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Research article

Conformable integral version of Hermite-Hadamard-Fejér inequalities via η-convex functions

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Abstract: The purpose of the article is to use symmetric η-convex functions to develop Hermite-Hadamard-Fejér inequality for conformable integral. We establish several conformable integral versions of Hermite-Hadamard-Fejér type inequality for the η-convex functions by use of an identity linked with Hermite-Hadamard inequality.

Keywords: η-convex function; Hermite-Hadamard inequality; Hermite-Hadamard-Fejér inequality; conformable derivative; conformable integral
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1. Introduction

A real-valued function $h : I \to \mathbb{R}$ is said to be a convex (concave) function of the interval $I \subseteq \mathbb{R}$ if the inequality

$$h(\lambda\kappa_1 + (1-\lambda)\kappa_2) \leq (\geq)\lambda h(\kappa_1) + (1-\lambda)h(\kappa_2)$$

takes place whenever $\kappa_1, \kappa_2 \in I$ and $\lambda \in [0, 1]$.

It is well known that convex (concave) function plays an important role in mathematics due to convexity (concavity) is widely used in all branches of pure and applied mathematics [1–7]. Recently, the generalizations, extensions and invariants for the convex (concave) function have attracted the attention of many researchers, for instance, the quasi-convex function [8], harmonic convex function [9,10], strongly convex function [11–13], two-parameter Hölder mean convex function [14,15], exponentially convex function [16,17], *GG* and *GA* convex functions [18], and s-convex function

[19,20]. In particular, many inequalities can be found in the literature [21–36] by use of properties of the convex (concave) function.

Let $h : I \to \mathbb{R}$ be a convex (concave) function. Then the well known $\mathcal{HH}$ (Hermite-Hadamard) inequality [37–39] states that the double inequality

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right) \le (\ge) \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} h(x)dx \le (\ge) \frac{h(\kappa_1)+h(\kappa_2)}{2} \tag{1.1}$$

is valid for all $\kappa_1, \kappa_2 \in I$ with $\kappa_1 \neq \kappa_2$.

Fejér generalized the $\mathcal{HH}$ inequality (1.1) to the Hermite-Hadamard-Fejér inequality (1.2) as follows:

$$\frac{h(\kappa_1)+h(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2} g(x)dx \ge (\le) \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} g(x)h(x)dx \ge (\le) h\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\kappa_1}^{\kappa_2} g(x)dx \tag{1.2}$$

if $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex (concave) function and $g : [\kappa_1, \kappa_2] \to \mathbb{R}^+$ is symmetric with respect to $(\kappa_1 + \kappa_2)/2$.

The following definition for the η-convex function was introduced by Eshaghi Gordji et al. in [40].

Definition 1.1. (See [40]) Let $\kappa_1, \kappa_2 \in \mathbb{R}$ with $\kappa_1 < \kappa_2$, $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a real-valued function and $\eta : h([\kappa_1, \kappa_2]) \times h([\kappa_1, \kappa_2]) \to \mathbb{R}$ be a two bivariate real-valued function. Then h is said to be η-convex (or convex with respect to η) if the inequality

$$h[s\mu_1 + (1-s)\mu_2] \le h(\mu_2) + s\eta[h(\mu_1), h(\mu_2)] \tag{1.3}$$

holds for all $\mu_1, \mu_2 \in [\kappa_1, \kappa_2]$ and $s \in [0, 1]$.

Let $\eta(\mu_1, \mu_2) = \mu_1 - \mu_2$ in (1.3). Then Definition 1.1 reduces to the definition of usual convex function.

Eshaghi Gordji et al. [40] established a $\mathcal{HH}$ type inequality for the η-convex function.

Theorem 1.1. (See [40]) Let $\kappa_1, \kappa_2 \in \mathbb{R}$ with $\kappa_1 < \kappa_2$, $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a real-valued function and $\eta : h([\kappa_1, \kappa_2]) \times h([\kappa_1, \kappa_2]) \to \mathbb{R}$ be a two bivariate bounded real-valued function. Then one has

$$\begin{aligned} h\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{M_\eta}{2} &\le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} h(x)dx \\ &\le \frac{h(\kappa_1)+h(\kappa_2)}{2} + \frac{\eta(h(\kappa_1), h(\kappa_2)) + \eta(h(\kappa_2), h(\kappa_1))}{4} \\ &\le \frac{h(\kappa_1)+h(\kappa_2)}{2} + \frac{M_\eta}{2} \end{aligned} \tag{1.4}$$

if h is η-convex, where M_η is the upper bound of η on $h([\kappa_1, \kappa_2]) \times h([\kappa_1, \kappa_2])$.

Let $0 < \beta \le 1$, $r > 0$ and $g : [0, \infty) \to \mathbb{R}$ be a real-valued function. Then the conformable derivative $D_\beta(g)(r)$ of order β is defined by

$$D_\beta(g)(r) = \frac{d_\beta g(r)}{d_\beta r} = \lim_{\epsilon\to 0} \frac{g(r + \epsilon r^{1-\beta}) - g(r)}{\epsilon}, \tag{1.5}$$

g is said to be conformable differentiable at r if the limit of (1.5) exists and is finite. The conformal derivative at 0 is defined by $D_\beta(g)(0) = \lim_{r\to 0^+} D_\beta(g)(r)$.

Let $\kappa_1, \kappa_2, \lambda, c \in \mathbb{R}$ be the constants, and h_1 and h_2 be differentiable at $r > 0$. Then the following formulas can be found in the literature [41]

$$\frac{d_\beta}{d_\beta r}\left(r^\lambda\right) = \lambda r^{\lambda-\beta}, \quad \frac{d_\beta}{d_\beta r}(c) = 0,$$

$$\frac{d_\beta}{d_\beta r}\left(\kappa_1 h_1(r) + \kappa_2 h_2(r)\right) = \kappa_1 \frac{d_\beta}{d_\beta r}(h_1(r)) + \kappa_2 \frac{d_\beta}{d_\beta r}(h_2(r)),$$

$$\frac{d_\beta}{d_\beta r}\left(h_1(r)h_2(r)\right) = h_1(r)\frac{d_\beta}{d_\beta r}(h_2(r)) + h_2(r)\frac{d_\beta}{d_\beta r}(h_1(r)),$$

$$\frac{d_\beta}{d_\beta r}\left(\frac{h_1(r)}{h_2(r)}\right) = \frac{h_2(r)\frac{d_\beta}{d_\beta r}(h_1(r)) - h_1(r)\frac{d_\beta}{d_\beta r}(h_2(r))}{(h_2(r))^2},$$

$$\frac{d_\beta}{d_\beta r}\left(h_1(h_2(r))\right) = h_1'(h_2(r))\frac{d_\beta}{d_\beta r}(h_2(r))$$

if h_1 differentiable at $h_2(r)$. In addition,

$$\frac{d_\beta}{d_\beta r}\left(h_1(r)\right) = r^{1-\beta}\frac{d}{dr}(h_1(r))$$

if h_1 is differentiable.

Let $\beta \in (0, 1]$ and $0 \le \kappa_1 < \kappa_2$. Then the function $g : [\kappa_1, \kappa_2] \to \mathbb{R}$ is said to be conformable integrable if

$$\int_{\kappa_1}^{\kappa_2} g(x) d_\beta x = \int_{\kappa_1}^{\kappa_2} g(x) x^{\beta-1} dx$$

exists and is finite. The set of all conformable integrable functions on $[\kappa_1, \kappa_2]$ is denoted by $L_\beta([\kappa_1, \kappa_2])$. Note that

$$I_\beta^{\kappa_1}(h_1)(r) = I_1^{\kappa_1}(r^{\beta-1}h_1) = \int_{\kappa_1}^{r} \frac{h_1(x)}{x^{1-\beta}} dx$$

for all $\beta \in (0, 1]$, where the integral is the usual Riemann improper integral.

For the theory and applications of the conformable integrals and derivatives we recommend the readers to refer the literature [42–50].

Anderson [51] established the conformable integral version of the $\mathcal{HH}$ type inequality

$$\frac{\beta}{\kappa_2^\beta - \kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x) d_\beta x \le \frac{h(\kappa_1) + h(\kappa_2)}{2}$$

if $\beta \in (0, 1]$ and $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ is conformable differentiable such that $D_\beta(h)$ is increasing. Moreover, if h is decreasing on $[\kappa_1, \kappa_2]$, then

$$h\left(\frac{\kappa_1 + \kappa_2}{2}\right) \le \frac{\beta}{\kappa_2^\beta - \kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x) d_\beta x.$$

It is the aim of the article to establish new Hermite-Hadamard-Fejér type inequalities for the η-convex functions via conformable integrals.

2. Hermite-Hadamard-Fejér type inequalities for conformable integrals by using η-convex functions

Theorem 2.1. Let $\kappa_1, \kappa_2 \in \mathbb{R}^+$ with $\kappa_1 < \kappa_2$, $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an η-convex function and symmetric with respect to $\frac{\kappa_1+\kappa_2}{2}$, $\xi : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a nonnegative integrable function. Then the inequality

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x - \frac{M_\eta}{2}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x \le \int_{\kappa_1}^{\kappa_2} h(x)\xi(x)d_\beta x$$

$$\le \frac{h(\kappa_1)+h(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x + \frac{\eta(h(\kappa_1),h(\kappa_2))+\eta(h(\kappa_2),h(\kappa_1))}{4}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x$$

$$\le \frac{h(\kappa_1)+h(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x + \frac{M_\eta}{2}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x \tag{2.1}$$

holds for any $\beta \in (0, 1]$ if η is bounded on $h([\kappa_1, \kappa_2]) \times h([\kappa_1, \kappa_2])$, where M_η is the upper bound of η on $h([\kappa_1, \kappa_2]) \times h([\kappa_1, \kappa_2])$.

Proof. Let $s \in [0, 1]$. Then it follows from the η-convexity and symmetry of h that

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right) = h\left(\frac{s\kappa_1 - s\kappa_1 + \kappa_1 + \kappa_2 + s\kappa_2 - s\kappa_2}{2}\right)$$

$$= h\left(\frac{s\kappa_1 + (1-s)\kappa_2 + s\kappa_2 + (1-s)\kappa_1}{2}\right)$$

$$\le h(s\kappa_2 + (1-s)\kappa_1) + \frac{1}{2}\eta(h(s\kappa_1 + (1-s)\kappa_2)), h(s\kappa_2 + (1-s)\kappa_1)$$

$$\le h(s\kappa_2 + (1-s)\kappa_1) + \frac{1}{2}M_\eta.$$

Let $x = s\kappa_2 + (1-s)\kappa_1$. Then we get

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x = (\kappa_2-\kappa_1)h\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_0^1 \xi(s\kappa_2+(1-s)\kappa_1)(s\kappa_2+(1-s)\kappa_1)^{\beta-1}ds$$

$$\le \int_0^1 h(s\kappa_2+(1-s)\kappa_1)\xi(s\kappa_2+(1-s)\kappa_1)(\kappa_2-\kappa_1)(s\kappa_2+(1-s)\kappa_1)^{\beta-1}ds$$

$$+\frac{M_\eta}{2}\int_0^1 \xi(s\kappa_2+(1-s)\kappa_1)(\kappa_2-\kappa_1)(s\kappa_2+(1-s)\kappa_1)^{\beta-1}ds$$

$$= \int_{\kappa_1}^{\kappa_2} h(x)\xi(x)d_\beta x + \frac{M_\eta}{2}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x,$$

which gives the proof of the first inequality of (2.1).

Next, we prove the second and third inequalities of (2.1). From the η-convexity of h we know that

$$h(s\kappa_1 + (1-s)\kappa_2) \leq h(\kappa_2) + s\eta(h(\kappa_1), h(\kappa_2)) \tag{2.2}$$

and

$$h(s\kappa_2 + (1-s)\kappa_1) \leq h(\kappa_1) + s\eta(h(\kappa_2), h(\kappa_1)). \tag{2.3}$$

Let $x = s\kappa_1 + (1-s)\kappa_2$. Then from the symmetry of h, inequalities (2.2) and (2.3) lead to

$$\int_{\kappa_1}^{\kappa_2} h(x)\xi(x)x^{\beta-1}dx = (\kappa_2-\kappa_1)\int_0^1 h(s\kappa_1+(1-s)\kappa_2)\xi(s\kappa_1+(1-s)\kappa_2)(s\kappa_1+(1-s)\kappa_2)^{\beta-1}ds$$

$$\leq (\kappa_2-\kappa_1)\left[h(\kappa_2)+\frac{1}{2}\eta(h(\kappa_1),h(\kappa_2))\right]\int_0^1 \xi(s\kappa_1+(1-s)\kappa_2)(s\kappa_1+(1-s)\kappa_2)^{\beta-1}ds \tag{2.4}$$

and

$$\int_{\kappa_1}^{\kappa_2} h(x)\xi(x)x^{\beta-1}dx = (\kappa_2-\kappa_1)\int_0^1 h(s\kappa_1+(1-s)\kappa_2)\xi(s\kappa_1+(1-s)\kappa_2)(s\kappa_1+(1-s)\kappa_2)^{\beta-1}ds$$

$$= (\kappa_2-\kappa_1)\int_0^1 h(s\kappa_2+(1-s)\kappa_1)\xi(s\kappa_1+(1-s)\kappa_2)(s\kappa_1+(1-s)\kappa_2)^{\beta-1}ds$$

$$\leq (\kappa_2-\kappa_1)\left[h(\kappa_1)+\frac{1}{2}\eta(h(\kappa_2),h(\kappa_1))\right]\int_0^1 \xi(s\kappa_1+(1-s)\kappa_2)(s\kappa_1+(1-s)\kappa_2)^{\beta-1}ds \tag{2.5}$$

Adding (2.4) and (2.5), and letting $x = s\kappa_1 + (1-s)\kappa_2$, we obtain

$$2\int_{\kappa_1}^{\kappa_2} h(x)\xi(x)d_\beta x \leq (\kappa_2-\kappa_1)(h(\kappa_1)+h(\kappa_2))\int_0^1 \xi(s\kappa_1+(1-s)\kappa_2)(s\kappa_1+(1-s)\kappa_2)^{\beta-1}ds$$

$$+(\kappa_2-\kappa_1)\frac{(\eta(h(\kappa_1),h(\kappa_2))+\eta(h(\kappa_2),h(\kappa_1)))}{2}\int_0^1 \xi(s\kappa_1+(1-s)\kappa_2)(s\kappa_1+(1-s)\kappa_2)^{\beta-1}ds$$

and

$$\int_{\kappa_1}^{\kappa_2} h(x)\xi(x)d_\beta x \leq \frac{h(\kappa_1)+h(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x + \frac{\eta(h(\kappa_1),h(\kappa_2))+\eta(h(\kappa_2),h(\kappa_1))}{4}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x$$

$$\leq \frac{h(\kappa_1)+h(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x + \frac{M_\eta}{2}\int_{\kappa_1}^{\kappa_2}\xi(x)d_\beta x.$$

□

Corollary 2.1. Let $\xi(x) = 1$. Then inequality (2.1) becomes

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{M_\eta}{2} \le \frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int\limits_{\kappa_1}^{\kappa_2} h(x)d_\beta x$$

$$\le \frac{h(\kappa_1)+h(\kappa_2)}{2}+\frac{\eta(h(\kappa_1),h(\kappa_2))+\eta(h(\kappa_2),h(\kappa_1))}{4}$$

$$\le \frac{h(\kappa_1)+h(\kappa_2)}{2}+\frac{M_\eta}{2}.$$

3. Further generalizations of $\mathcal{HH}$ inequality

In order to establish our main results, we need a key lemma which we present in this section.

Lemma 3.1. Let $\kappa_1, \kappa_2 \in \mathbb{R}^+$ with $\kappa_1 < \kappa_2$, and $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a differentiable function on (κ_1, κ_2) such that $D_\beta(h) \in L_\beta([\kappa_1, \kappa_2])$. Then the identity

$$\frac{h(\kappa_1)+h(\kappa_2)}{2}-\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x$$

$$= \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\left[\int_0^1 \left(((1-s)\kappa_1+s\kappa_2)^{2\beta-1}-\kappa_1^\beta((1-s)\kappa_1+s\kappa_2)^{\beta-1}\right)\right.$$

$$\times D_\beta(h)((1-s)\kappa_1+s\kappa_2)s^{1-\beta}d_\beta s+\int_0^1 \left(((1-s)\kappa_2+s\kappa_1)^{2\beta-1}-d_2^\beta((1-s)\kappa_2+s\kappa_1)^{\beta-1}\right)$$

$$\left.\times D_\beta(h)((1-s)\kappa_2+s\kappa_1)s^{1-\beta}d_\beta s\right].$$

holds for $\beta \in (0, 1]$.

Proof. Integration by parts, we have

$$\int_0^1 (((1-s)\kappa_1+s\kappa_2)^{2\beta-1}-\kappa_1^\beta((1-s)\kappa_1+s\kappa_2)^{\beta-1})D_\beta(h)((1-s)\kappa_1+s\kappa_2)ds$$

$$+\int_0^1 \left(((1-s)\kappa_2+s\kappa_1)^{2\beta-1}-\kappa_2^\beta((1-s)\kappa_2+s\kappa_1)^{\beta-1}\right)D_\beta(h)((1-s)\kappa_2+s\kappa_1)ds$$

$$=\int_0^1 \left(((1-s)\kappa_1+s\kappa_2)^\beta-\kappa_1^\beta\right)h'((1-s)\kappa_1+s\kappa_2)ds$$

$$+\int_0^1 \left(((1-s)\kappa_2+s\kappa_1)^\beta-\kappa_2^\beta\right)h'((1-s)\kappa_2+s\kappa_1)ds$$

$$=\left[\left(((1-s)\kappa_1+s\kappa_2)^\beta-\kappa_1^\beta\right)\frac{h((1-s)\kappa_1+s\kappa_2)}{\kappa_2-\kappa_1}\right|_0^1$$

$$-\int_0^1 \beta((1-s)\kappa_1+\kappa_2)^{\beta-1}(\kappa_2-a_1)\frac{h((1-s)\kappa_1+s\kappa_2)}{\kappa_2-\kappa_1}ds\Bigg]$$

$$+\Bigg[\left(((1-s)\kappa_2+s\kappa_1)^\beta-\kappa_2^\beta\right)\frac{h((1-s)\kappa_2+s\kappa_1)}{\kappa_1-\kappa_2}\Bigg|_0^1$$

$$-\int_0^1 \beta((1-s)\kappa_2+s\kappa_1)^{\beta-1}(\kappa_1-\kappa_2)\frac{h((1-s)\kappa_2+s\kappa_1)}{\kappa_1-\kappa_2}ds\Bigg]$$

$$=\left[\frac{\kappa_2^\beta-\kappa_1^\beta}{\kappa_2-\kappa_1}h(\kappa_2)-\frac{\beta}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right]$$

$$+\left[\frac{\kappa_2^\beta-\kappa_1^\beta}{\kappa_2-\kappa_1}h(\kappa_1)-\frac{\beta}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right]$$

$$=\frac{\kappa_2^\beta-\kappa_1^\beta}{\kappa_2-\kappa_1}(h(\kappa_1)+h(\kappa_2))-\frac{2\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x.$$

□

Theorem 3.1. Let $\kappa_1,\kappa_2\in\mathbb{R}^+$ with $\kappa_1<\kappa_2$, and $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a differentiable function on (κ_1,κ_2) such that $D_\beta(h)\in L_\beta([\kappa_1,\kappa_2])$. Then the inequality

$$\left|\frac{h(\kappa_1)+h(\kappa_2)}{2}-\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|$$

$$\leq\frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\Bigg[\left(\frac{\kappa_1^{\beta-1}\kappa_2+\kappa_2^{\beta-1}\kappa_1+2\kappa_2^\beta-4\kappa_1^\beta}{6}\right)|h'(\kappa_1)|+\eta(|h'(\kappa_2)|,|h'(\kappa_1)|)$$

$$\times\left(\frac{\kappa_1^{\beta-1}\kappa_2+\kappa_2^{\beta-1}\kappa_1+3\kappa_2^\beta-5\kappa_1^\beta}{12}\right)+|h'(\kappa_2)|\left(\frac{\kappa_2^\beta-\kappa_1^\beta}{2}\right)+\eta(|h'(\kappa_1)|,|h'(\kappa_2)|)\left(\frac{\kappa_2^\beta-\kappa_1^\beta}{6}\right)\Bigg]\tag{3.1}$$

holds for $\beta\in(0,1]$ if $|h'|$ is η-convex.

Proof. Let $y>0$, $\varphi_1(y)=y^{\beta-1}$ and $\varphi_2(y)=-y^\beta$. Then we clearly see that both the functions φ_1 and φ_2 are convex. It follows from Lemma 3.1 and the convexity of φ_1 and φ_2 together with the η-convexity of $|h'|$ that

$$\left|\frac{h(\kappa_1)+h(\kappa_2)}{2}-\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|$$

$$\leq\frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\Bigg[\int_0^1\left(((1-s)\kappa_1+s\kappa_2)^\beta-\kappa_1^\beta\right)|h'((1-s)\kappa_1+s\kappa_2)|ds$$

$$+\int_0^1\left(((1-s)\kappa_2+s\kappa_1)^\beta-\kappa_2^\beta\right)|h'((1-s)\kappa_2+s\kappa_1)|ds\Bigg]$$

$$=\frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\Bigg[\int_0^1\left(((1-s)\kappa_1+s\kappa_2)^{\beta+1-1}-\kappa_1^\beta\right)|h'((1-s)\kappa_1+s\kappa_2)|ds$$

$$+\int_0^1 \left(\kappa_2^\beta - ((1-s)\kappa_2 + s\kappa_1)^\beta\right)|h'((1-s)\kappa_2 + s\kappa_1)|ds\Bigg]$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta - \kappa_1^\beta)}\Bigg[\int_0^1 \left(((1-s)\kappa_1 + s\kappa_2)^{\beta-1}((1-s)\kappa_1 + s\kappa_2) - \kappa_1^\beta\right)|h'((1-s)\kappa_1 + s\kappa_2)|ds$$

$$+\int_0^1 \left(\kappa_2^\beta - \left((1-s)\kappa_2^\beta + s\kappa_1^\beta\right)\right)|h'((1-s)\kappa_2 + s\kappa_1)|ds\Bigg]$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta - \kappa_1^\beta)}\Bigg[\int_0^1 \left(\left((1-s)\kappa_1^{\beta-1} + s\kappa_2^{\beta-1}\right)((1-s)\kappa_1 + s\kappa_2) - \kappa_1^\beta\right)|h'((1-s)\kappa_1 + s\kappa_2)|ds$$

$$+\int_0^1 \left(\kappa_2^\beta - \left((1-s)\kappa_2^\beta + s\kappa_1^\beta\right)\right)|h'((1-s)\kappa_2 + s\kappa_1)|ds\Bigg]$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta - \kappa_1^\beta)}\Bigg[\int_0^1 \left(\left((1-s)\kappa_1^{\beta-1} + s\kappa_2^{\beta-1}\right)((1-s)\kappa_1 + s\kappa_2) - \kappa_1^\beta\right)$$

$$\times\left[|h'(\kappa_1)| + s\eta(|h'(\kappa_2)|, |h'(\kappa_1)|)\right]ds + \int_0^1 \left(\kappa_2^\beta - \left((1-s)\kappa_2^\beta + s\kappa_1^\beta\right)\right)$$

$$\times\left[|h'(\kappa_2)| + s\eta(|h'(\kappa_1)|, |h'(\kappa_2)|)\right]ds\Bigg]$$

and

$$\left|\frac{h(\kappa_1)+h(\kappa_2)}{2} - \frac{\beta}{\kappa_2^\beta - \kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x\right|$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta - \kappa_1^\beta)}\Bigg[\left(\frac{\kappa_1^{\beta-1}\kappa_2 + \kappa_2^{\beta-1}\kappa_1 + 2\kappa_2^\beta - 4\kappa_1^\beta}{6}\right)|h'(\kappa_1)|$$

$$+\eta(|h'(\kappa_2)|, |h'(\kappa_1)|)\left(\frac{\kappa_1^{\beta-1}\kappa_2 + \kappa_2^{\beta-1}\kappa_1 + 3\kappa_2^\beta - 5\kappa_1^\beta}{12}\right)$$

$$+|h'(\kappa_2)|\left(\frac{\kappa_2^\beta - \kappa_1^\beta}{2}\right) + \eta(|h'(\kappa_1)|, |h'(\kappa_2)|)\left(\frac{\kappa_2^\beta - \kappa_1^\beta}{6}\right)\Bigg].$$

□

Corollary 3.1. Let $\eta(\kappa_2, \kappa_1) = \kappa_2 - \kappa_1$. Then inequality (3.1) reduces to

$$\left|\frac{h(\kappa_1)+h(\kappa_2)}{2} - \frac{\beta}{\kappa_2^\beta - \kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x\right|$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta - \kappa_1^\beta)}\Bigg[\left(\frac{\kappa_1^{\beta-1}\kappa_2 + \kappa_2^{\beta-1}\kappa_1 + 3a_2^\beta - 5\kappa_1^\beta}{6}\right)|h'(\kappa_1)|$$

$$+|h'(\kappa_2)|\left(\frac{7\kappa_2^\beta - 9\kappa_1^\beta + \kappa_1^{\beta-1}\kappa_2 + \kappa_2^{\beta-1}\kappa_1}{12}\right)\Bigg].$$

Theorem 3.2. Let $q > 1$, $\kappa_1, \kappa_2 \in \mathbb{R}^+$ with $\kappa_1 < \kappa_2$, and $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a differentiable function on (κ_1, κ_2) such that $D_\beta(h) \in L_\beta([\kappa_1, \kappa_2])$. Then the inequality

$$\left|\frac{h(\kappa_1)+h(\kappa_2)}{2} - \frac{\beta}{\kappa_2^\beta - \kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x\right|$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\Bigg[(A_1(\beta))^{1-\frac{1}{q}}\left(|h'(\kappa_1)|^q\left(\frac{\kappa_1^{\beta-1}\kappa_2+\kappa_2^{\beta-1}\kappa_1+2\kappa_2^\beta-4\kappa_1^\beta}{6}\right)\right.$$

$$\left.+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)\left(\frac{\kappa_1^{\beta-1}\kappa_2+\kappa_2^{\beta-1}\kappa_1+3\kappa_2^\beta-5\kappa_1^\beta}{12}\right)\right)+(B_1(\beta))^{1-\frac{1}{q}}$$

$$\times\left(|h'(\kappa_2)|^q\left(\frac{\kappa_2^\beta-\kappa_1^\beta}{2}\right)+\eta(|h'(\kappa_1)|^q,|h'(\kappa_2)|^q)\left(\frac{\kappa_2^\beta-\kappa_1^\beta}{3}\right)\right)\Bigg]$$

takes place for $\beta \in (0, 1]$ if $|h'|^q$ is η-convex, where

$$A_1(\beta) = \frac{2\kappa_1^\beta+\kappa_1^{\beta-1}\kappa_2+\kappa_2^{\beta-1}\kappa_1+2\kappa_2^\beta-6\kappa_1^\beta}{6}, \quad B_1(\beta) = \frac{\kappa_2^\beta-\kappa_1^\beta}{2}.$$

Proof. We clearly see that

$$\left|\frac{h(\kappa_1)+h(\kappa_2)}{2} - \frac{\beta}{\kappa_2^\beta - \kappa_1^\beta}\int_{\kappa_1}^{\kappa_2} h(x)d_\beta x\right|$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\Bigg[\int_0^1\left(((1-s)\kappa_1+s\kappa_2)^{\beta-1}((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)|h'((1-s)\kappa_1+s\kappa_2)|ds$$

$$+\int_0^1\left(\kappa_2^\beta-\left((1-s)\kappa_2^\beta+s\kappa_1^\beta\right)\right)|h'((1-s)\kappa_2+s\kappa_1)|ds\Bigg].$$

It follows from the power-mean inequality that

$$\int_0^1\left(\left((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}\right)((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)|h'((1-s)\kappa_1+s\kappa_2)|ds$$

$$\leq\left(\int_0^1\left(\left((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}\right)((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)ds\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left(\left((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}\right)((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)|h'\left((1-s)\kappa_1+s\kappa_2\right)|^q ds\right)^{\frac{1}{q}}$$

and

$$\int_0^1\left(\kappa_2^\beta\left((1-s)\kappa_2^\beta+s\kappa_1^\beta\right)\right)|h'((1-s)\kappa_2+s\kappa_1)|\,ds$$

$$\leq\left(\int_0^1\left(\kappa_2^\beta-\left((1-s)\kappa_2^\beta+s\kappa_1^\beta\right)\right)ds\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left(\kappa_2^\beta - \left((1-s)\kappa_2^\beta + s\kappa_1^\beta\right)\right) |h'((1-s)\kappa_2 + s\kappa_1)|^q \, ds\right)^{\frac{1}{q}}.$$

Making use of the η-convexity of $|h'|^q$ and the facts that

$$\int_0^1 \left(\left((1-s)\kappa_1^{\beta-1} + s\kappa_2^{\beta-1}\right)((1-s)\kappa_1 + s\kappa_2) - \kappa_1^\beta\right) dt$$

$$= A_1(\beta) = \frac{2\kappa_1^\beta + \kappa_1^{\beta-1}\kappa_2 + \kappa_2^{\beta-1}\kappa_1 + 2\kappa_2^\beta - 6\kappa_1^\beta}{6}$$

and

$$\int_0^1 \left(\kappa_2^\beta - \left((1-s)\kappa_2^\beta + s\kappa_1^\beta\right)\right) ds = B_1(\beta) = \frac{\kappa_2^\beta - \kappa_1^\beta}{2},$$

we get

$$\int_0^1 \left(\left((1-s)\kappa_1^{\beta-1} + s\kappa_2^{\beta-1}\right)((1-s)\kappa_1 + s\kappa_2) - \kappa_1^\beta\right) |h'((1-s)\kappa_1 + s\kappa_2)|^q ds$$

$$\leq \int_0^1 \left(\left((1-s)\kappa_1^{\beta-1} + s\kappa_2^{\beta-1}\right)((1-s)\kappa_1 + s\kappa_2) - \kappa_1^\beta\right) \left[|h'(\kappa_1)|^q + s\eta(|h'(\kappa_2)|^q, |h'(\kappa_1)|^q)\right] ds$$

$$= |h'(\kappa_1)|^q \int_0^1 \left(\left((1-s)\kappa_1^{\beta-1} + s\kappa_2^{\beta-1}\right)((1-s)\kappa_1 + s\kappa_2) - \kappa_1^\beta\right) ds$$

$$+\eta(|h'(\kappa_2)|^q, |h'(\kappa_1)|^q) \int_0^1 \left(\left((1-s)\kappa_1^{\beta-1} + s\kappa_2^{\beta-1}\right)((1-s)\kappa_1 + s\kappa_2) - \kappa_1^\beta\right) s ds$$

$$= |h'(\kappa_1)|^q \left(\frac{\kappa_1^{\beta-1}\kappa_2 + \kappa_2^{\beta-1}\kappa_1 + 2\kappa_2^\beta - 4\kappa_1^\beta}{6}\right) + \eta(|h'(\kappa_2)|^q, |h'(\kappa_1)|^q)$$

$$\times\left(\frac{\kappa_1^{\beta-1}\kappa_2 + \kappa_2^{\beta-1}\kappa_1 + 3\kappa_2^\beta - 5\kappa_1^\beta}{12}\right)$$

and

$$\int_0^1 \left(\kappa_2^\beta - \left((1-s)\kappa_2^\beta + s\kappa_1^\beta\right)\right) |h'((1-s)\kappa_2 + s\kappa_1)|^q \, ds$$

$$\leq \int_0^1 \left(\kappa_2^\beta - \left((1-s)\kappa_2^\beta + s\kappa_1^\beta\right)\right) \left[|h'(\kappa_2)|^q + s\eta\left(|h'(\kappa_1)|^q, |h'(\kappa_2)|^q\right)\right] ds$$

$$= |h'(\kappa_2)|^q \left(\frac{\kappa_2^\beta - \kappa_1^\beta}{2}\right) + \eta\left(|h'(\kappa_1)|^q, |h'(\kappa_2)|^q\right) \left(\frac{\kappa_2^\beta - \kappa_1^\beta}{3}\right),$$

which completes the proof of Theorem 3.2. □

Corollary 3.2. Let $\eta(\kappa_2, \kappa_1) = \kappa_2 - \kappa_1$. Then Theorem 3.2 leads to the conclusion that

$$\left|\frac{h(\kappa_1) + h(\kappa_2)}{2} - \frac{\beta}{\kappa_2^\beta - \kappa_1^\beta} \int_{\kappa_1}^{\kappa_2} h(x) d_\beta x\right|$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\Bigg[(A_1(\beta))^{1-\frac{1}{q}}\left(|h'(\kappa_1)|^q\left(\frac{\kappa_1^{\beta-1}\kappa_2+\kappa_2^{\beta-1}\kappa_1+\kappa_2^\beta-3\kappa_1^\beta}{6}\right)+|h'(\kappa_2)|^q\right.$$

$$\left.\times\left(\frac{\kappa_1^{\beta-1}\kappa_2+\kappa_2^{\beta-1}\kappa_1+3\kappa_2^\beta-5\kappa_1^\beta}{12}\right)\right)+(B_1(\beta))^{1-\frac{1}{q}}\left(|h'(\kappa_2)|^q\left(\frac{\kappa_2^\beta-\kappa_1^\beta}{6}\right)+|h'(\kappa_1)|^q\left(\frac{\kappa_2^\beta-\kappa_1^\beta}{3}\right)\right)\Bigg].$$

Theorem 3.3. Let $p,q>1$ with $1/p+1/q=1$, $\kappa_1,\kappa_2\in\mathbb{R}^+$ with $\kappa_1<\kappa_2$, and $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a differentiable function on (κ_1,κ_2) such that $D_\beta(h)\in L_\beta([\kappa_1,\kappa_2])$. Then the inequality

$$\left|\frac{h(\kappa_1)+h(\kappa_2)}{2}-\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\Bigg[(\mathcal{A}_1(\beta,p))^{\frac{1}{p}}\left(\frac{2|h'(\kappa_1)|^q+\eta(|h'(\kappa_2)|^q,|h'(\kappa_1)|^q)}{2}\right)^{\frac{1}{q}}$$

$$+(\mathcal{A}_2(\beta,p))^{\frac{1}{p}}\left(\frac{2|h'(\kappa_2)|^q+\eta(|h'(\kappa_1)|^q,|h'(\kappa_2)|^q)}{2}\right)^{\frac{1}{q}}\Bigg]$$

is valid for $\beta\in(0,1]$ if $|h'|^q$ is η-convex, where

$$\mathcal{A}_1(\beta,p)=\int_0^1\left(\left((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}\right)((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)^p ds$$

and

$$\mathcal{A}_2(\beta,p)=\int_0^1\left(\kappa_2^\beta-\left((1-s)\kappa_2^\beta+s\kappa_1^\beta\right)\right)^p ds.$$

Proof. We clearly see that

$$\left|\frac{h(\kappa_1)+h(\kappa_2)}{2}-\frac{\beta}{\kappa_2^\beta-\kappa_1^\beta}\int_{\kappa_1}^{\kappa_2}h(x)d_\beta x\right|$$

$$\leq \frac{(\kappa_2-\kappa_1)}{2(\kappa_2^\beta-\kappa_1^\beta)}\Bigg[\int_0^1\left(((1-s)\kappa_1+s\kappa_2)^{\beta-1}((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)|h'((1-s)\kappa_1+s\kappa_2)|ds$$

$$+\int_0^1\left(\kappa_2^\beta-\left((1-s)\kappa_2^\beta+s\kappa_1^\beta\right)\right)|h'((1-s)\kappa_2+s\kappa_1)|ds\Bigg].$$

Making use of Hölder inequality, we have

$$\int_0^1\left(\left((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}\right)((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)|h'((1-s)\kappa_1+s\kappa_2)|ds$$

$$\leq\left(\int_0^1\left(\left((1-s)\kappa_1^{\beta-1}+s\kappa_2^{\beta-1}\right)((1-s)\kappa_1+s\kappa_2)-\kappa_1^\beta\right)^p ds\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1|h'((1-s)\kappa_1+s\kappa_2)|^q ds\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \left(\left((1-s)\kappa_1^{\beta-1} + s\kappa_2^{\beta-1} \right) ((1-s)\kappa_1 + s\kappa_2) - \kappa_1^{\beta} \right)^p ds \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 |h'(\kappa_1)|^q + s\eta(|h'(\kappa_2)|^q, |h'(\kappa_1)|^q) ds \right)^{\frac{1}{q}}$$

$$= (\mathcal{A}_1(\beta, p))^{\frac{1}{p}} \left(\frac{2|h'(\kappa_1)|^q + \eta(|h'(\kappa_2)|^q, |h'(\kappa_1)|^q)}{2} \right)^{\frac{1}{q}}$$

and

$$\int_0^1 \left(\kappa_2^{\beta} - \left((1-s)\kappa_2^{\beta} + s\kappa_1^{\beta} \right) \right) |h'((1-s)\kappa_2 + s\kappa_1)| \, ds$$

$$\leq \left(\int_0^1 \left(\kappa_2^{\beta} - \left((1-s)\kappa_2^{\beta} + s\kappa_1^{\beta} \right) \right)^p ds \right)^{\frac{1}{p}} (|h'((1-s)\kappa_2 + s\kappa_1)|^q \, ds)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \left(\kappa_2^{\beta} - \left((1-s)\kappa_2^{\beta} + s\kappa_1^{\beta} \right) \right)^p ds \right)^{\frac{1}{p}} \left(\int_0^1 (|h'(\kappa_2)|^q + s\eta \left(|h'(\kappa_1)|^q, |h'(\kappa_2)|^q \right)) \, ds \right)^{\frac{1}{q}}$$

$$= (\mathcal{A}_2(\beta, p))^{\frac{1}{p}} \left(\frac{2|h'(\kappa_2)|^q + \eta(|h'(\kappa_1)|^q, |h'(\kappa_2)|^q)}{2} \right)^{\frac{1}{q}}.$$

□

Corollary 3.3. Let $\eta(\kappa_2, \kappa_1) = \kappa_2 - \kappa_1$. Then Theorem 3.3 leads to

$$\left| \frac{h(\kappa_1) + h(\kappa_2)}{2} - \frac{\beta}{\kappa_2^{\beta} - \kappa_1^{\beta}} \int_{\kappa_1}^{\kappa_2} h(x) d_{\beta} x \right|$$

$$\leq \frac{(\kappa_2 - \kappa_1)}{2(\kappa_2^{\beta} - \kappa_1^{\beta})} \left[(\mathcal{A}_1(\beta, p))^{\frac{1}{p}} \left(\frac{|h'(\kappa_1)|^q + |h'(\kappa_2)|^q}{2} \right)^{\frac{1}{q}} + (\mathcal{A}_2(\beta, p))^{\frac{1}{p}} \left(\frac{|h'(\kappa_2)|^q + |h'(\kappa_1)|^q}{2} \right)^{\frac{1}{q}} \right].$$

4. Conclusion

We have generalized the Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities for convex functions to the η-convex functions via the conformable integral. Our obtained results are the improvements and generalizations of some previous known results, our ideas and approach may lead to a lot of follow-up research.

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Conflict of interest

The authors declare no conflict of interest.

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