the topological degree of G-shift of finite type is achieved as the maximal spectral radius of a collection of matrices corresponding to the shift itself.

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Conflict of interest

The authors declare that there are no conflicts of interest related to this study.

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