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Research article

Oscillatory and asymptotic behavior of third-order neutral delay differential equations with distributed deviating arguments

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Abstract: This paper examines the oscillatory and asymptotic properties of a class of third-order neutral delay differential equations with distributed deviating arguments. A series of new oscillation criteria are presented under some more relaxed conditions by using the Riccati transformation technique. The results obtained here improve and complement that in the literature. At last, two examples are provided to illustrate the main results.

Keywords: oscillatory property; asymptotic behavior; neutral delay differential equations; distributed deviating arguments; Riccati transformation
Mathematics Subject Classification: 34K11

1. Introduction

The purpose of this work is to investigate the oscillatory and asymptotic behavior of the third-order neutral delay differential equations with distributed deviating arguments

$$[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}]' + \int_a^b q(t,\xi)f(x(\sigma(t,\xi)))d\xi = 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $t \geq t_0 > 0$, $0 \leq a < b$. We also assume that the following conditions are satisfied:

- (H1) $r_1(t)$, $r_2(t)$, $p(t) \in C([t_0,\infty),\mathbb{R})$, $q(t,\xi) \in C([t_0,\infty) \times [a,b], [0,\infty))$, $r_1(t) > 0$, $r_2(t) > 0$ and $p(t) \geq 1$ with $p(t) \not\equiv 1$;
- (H2) $\tau(t) \in C([t_0,\infty),\mathbb{R})$ is invertible, $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$;
- (H3) $\sigma(t,\xi) \in C([t_0,\infty)\times[a,b],\mathbb{R})$ is non-increasing for ξ and $\liminf_{t\to\infty} \sigma(t,\xi) = \infty$ for $\xi \in [a,b]$;
- (H4) $f(x) \in C(\mathbb{R},\mathbb{R})$ is assumed to satisfy $xf(x) > 0$ and there exists a positive constant K such that

$$\frac{f(x)}{x^{\alpha_3}} \geq K \text{ for any variable } x \neq 0;$$

- (H5) α_i, $i = 1,\ 2,\ 3$ are ratios of positive odd integers.

By a solution of Eq. (1.1) we mean a function $x(t) \in C([T_y, \infty), \mathbb{R})$, $T_y \geq t_0$, which has $z(t)$, $r_2(t)(z'(t))^{\alpha_2}$, $r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1} \in C^1([T_y, \infty), \mathbb{R})$ and satisfies (1.1) on $[T_y, \infty)$. A solution $x(t)$ of (1.1) is said to be *proper* if it exists on the interval $[T_y, \infty)$ and satisfies the condition

$$\sup\{|x(t)| : T \leq t < \infty\} > 0 \text{ for any } T \geq T_y.$$

Our attention is restricted to these solutions and we make the standing hypothesis that (1.1) admits such a solution. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros on $[T_y, \infty)$ and otherwise it is called *non-oscillatory*, i.e., the solution is positive or negative eventually. Eq. (1.1) is said to be *oscillatory* if all its solutions are oscillatory.

Main results of this paper are organized into three parts in accordance with different assumptions on the coefficients $r_1(t)$ and $r_2(t)$. In Section 2, oscillation results of (1.1) are established in the case

$$\int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha_1}}(t)dt = \infty, \quad \int_{t_0}^{\infty} r_2^{-\frac{1}{\alpha_2}}(t)dt = \infty. \tag{1.2}$$

In Section 3, some new oscillation criteria for (1.1) are obtained in the case

$$\int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha_1}}(t)dt < \infty, \quad \int_{t_0}^{\infty} r_2^{-\frac{1}{\alpha_2}}(t)dt = \infty. \tag{1.3}$$

By assuming that

$$\int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha_1}}(t)dt < \infty, \quad \int_{t_0}^{\infty} r_2^{-\frac{1}{\alpha_2}}(t)dt < \infty. \tag{1.4}$$

Some oscillation theorems of (1.1) are given in Section 4. In order to illustrate the results reported in Sections 2, 3 and 4, we present some examples in Section 5.

During the last few decades, analysis of the oscillation and asymptotic behavior of solutions of third-order differential equations, difference equations and dynamic equations on time scales have experienced long-term interest and we refer the reader to the papers [1–12]. Due to the huge advantage of neutral differential equations in describing several neutral phenomena, there is of great scientific and academic values theoretically and practically for studying neutral differential equations. Hence, a large amount of research attention has been focused on the oscillation problem of third-order linear and nonlinear neutral differential equations in resent years; see, for example [13–17], and the references are cited therein.

The third-order neutral differential equation

$$\left[r(t)((x(t) + p(t)x(\tau(t)))'')^{\gamma}\right]' + q(t)f(x(\sigma(t))) = 0,$$

and its special cases have been studied by Şenel and Utku [8], Baculíková and Džurina [13], Jiang et al. [14, 15], where $\int_{t_0}^{\infty} r^{-\frac{1}{\gamma}}(t)dt = \infty$, $0 \leq p(t) \leq P < 1$. Candan [11], Došlá and Liška [16], and Li et al. [17] established some sufficient conditions for oscillation of the following class of third-order neutral differential equations

$$\left[a(t)(b(t)(x(t) + p(t)x(\tau(t)))')'\right]' + q(t)f(x(\sigma(t))) = 0,$$

where [11] and [16] only considered the conditions $0 \le p(t) \le P < 1$ and

$$\int_{t_0}^{\infty} \frac{1}{a(t)}dt = \int_{t_0}^{\infty} \frac{1}{b(t)}dt = \infty,$$

and [17] also studied the cases

$$\int_{t_0}^{\infty} \frac{1}{a(t)}dt < \infty,\ \int_{t_0}^{\infty} \frac{1}{b(t)}dt = \infty \text{ and } \int_{t_0}^{\infty} \frac{1}{a(t)}dt = \int_{t_0}^{\infty} \frac{1}{b(t)}dt < \infty.$$

Recently, there has been an increasing interest in studying the oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments [18–24]. Elabbasy and Moaaz [19] considered the special cases of (1.1) and obtained several oscillation results under the assumption (1.2) and $0 \le p(t) \le P < 1$. By using a new method different from the existing results, Tunç [22] established some new oscillation criteria for

$$\Big(r(t)\Big(\big(x(t) + p(t)x(\tau(t))\big)''\Big)^{\alpha}\Big)' + \int_a^b q(t,\xi)x^{\alpha}(\phi(t,\xi))d\xi = 0,$$

where $p(t) \ge 1$, and the obtained results greatly enriched the oscillation theory.

It is clear that the above introduced equations are the special cases of (1.1), i.e., (1.1) can be transformed into these equations by letting the corresponding parameters being 1. To the best of our knowledge, there are few results in the literature which ensure that all solutions are either oscillatory or tends to zero monotonically for the third-order neutral differential equations with distributed deviating arguments under the conditions $p(t) \ge 1$, (1.3) or (1.4) holds, and the above mentioned results are inapplicable to these conditions. Motivated by Li et al. [17], Elabbasy and Moaaz [19] and Tunç [22], we consider Eq. (1.1) which is not studied in the past and utilize the Riccati transformation technique to establish several oscillation criteria for (1.1) by assume that $p(t) \ge 1$, (1.2), (1.3) or (1.4) holds. The results obtained in this paper improve and complement the related criteria reported in [13, 16, 17, 19, 22]. All functional inequalities considered here are assumed to hold eventually, that is, they are satisfied for all t large enough.

In the sequel, we use the following notations for a compact presentation of our results:

$$\begin{aligned}
&\sigma_1(t) = \sigma(t,a),\ \sigma_2(t) = \sigma(t,b),\\
&z'(\tau(t)) = (z(\tau(t)))',\ \rho'_+(t) = \max\{0, \rho'(t)\},\\
&\delta_1(t,t_1) = \int_{t_1}^{t} r_1^{-\frac{1}{\alpha_1}}(s)ds,\ \delta_2(t,t_1) = \Big(\frac{\delta_1(t,t_1)}{r_2(t)}\Big)^{\frac{1}{\alpha_2}},\\
&\delta_3(t,t_1) = \int_{t_1}^{t} \delta_2(s,t_1)ds,\ t \ge t_1,
\end{aligned}$$

where $\rho(t)$ will be explained later and t_1 is sufficiently large with $t_1 \ge t_0$. Furthermore, assume that

$$p_1(t) = \frac{1}{p(\tau^{-1}(t))}\Big(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\Big) > 0, \tag{1.5}$$

$$p_2(t) = \frac{1}{p(\tau^{-1}(t))}\Big(1 - \frac{\delta_3(\tau^{-1}(\tau^{-1}(t)), t_1)}{p(\tau^{-1}(\tau^{-1}(t)))\delta_3(\tau^{-1}(t), t_1)}\Big) > 0, \tag{1.6}$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. Then let

$$q_1(t) = K\int_a^b q(t,\xi)p_1^{\alpha_3}(\sigma(t,\xi))d\xi,\ \ q_2(t) = K\int_a^b q(t,\xi)p_2^{\alpha_3}(\sigma(t,\xi))d\xi.$$

We will present the main contribution of this paper as follows.

2. Oscillation criteria for the case (1.2)

In this section, we respectively consider the following two cases

$$\sigma(t,\xi)\le\tau(t),\ \ \xi\in[a,b], \tag{2.1}$$

and

$$\sigma(t,\xi)\ge\tau(t),\ \ \xi\in[a,b]. \tag{2.2}$$

We now begin with the case when (2.1) holds.

Theorem 2.1. *Assume that conditions (H1)–(H5),* (1.2), (1.5), (1.6) *and* (2.1) *hold. Furthermore, assume that there exists a function* $\rho(t)\in C^1([t_0,\infty),(0,\infty))$ *such that for sufficiently large* $t_*>t_2>t_1\ge t_0$,

$$\limsup_{t\to\infty}\int_{t_*}^t\Big[\Big(\frac{\delta_3(\tau^{-1}(\sigma_2(s)),t_1)}{\delta_2(\tau^{-1}(\sigma_2(s)),t_1)}\Big)^{\alpha_1\alpha_2}\frac{\rho(s)q_2(s)\gamma(\tau^{-1}(\sigma_2(s)))}{r_2^{\alpha_1}(\tau^{-1}(\sigma_2(s)))}-\frac{(\rho'_+(s))^{\alpha_1+1}r_1(\tau^{-1}(\sigma_2(s)))}{(\alpha_1+1)^{\alpha_1+1}\rho^{\alpha_1}(s)}\Big]ds=\infty, \tag{2.3}$$

and

$$\int_{t_0}^\infty\Big[\frac{1}{r_2(u)}\int_u^\infty\Big(\frac{1}{r_1(v)}\int_v^\infty q_1(s)ds\Big)^{\frac{1}{\alpha_1}}dv\Big]^{\frac{1}{\alpha_2}}du=\infty, \tag{2.4}$$

where

$$\gamma(t)=\begin{cases} m_1(\delta_3(t,t_1))^{\alpha_3-\alpha_1\alpha_2}, & m_1 \textit{ is any positive constant}, \ \ if\ \alpha_1\alpha_2>\alpha_3,\\ m_2, & m_2 \textit{ is any positive constant}, \ \ if\ \alpha_1\alpha_2\le\alpha_3.\end{cases}$$

Then every solution of (1.1) *is either oscillatory or tends to zero as* $t\to\infty$.

Proof. Suppose that (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, we assume that there exists $t_1\ge t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t,\xi))>0$ for $\xi\in[a,b]$ and $t\ge t_1$. Then from the definition of $z(t)$, we have $z(t)>0$. Based on the condition (1.2), $z(t)$ satisfies the following two cases (see, for example [18, 19]):

(I) $z(t)>0$, $z'(t)>0$, $(r_2(t)(z'(t))^{\alpha_2})'>0$ and $\big[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\big]'\le 0$;

(II) $z(t)>0$, $z'(t)<0$, $(r_2(t)(z'(t))^{\alpha_2})'>0$ and $\big[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\big]'\le 0$, for $t\ge t_1$.

Assume first that Case (I) holds. Then we get

$$x(t)=\frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}-\frac{z(\tau^{-1}(\tau^{-1}(t)))-x(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}\ge\frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}-\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}. \tag{2.5}$$

(2.5) can also be seen in [22]. Since $r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}$ is non-increasing for $t\ge t_1$, it indicates that

$$r_2(t)(z'(t))^{\alpha_2}\ge\int_{t_1}^t\frac{r_1^{\frac{1}{\alpha_1}}(s)(r_2(s)(z'(s))^{\alpha_2})'}{r_1^{\frac{1}{\alpha_1}}(s)}ds\ge\delta_1(t,t_1)r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})'. \tag{2.6}$$

We deduce from (2.6) that

$$\Big(\frac{r_2(t)(z'(t))^{\alpha_2}}{\delta_1(t,t_1)}\Big)' \le 0,$$

and $z'(t)/\delta_2(t,t_1)$ is non-increasing for $t \ge t_1$. Therefore, we obtain

$$z(t) \ge \int_{t_1}^{t} \frac{z'(s)}{\delta_2(s,t_1)}\delta_2(s,t_1)ds \ge \frac{\delta_3(t,t_1)}{\delta_2(t,t_1)}z'(t), \tag{2.7}$$

and

$$\Big(\frac{z(t)}{\delta_3(t,t_1)}\Big)' \le 0, \tag{2.8}$$

which yields that

$$z(\tau^{-1}(\tau^{-1}(t))) \le \frac{\delta_3(\tau^{-1}(\tau^{-1}(t)),t_1)}{\delta_3(\tau^{-1}(t),t_1)}z(\tau^{-1}(t)),\ t \ge t_1, \tag{2.9}$$

for $\tau(t) \le t$. Substituting (2.9) into (2.5), we have

$$x(t) \ge p_2(t)z(\tau^{-1}(t)).$$

Then there exists $t_2 > t_1$ such that $\sigma(t,\xi) \ge t_1$ and

$$x(\sigma(t,\xi)) \ge p_2(\sigma(t,\xi))z(\tau^{-1}(\sigma(t,\xi))),\ t \ge t_2. \tag{2.10}$$

Combining (1.1), (H3), (H4) and (2.10), we conclude that

$$\big[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\big]' + q_2(t)z^{\alpha_3}(\tau^{-1}(\sigma_2(t))) \le 0. \tag{2.11}$$

Define a Riccati transformation $\omega(t)$ by

$$\omega(t) = \rho(t)\frac{r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}}{\big(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\big)^{\alpha_1}},\ t \ge t_2. \tag{2.12}$$

Clearly, $\omega(t) > 0$, and

$$\begin{aligned}\omega'(t) =&\frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\big[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\big]'}{\big(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\big)^{\alpha_1}}\\ &- \alpha_1\rho(t)\frac{r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\big(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\big)'}{\big(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\big)^{\alpha_1+1}}.\end{aligned} \tag{2.13}$$

Since $\sigma(t,\xi) \le \tau(t)$ and $\big[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\big]' \le 0$, we get $\tau^{-1}(\sigma_2(t)) \le t$ and

$$r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})' \le r_1^{\frac{1}{\alpha_1}}(\tau^{-1}(\sigma_2(t)))\big(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\big)'. \tag{2.14}$$

Combining (2.11), (2.13) and (2.14), we have

$$\omega'(t) \le \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \frac{\rho(t)q_2(t)z^{\alpha_3}(\tau^{-1}(\sigma_2(t)))}{\big(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\big)^{\alpha_1}}$$

$$-\frac{\alpha_1\rho(t)r_1^{\frac{1}{\alpha_1}+1}(t)}{r_1^{\frac{1}{\alpha_1}}(\tau^{-1}(\sigma_2(t)))}\Big(\frac{(r_2(t)(z'(t))^{\alpha_2})'}{r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}}\Big)^{\alpha_1+1}. \tag{2.15}$$

From (2.7), we get

$$\frac{z^{\alpha_1\alpha_2}(t)}{(r_2(t)(z'(t))^{\alpha_2})^{\alpha_1}} \geq \Big(\frac{\delta_3(t,t_1)}{\delta_2(t,t_1)}\Big)^{\alpha_1\alpha_2} r_2^{-\alpha_1}(t). \tag{2.16}$$

Combining (2.12), (2.15) and (2.16), we obtain

$$\omega'(t) \leq \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \Big(\frac{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)}{\delta_2(\tau^{-1}(\sigma_2(t)),t_1)}\Big)^{\alpha_1\alpha_2}\frac{\rho(t)q_2(t)z^{\alpha_3-\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(t)))}{r_2^{\alpha_1}(\tau^{-1}(\sigma_2(t)))}$$
$$-\frac{\alpha_1}{(\rho(t)r_1(\tau^{-1}(\sigma_2(t))))^{\frac{1}{\alpha_1}}}\omega^{\frac{1}{\alpha_1}+1}(t). \tag{2.17}$$

In order to compute $z^{\alpha_3-\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(t)))$, we consider the following two cases:

(1) $\alpha_1\alpha_2 > \alpha_3$. From (2.8), there exist $t_3 > t_2$ and $h_1 > 0$ such that $\tau^{-1}(\sigma_2(t)) \geq t_2$ and

$$\frac{z(\tau^{-1}(\sigma_2(t)))}{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)} \leq \frac{z(t_2)}{\delta_3(t_2,t_1)} = h_1,\ t \geq t_3,$$

which implies that

$$z^{\alpha_3-\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(t))) \geq m_1\big(\delta_3(\tau^{-1}(\sigma_2(t)),t_1)\big)^{\alpha_3-\alpha_1\alpha_2}, \tag{2.18}$$

where $m_1 = h_1^{\alpha_3-\alpha_1\alpha_2}$.

(2) $\alpha_1\alpha_2 \leq \alpha_3$. Based on the fact that $z'(t) > 0$, there exists $h_2 > 0$ such that

$$z(\tau^{-1}(\sigma_2(t))) \geq z(t_2) = h_2,\ t \geq t_3,$$

which yields that

$$z^{\alpha_3-\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(t))) \geq m_2, \tag{2.19}$$

where $m_2 = h_2^{\alpha_3-\alpha_1\alpha_2}$.

Substituting (2.18) and (2.19) into (2.17), we have

$$\omega'(t) \leq \frac{\rho'_+(t)}{\rho(t)}\omega(t) - \Big(\frac{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)}{\delta_2(\tau^{-1}(\sigma_2(t)),t_1)}\Big)^{\alpha_1\alpha_2}\frac{\rho(t)q_2(t)\gamma(\tau^{-1}(\sigma_2(t)))}{r_2^{\alpha_1}(\tau^{-1}(\sigma_2(t)))} - \frac{\alpha_1}{(\rho(t)r_1(\tau^{-1}(\sigma_2(t))))^{\frac{1}{\alpha_1}}}\omega^{\frac{1}{\alpha_1}+1}(t). \tag{2.20}$$

Let

$$C = \frac{\rho'_+(t)}{\rho(t)},\ D = \frac{\alpha_1}{(\rho(t)r_1(\tau^{-1}(\sigma_2(t))))^{\frac{1}{\alpha_1}}},\ \alpha = \alpha_1,\ u = \omega(t).$$

Applying the inequality (see [13])

$$Cu - Du^{\frac{1}{\alpha}+1} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{C^{\alpha+1}}{D^\alpha},\ D > 0,\ u > 0, \tag{2.21}$$

together with (2.20), we get

$$\omega'(t) \leq -\Big(\frac{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)}{\delta_2(\tau^{-1}(\sigma_2(t)),t_1)}\Big)^{\alpha_1\alpha_2}\frac{\rho(t)q_2(t)\gamma(\tau^{-1}(\sigma_2(t)))}{r_2^{\alpha_1}(\tau^{-1}(\sigma_2(t)))} + \frac{(\rho'_+(t))^{\alpha_1+1}r_1(\tau^{-1}(\sigma_2(t)))}{(\alpha_1+1)^{\alpha_1+1}\rho^{\alpha_1}(t)}. \tag{2.22}$$

Integrating (2.22) from t_3 to t, we obtain

$$\int_{t_3}^{t}\Big[\Big(\frac{\delta_3(\tau^{-1}(\sigma_2(s)),t_1)}{\delta_2(\tau^{-1}(\sigma_2(s)),t_1)}\Big)^{\alpha_1\alpha_2}\frac{\rho(s)q_2(s)\gamma(\tau^{-1}(\sigma_2(s)))}{r_2^{\alpha_1}(\tau^{-1}(\sigma_2(s)))}-\frac{(\rho'_+(s))^{\alpha_1+1}r_1(\tau^{-1}(\sigma_2(s)))}{(\alpha_1+1)^{\alpha_1+1}\rho^{\alpha_1}(s)}\Big]ds<\omega(t_3),$$

for all sufficiently large t, which contradicts (2.3).

Secondly, assume that Case (II) holds. Since $z'(t)<0$ and $\tau(t)\le t$, (2.5) yields that

$$x(t)\ge p_1(t)z(\tau^{-1}(t)),$$

which indicates that

$$x(\sigma(t,\xi))\ge p_1(\sigma(t,\xi))z(\tau^{-1}(\sigma(t,\xi))), \tag{2.23}$$

for $t\ge t_2$. Using (1.1), (H4) and (2.23), we conclude that

$$[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}]'\le -q_1(t)z^{\alpha_3}(\tau^{-1}(\sigma_1(t))). \tag{2.24}$$

By using a similar proof of [19, Lemma 2.2], we can obtain $\lim_{t\to\infty}x(t)=0$ due to the condition (2.4). This completes the proof of Theorem 2.1. □

Next, we turn our attention to the case when (2.2) holds.

Theorem 2.2. *Assume that conditions (H1)–(H5),* (1.2)*,* (1.5)*,* (1.6)*,* (2.2) *and* (2.4) *hold. Moreover, assume that there exists a function $\rho(t)\in C^1([t_0,\infty),(0,\infty))$ such that for sufficiently large $t_*>t_2>t_1\ge t_0$,*

$$\limsup_{t\to\infty}\int_{t_*}^{t}\Big[\Big(\frac{\delta_3(\tau(s),t_1)}{\delta_2(\tau(s),t_1)}\Big)^{\alpha_1\alpha_2}\frac{\rho(s)q_2(s)\gamma(\tau(s))}{r_2^{\alpha_1}(\tau(s))}-\frac{(\rho'_+(s))^{\alpha_1+1}r_1(\tau(s))}{(\alpha_1+1)^{\alpha_1+1}\rho^{\alpha_1}(s)}\Big]ds=\infty, \tag{2.25}$$

where $\gamma(t)$ is defined as in Theorem 2.1. Then every solution of (1.1) *is either oscillatory or tends to zero as $t\to\infty$.*

Proof. Suppose that (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, we assume that there exists $t_1\ge t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t,\xi))>0$ for $\xi\in[a,b]$ and $t\ge t_1$. As in the proof of Theorem 2.1, we have (2.11) and (2.16), and there exist two possible cases (I) and (II) for $z(t)$ (as those in the proof of Theorem 2.1).

Assume that Case (I) holds. Define a Riccati transformation $\tilde\omega(t)$ by

$$\tilde\omega(t)=\rho(t)\frac{r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}}{(r_2(\tau(t))((z(\tau(t)))')^{\alpha_2})^{\alpha_1}},\ t\ge t_2>t_1. \tag{2.26}$$

Clearly, $\tilde\omega(t)>0$, and since $\tau(t)\le t$,

$$\begin{aligned}\tilde\omega'(t)=&\frac{\rho'(t)}{\rho(t)}\tilde\omega(t)+\rho(t)\frac{[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}]'}{(r_2(\tau(t))((z(\tau(t)))')^{\alpha_2})^{\alpha_1}}-\alpha_1\rho(t)\frac{r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}(r_2(\tau(t))((z(\tau(t)))')^{\alpha_2})'}{(r_2(\tau(t))((z(\tau(t)))')^{\alpha_2})^{\alpha_1+1}}\\ \le&\frac{\rho'_+(t)}{\rho(t)}\tilde\omega(t)-\frac{\rho(t)q_2(t)z^{\alpha_3}(\tau^{-1}(\sigma_2(t)))}{(r_2(\tau(t))((z(\tau(t)))')^{\alpha_2})^{\alpha_1}}-\frac{\alpha_1\rho(t)r_1^{\frac{1}{\alpha_1}+1}(t)}{r_1^{\frac{1}{\alpha_1}}(\tau(t))}\Big(\frac{(r_2(t)(z'(t))^{\alpha_2})'}{r_2(\tau(t))((z(\tau(t)))')^{\alpha_2}}\Big)^{\alpha_1+1}.\end{aligned} \tag{2.27}$$

From (2.2), we get $\tau^{-1}(\sigma_2(t)) \geq t \geq \tau(t)$, and then $z(\tau^{-1}(\sigma_2(t))) \geq z(\tau(t))$. Therefore, based on (2.16), we obtain

$$\frac{z^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(t)))}{(r_2(\tau(t))((z(\tau(t)))')^{\alpha_2})^{\alpha_1}} \geq \Big(\frac{\delta_3(\tau(t),t_1)}{\delta_2(\tau(t),t_1)}\Big)^{\alpha_1\alpha_2} r_2^{-\alpha_1}(\tau(t)).$$

Substituting the later inequality, (2.18) and (2.19) into (2.27), we deduce that

$$\tilde{\omega}'(t) \leq \frac{\rho'_+(t)}{\rho(t)}\tilde{\omega}(t) - \Big(\frac{\delta_3(\tau(t),t_1)}{\delta_2(\tau(t),t_1)}\Big)^{\alpha_1\alpha_2} \frac{\rho(t)q_2(t)\gamma(\tau(t))}{r_2^{\alpha_1}(\tau(t))} - \frac{\alpha_1}{(\rho(t)r_1(\tau(t)))^{\frac{1}{\alpha_1}}}\tilde{\omega}^{\frac{1}{\alpha_1}+1}(t). \tag{2.28}$$

By using the inequality (2.21) and (2.28), we conclude that

$$\tilde{\omega}'(t) \leq -\Big(\frac{\delta_3(\tau(t),t_1)}{\delta_2(\tau(t),t_1)}\Big)^{\alpha_1\alpha_2} \frac{\rho(t)q_2(t)\gamma(\tau(t))}{r_2^{\alpha_1}(\tau(t))} + \frac{(\rho'_+(t))^{\alpha_1+1}r_1(\tau(t))}{(\alpha_1+1)^{\alpha_1+1}\rho^{\alpha_1}(t)}. \tag{2.29}$$

An integration of (2.29) from t_3 ($t_3 > t_2$) to t leads to a contradiction to (2.25).

Secondly, assume that Case (II) holds. Proceeding as in the proof of Case (II) in Theorem 2.1, we arrive at the conclusion $\lim_{t\to\infty} x(t) = 0$. This completes the proof of Theorem 2.2. □

Remark 2.1. *With different choices of the function $\rho(t)$, one can derive a number of oscillation criteria for* (1.1) *from Theorems 2.1 and 2.2.*

Remark 2.2. *Our results in this section extend and improve those obtained by Tunç [22], and we can get some relevant results by using the technique presented in [22]. The established results here also complement and improve those in [13, 16, 17, 19], since the considered equations in these papers are special cases of* (1.1) *and our results can be applied to* (1.1) *in the case where $p(t) \geq 1$.*

3. Oscillation criteria for the case (1.3)

In this section, we will establish some oscillation criteria for (1.1) under the assumption that (1.3) holds. Similarly as in Section 2, we start with the case when (2.1) is satisfied. Firstly, we define the following notations:

$$\tilde{\delta}_1(t) = \int_t^\infty r_1^{-\frac{1}{\alpha_1}}(s)ds,\ \tilde{\delta}_2(t,t_1) = \int_{t_1}^t r_2^{-\frac{1}{\alpha_2}}(s)ds.$$

Furthermore, assume that

$$\tilde{p}_2(t) = \frac{1}{p(\tau^{-1}(t))}\Big(1 - \frac{\tilde{\delta}_2(\tau^{-1}(\tau^{-1}(t)),t_1)}{p(\tau^{-1}(\tau^{-1}(t)))\tilde{\delta}_2(\tau^{-1}(t),t_1)}\Big) > 0. \tag{3.1}$$

Then let

$$\tilde{q}_2(t) = K\int_a^b q(t,\xi)\tilde{p}_2^{\alpha_3}(\sigma(t,\xi))d\xi.$$

Theorem 3.1. *Assume that conditions (H1)–(H5),* (1.3), (1.5), (1.6), (2.1), (2.4) *and* (3.1) *hold. Furthermore, assume that there exists $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ such that* (2.3) *is satisfied for sufficiently large $t_* > t_2 > t_1 \geq t_0$. If*

$$\limsup_{t\to\infty}\int_{t_*}^t \Big[\tilde{\delta}_1^{\alpha_1}(s)\tilde{q}_2(s)\tilde{\gamma}(s)\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(s)),t_1) - \Big(\frac{\alpha_1}{\alpha_1+1}\Big)^{\alpha_1+1}\frac{1}{\tilde{\delta}_1(s)r_1^{\frac{1}{\alpha_1}}(s)}\Big]ds = \infty, \tag{3.2}$$

where

$$\tilde{\gamma}(t)=\begin{cases} m_3(\tilde{\delta}_2(t,t_1))^{\alpha_3-\alpha_1\alpha_2}, & m_3 \text{ is any positive constant}, \quad if\ \alpha_1\alpha_2>\alpha_3,\\ m_2, & m_2 \text{ is any positive constant}, \quad if\ \alpha_1\alpha_2\le\alpha_3,\end{cases}$$

then every solution of (1.1) *is either oscillatory or tends to zero as* $t\to\infty$.

Proof. Suppose that (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, we assume that there exists $t_1\ge t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t,\xi))>0$ for $\xi\in[a,b]$ and $t\ge t_1$. Based on the condition (1.3), there exist three possible cases (I), (II) (as those in the proof of Theorem 2.1) and

(III) $z(t)>0$, $z'(t)>0$, $(r_2(t)(z'(t))^{\alpha_2})'<0$ and $[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}]'\le 0$, for $t\ge t_1$.

We firstly prove that Case (III) holds. It is easy to verify that

$$(r_2(t)(z'(t))^{\alpha_2})'>0 \text{ or } (r_2(t)(z'(t))^{\alpha_2})'<0$$

holds under the condition

$$\int_{t_0}^{\infty} r_1^{-\frac{1}{\alpha_1}}(t)dt<\infty,$$

and in the proof of Theorem 2.1, we can see that $z(t)$ has properties (I) and (II). If $(r_2(t)(z'(t))^{\alpha_2})'<0$, then we claim that $z'(t)>0$. Otherwise, there exists a constant M_1 such that

$$r_2(t)(z'(t))^{\alpha_2}\le M_1<0,\ t\ge t_1.$$

Integrating the last inequality from t_1 to t, we have

$$z(t)\le z(t_1)+M_1\int_{t_1}^{t} r_2^{-\frac{1}{\alpha_2}}(s)ds.$$

Letting $t\to\infty$, we get $z(t)\to-\infty$, which contradicts the fact that $z(t)>0$. Hence we conclude that $z'(t)>0$ here.

Assume now that Cases (I) and (II) hold. Then we can obtain the conclusion of Theorem 3.1 by using the proof of Theorem 2.1.

Assume that Case (III) holds. Since $r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}$ is nonincreasing for $t\ge t_1$ and $(r_2(t)(z'(t))^{\alpha_2})'<0$, we have

$$r_2(l)(z'(l))^{\alpha_2}-r_2(t)(z'(t))^{\alpha_2}=\int_t^l \frac{r_1^{\frac{1}{\alpha_1}}(s)(r_2(s)(z'(s))^{\alpha_2})'}{r_1^{\frac{1}{\alpha_1}}(s)}ds\le r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})'\int_t^l r_1^{-\frac{1}{\alpha_1}}(s)ds<0.$$

Letting $l\to\infty$, we get

$$r_2(t)(z'(t))^{\alpha_2}\ge-\tilde{\delta}_1(t)r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})',$$

that is

$$0<-\tilde{\delta}_1(t)\frac{r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})'}{r_2(t)(z'(t))^{\alpha_2}}\le 1. \tag{3.3}$$

In view of $r_2^{\frac{1}{\alpha_2}}(t)z'(t)$ is non-increasing for $t\ge t_1$, we see that

$$z(t)\ge\tilde{\delta}_2(t,t_1)r_2^{\frac{1}{\alpha_2}}(t)z'(t), \tag{3.4}$$

and

$$\Big(\frac{z(t)}{\tilde{\delta}_2(t,t_1)}\Big)' \le 0, \tag{3.5}$$

which yields that

$$z(\tau^{-1}(\tau^{-1}(t))) \le \frac{\tilde{\delta}_2(\tau^{-1}(\tau^{-1}(t)),t_1)}{\tilde{\delta}_2(\tau^{-1}(t),t_1)} z(\tau^{-1}(t)),\ t \ge t_1. \tag{3.6}$$

Substituting (3.6) into (2.5), we get

$$x(t) \ge \tilde{p}_2(t) z(\tau^{-1}(t)).$$

Then there exists $t_2 > t_1$ such that $\sigma(t,\xi) \ge t_1$ and

$$\big[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\big]' \le -\tilde{q}_2(t) z^{\alpha_3}(\tau^{-1}(\sigma_2(t))),\ t \ge t_2. \tag{3.7}$$

Define the function $\nu(t)$ by

$$\nu(t) = \frac{r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}}{(r_2(t)(z'(t))^{\alpha_2})^{\alpha_1}},\ t \ge t_2. \tag{3.8}$$

Then $\nu(t) < 0$ for $t \ge t_2$. From (3.3) and (3.8), we obtain

$$-\tilde{\delta}_1^{\alpha_1}(t)\nu(t) \le 1. \tag{3.9}$$

Differentiating (3.8) and using (3.7) and (3.8), we get

$$\begin{aligned} \nu'(t) =& \frac{\big[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\big]'}{(r_2(t)(z'(t))^{\alpha_2})^{\alpha_1}} - \frac{\alpha_1 r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}(r_2(t)(z'(t))^{\alpha_2})'}{(r_2(t)(z'(t))^{\alpha_2})^{\alpha_1+1}} \\ \le& -\frac{\tilde{q}_2(t) z^{\alpha_3}(\tau^{-1}(\sigma_2(t)))}{(r_2(t)(z'(t))^{\alpha_2})^{\alpha_1}} - \frac{\alpha_1}{r_1^{\frac{1}{\alpha_1}}(t)} \nu^{\frac{1}{\alpha_1}+1}(t). \end{aligned} \tag{3.10}$$

From (1.5) and (3.5), we obtain

$$z(\tau^{-1}(\sigma_2(t))) \ge \frac{\tilde{\delta}_2(\tau^{-1}(\sigma_2(t)),t_1)}{\tilde{\delta}_2(t,t_1)} z(t). \tag{3.11}$$

(3.4) implies that

$$\frac{z^{\alpha_1\alpha_2}(t)}{(r_2(t)(z'(t))^{\alpha_2})^{\alpha_1}} \ge \tilde{\delta}_2^{\alpha_1\alpha_2}(t,t_1). \tag{3.12}$$

Combining (3.10), (3.11) and (3.12), we get

$$\nu'(t) \le -\tilde{q}_2(t)\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(t)),t_1) z^{\alpha_3-\alpha_1\alpha_2}(t) - \frac{\alpha_1}{r_1^{\frac{1}{\alpha_1}}(t)} \nu^{\frac{1}{\alpha_1}+1}(t), \tag{3.13}$$

due to $\tilde{\delta}_2(\tau^{-1}(\sigma_2(t)),t_1) \le \tilde{\delta}_2(t,t_1)$. In order to compute $z^{\alpha_3-\alpha_1\alpha_2}(t)$, the case $\alpha_1\alpha_2 \le \alpha_3$ is the same as that in the proof of Theorem 2.1. We now compute the case $\alpha_1\alpha_2 > \alpha_3$. Applying the monotonicity of $z(t)/\tilde{\delta}_2(t,t_1)$ for $t \ge t_2$ derived from (3.5), there exists $h_3 > 0$ such that

$$\frac{z(t)}{\tilde{\delta}_2(t,t_1)} \le \frac{z(t_2)}{\tilde{\delta}_2(t_2,t_1)} = h_3,$$

which implies that
$$z^{\alpha_3-\alpha_1\alpha_2}(t) \geq m_3(\tilde{\delta}_2(t,t_1))^{\alpha_3-\alpha_1\alpha_2}, \tag{3.14}$$
where $m_3 = h_3^{\alpha_3-\alpha_1\alpha_2}$. Combining (2.19), (3.13) and (3.14), we conclude that
$$\nu'(t) \leq -\tilde{q}_2(t)\tilde{\gamma}(t)\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(t)),t_1) - \frac{\alpha_1}{r_1^{\frac{1}{\alpha_1}}(t)}\nu^{\frac{1}{\alpha_1}+1}(t). \tag{3.15}$$

Multiplying (3.15) by $\tilde{\delta}_1^{\alpha_1}(t)$ and integrating it from t_2 to t, we obtain
$$\tilde{\delta}_1^{\alpha_1}(t)\nu(t) - \tilde{\delta}_1^{\alpha_1}(t_2)\nu(t_2) + \int_{t_2}^{t} \tilde{\delta}_1^{\alpha_1}(s)\tilde{q}_2(s)\tilde{\gamma}(s)\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(s)),t_1)ds$$
$$+ \alpha_1\int_{t_2}^{t}\Big[\frac{\tilde{\delta}_1^{\alpha_1}(s)}{r_1^{\frac{1}{\alpha_1}}(s)}|\nu(s)|^{\frac{1}{\alpha_1}+1} - \frac{\tilde{\delta}_1^{\alpha_1-1}(s)}{r_1^{\frac{1}{\alpha_1}}(s)}|\nu(s)|\Big]ds \leq 0. \tag{3.16}$$
By using the inequality (2.21) and (3.16) with
$$C = \frac{\tilde{\delta}_1^{\alpha_1-1}(s)}{r_1^{\frac{1}{\alpha_1}}(s)},\ D = \frac{\tilde{\delta}_1^{\alpha_1}(s)}{r_1^{\frac{1}{\alpha_1}}(s)},\ u = |\nu(s)|,$$
we conclude that
$$\int_{t_2}^{t}\Big[\tilde{\delta}_1^{\alpha_1}(s)\tilde{q}_2(s)\tilde{\gamma}(s)\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(s)),t_1) - \Big(\frac{\alpha_1}{\alpha_1+1}\Big)^{\alpha_1+1}\frac{1}{\tilde{\delta}_1(s)r_1^{\frac{1}{\alpha_1}}(s)}\Big]ds \leq \tilde{\delta}_1^{\alpha_1}(t_2)\nu(t_2)+1,$$
due to (3.9), which contradicts (3.2). This completes the proof of Theorem 3.1. □

With a similar proof to that of Theorems 2.2 and 3.1, we can obtain the following criteria for (1.1) by assuming that (2.2) is satisfied.

Theorem 3.2. *Assume that conditions (H1)–(H5),* (1.3)*,* (1.5)*,* (1.6)*,* (2.2)*,* (2.4) *and* (3.1) *hold. Furthermore, assume that there exists* $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ *such that* (2.25) *is satisfied for sufficiently large* $t_* > t_2 > t_1 \geq t_0$*. If*
$$\limsup_{t\to\infty}\int_{t_*}^{t}\Big[\tilde{\delta}_1^{\alpha_1}(s)\tilde{q}_2(s)\tilde{\gamma}(\tau^{-1}(\sigma_2(s)))\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(s)),t_1) - \Big(\frac{\alpha_1}{\alpha_1+1}\Big)^{\alpha_1+1}\frac{1}{\tilde{\delta}_1(s)r_1^{\frac{1}{\alpha_1}}(\tau^{-1}(\sigma_2(s)))}\Big]ds = \infty, \tag{3.17}$$
where $\tilde{\gamma}(t)$ *is defined as in Theorem 3.1, then every solution of* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$*.*

Proof. Suppose that (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$ for $\xi \in [a,b]$ and $t \geq t_1$. As in the proof of Theorem 3.1, we have (3.3) and (3.7), and there exist three possible cases (I), (II) and (III) for $z(t)$.

Assume first that Cases (I) and (II) hold. We can obtain the conclusion of Theorem 3.2 by using the proof of Theorem 2.2.

Assume that Case (III) holds. Define the function $\tilde{v}(t)$ by

$$\tilde{v}(t) = \frac{r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}}{\left(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\right)^{\alpha_1}},\ t \ge t_1. \tag{3.18}$$

Then $\tilde{v}(t) < 0$ for $t \ge t_1$. Since $\sigma_2(t) \ge \tau(t)$ and $(r_2(t)(z'(t))^{\alpha_2})' < 0$, we have

$$-\tilde{\delta}_1(t)\frac{r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})'}{r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}} \le -\tilde{\delta}_1(t)\frac{r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})'}{r_2(t)(z'(t))^{\alpha_2}}. \tag{3.19}$$

From (3.3), (3.18) and (3.19), we get

$$-\tilde{\delta}_1^{\alpha_1}(t)\tilde{v}(t) \le 1.$$

Since $r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})'$ is non-increasing for $t \ge t_1$, we obtain

$$\left(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\right)' \le \frac{r_1^{\frac{1}{\alpha_1}}(t)(r_2(t)(z'(t))^{\alpha_2})'}{r_1^{\frac{1}{\alpha_1}}(\tau^{-1}(\sigma_2(t)))}. \tag{3.20}$$

Differentiating (3.18) and using (3.7), (3.18) and (3.20), we deduce that

$$\begin{aligned}\tilde{v}'(t) =& \frac{[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}]'}{\left(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\right)^{\alpha_1}}\\ &- \frac{\alpha_1 r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}\left(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\right)'}{\left(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\right)^{\alpha_1+1}}\\ \le& -\frac{\tilde{q}_2(t)z^{\alpha_3}(\tau^{-1}(\sigma_2(t)))}{\left(r_2(\tau^{-1}(\sigma_2(t)))((z(\tau^{-1}(\sigma_2(t))))')^{\alpha_2}\right)^{\alpha_1}} - \frac{\alpha_1}{r_1^{\frac{1}{\alpha_1}}(\tau^{-1}(\sigma_2(t)))}\tilde{v}^{\frac{1}{\alpha_1}+1}(t).\end{aligned} \tag{3.21}$$

Furthermore, substituting (3.12) and the definition of $\tilde{\gamma}(t)$ into (3.21), we conclude that

$$\tilde{v}'(t) \le -\tilde{q}_2(t)\tilde{\gamma}(\tau^{-1}(\sigma_2(t)))\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(t)), t_1) - \frac{\alpha_1}{r_1^{\frac{1}{\alpha_1}}(\tau^{-1}(\sigma_2(t)))}\tilde{v}^{\frac{1}{\alpha_1}+1}(t).$$

The rest of the proof is similar to that of Theorem 3.1 and we can get a contradiction to (3.17). So we omit it here. This completes the proof of Theorem 3.2. □

4. Oscillation criteria for the case (1.4)

In this section, we will establish some oscillation criteria for (1.1) under the assumption that (1.4) holds. Similarly as in the previous sections, we start with the case when (2.1) is satisfied.

Theorem 4.1. *Assume that conditions (H1)–(H5),* (1.4)*,* (1.5)*,* (1.6)*,* (2.1)*,* (2.4)*,* (3.1) *and* (3.2) *hold. Furthermore, assume that there exists* $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ *such that for sufficiently large* $t_* > t_2 > t_1 \ge t_0$*, one has* (2.3)*. If*

$$\int_{t_*}^{\infty}\Big[\frac{1}{r_2(u)}\int_{t_*}^{u}\Big(\frac{1}{r_1(v)}\int_{t_*}^{v}q_1(s)ds\Big)^{\frac{1}{\alpha_1}}dv\Big]^{\frac{1}{\alpha_2}}du = \infty, \tag{4.1}$$

then every solution of (1.1) *is either oscillatory or tends to zero as* $t \to \infty$*.*

Proof. Suppose that (1.1) has a non-oscillatory solution $x(t)$. Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$ for $\xi \in [a,b]$ and $t \geq t_1$. Based on the condition (1.4), there exist three possible cases (I), (II), (III) (as those in the proof of Theorem 3.1) and (IV) $z(t) > 0$, $z'(t) < 0$, $(r_2(t)(z'(t))^{\alpha_2})' < 0$ and $[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}]' \leq 0$, for $t \geq t_1$.

Assume first that Cases (I), (II) and (III) hold. We can obtain the conclusion of Theorem 4.1 by using the proof of Theorem 3.1.

Assume that Case (IV) holds. Then there exists a constant $l \geq 0$ such that

$$\lim_{t\to\infty} z(t) = l.$$

We claim that $l = 0$. Otherwise, assume that $l > 0$. We see that there exists $t_2 > t_1$ such that $\tau^{-1}(\sigma_1(t)) \geq t_1$ and $z(\tau^{-1}(\sigma_1(t))) \geq l$, $t \geq t_2$. From (2.24), we obtain

$$[r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1}]' + l^{\alpha_3} q_1(t) \leq 0. \tag{4.2}$$

Integrating (4.2) from t_2 to t, we have

$$r_1(t)((r_2(t)(z'(t))^{\alpha_2})')^{\alpha_1} + l^{\alpha_3} \int_{t_2}^{t} q_1(s)ds \leq 0,$$

which can be rewritten as

$$(r_2(t)(z'(t))^{\alpha_2})' + \Big(\frac{l^{\alpha_3}}{r_1(t)} \int_{t_2}^{t} q_1(s)ds\Big)^{\frac{1}{\alpha_1}} \leq 0.$$

Integrating again from t_2 to t, we get

$$r_2(t)(z'(t))^{\alpha_2} + l^{\frac{\alpha_3}{\alpha_1}} \int_{t_2}^{t} \Big(\frac{1}{r_1(v)} \int_{t_2}^{v} q_1(s)ds\Big)^{\frac{1}{\alpha_1}} dv \leq 0,$$

which yields that

$$z'(t) + \Big[\frac{l^{\frac{\alpha_3}{\alpha_1}}}{r_2(t)} \int_{t_2}^{t} \Big(\frac{1}{r_1(v)} \int_{t_2}^{v} q_1(s)ds\Big)^{\frac{1}{\alpha_1}} dv\Big]^{\frac{1}{\alpha_2}} \leq 0.$$

Integrating the last inequality from t_2 to t, we obtain

$$l^{\frac{\alpha_3}{\alpha_1\alpha_2}} \int_{t_*}^{t} \Big[\frac{1}{r_2(u)} \int_{t_2}^{u} \Big(\frac{1}{r_1(v)} \int_{t_2}^{v} q_1(s)ds\Big)^{\frac{1}{\alpha_1}} dv\Big]^{\frac{1}{\alpha_2}} du \leq z(t_2),$$

which contradicts (4.1). This completes the proof of Theorem 4.1. □

With a similar proof to that of Theorems 3.2 and 4.1, we can obtain the following criteria for (1.1) assuming that (2.2) is satisfied.

Theorem 4.2. *Assume that conditions (H1)–(H5),* (1.4), (1.5), (1.6), (2.2), (2.4), (3.1), (3.17) *and* (4.1) *hold. Furthermore, assume that there exists a function* $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ *such that for sufficiently large* $t_* > t_2 > t_1 \geq t_0$*, one has* (2.25)*. Then every solution of* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$*.*

Remark 4.1. *The results in [13, 16, 19, 22] are obtained only in the case* (1.2) *and they are inapplicable to* (1.3) *and* (1.4)*. Hence, the main results of this paper complement and improve those in the literature.*

5. Examples

In this section, we will present some examples to illustrate the main results.

Example 4.1 For $t > k_1 \geq 1$, consider the third-order neutral delay differential equation with distributed deviating arguments

$$\Big[t\Big(\Big(t^{\frac{1}{2}}\Big(x(t)+\frac{6t+5k_1}{t+k_1}x\Big(\frac{t}{3}\Big)\Big)'\Big)'\Big)^{\frac{1}{3}}\Big]'+\int_{k_1}^{k_1+1}15^5(t+\xi)x^5\Big(\frac{t}{3}-\xi\Big)d\xi=0, \tag{5.1}$$

where $\alpha_1 = 1/3$, $\alpha_2 = 1$, $\alpha_3 = 5$, $a = k_1$, $b = k_1 + 1$, $f(x) = x^5$, $r_1(t) = t$, $r_2(t) = t^{\frac{1}{2}}$,

$$\tau(t)=\frac{t}{3},\ \sigma(t,\xi)=\frac{t}{3}-\xi,\ p(t)=\frac{6t+5k_1}{t+k_1},\ q(t,\xi)=15^5(t+\xi).$$

Choose $t_0 = t_1 = k_1$. Then we get $\alpha_1\alpha_2 < \alpha_3$, $5 \leq p(t) < 6$,

$$\sigma_2(t)=\sigma(t,k_1+1)=\frac{t}{3}-(k_1+1),\ \ \delta_1(t,t_1)=\int_{k_1}^{t}s^{-3}ds=\frac{1}{2k_1^2}-\frac{1}{2t^2},$$

$$\delta_2(t,t_1)=\frac{\delta_1(t,k_1)}{t^{\frac{1}{2}}}=\frac{1}{2k_1^2t^{\frac{1}{2}}}-\frac{1}{2t^{\frac{5}{2}}},$$

$$\delta_3(t,t_1)=\int_{k_1}^{t}\Big(\frac{1}{2k_1^2s^{\frac{1}{2}}}-\frac{1}{2s^{\frac{5}{2}}}\Big)ds=\frac{1}{k_1^2}(t^{\frac{1}{2}}-k_1^{\frac{1}{2}})+\frac{1}{3}(t^{-\frac{3}{2}}-k_1^{-\frac{3}{2}}),$$

$$\tilde{\delta}_1(t)=\int_{t}^{\infty}s^{-3}ds=\frac{1}{2t^2},\ \ \tilde{\delta}_2(t,t_1)=\int_{k_1}^{t}s^{-\frac{1}{2}}ds=2(t^{\frac{1}{2}}-k_1^{\frac{1}{2}}).$$

Furthermore, we deduce that

$$p_1(t)>\frac{1}{6}\Big(1-\frac{1}{5}\Big)=\frac{2}{15}>0,$$

$$p_2(t)=\frac{1}{6}\Big(1-\frac{1}{5}\cdot\frac{\frac{1}{k_1^2}((9t)^{\frac{1}{2}}-k_1^{\frac{1}{2}})+\frac{1}{3}((9t)^{-\frac{3}{2}}-k_1^{-\frac{3}{2}})}{\frac{1}{k_1^2}((3t)^{\frac{1}{2}}-k_1^{\frac{1}{2}})+\frac{1}{3}((3t)^{-\frac{3}{2}}-k_1^{-\frac{3}{2}})}\Big)>\frac{1}{10}>0,$$

$$\tilde{p}_2(t)=\frac{1}{6}\Big(1-\frac{1}{5}\cdot\frac{((9t)^{\frac{1}{2}}-k_1^{\frac{1}{2}})}{((3t)^{\frac{1}{2}}-k_1^{\frac{1}{2}})}\Big)>\frac{1}{15}>0,$$

$$q_1(t)>\int_{k_1}^{k_1+1}\Big(\frac{2}{15}\Big)^5\cdot15^5(t+\xi)d\xi=32\Big(t+k_1+\frac{1}{2}\Big),$$

$$q_2(t)>\int_{k_1}^{k_1+1}\Big(\frac{1}{10}\Big)^5\cdot15^5(t+\xi)d\xi=\Big(\frac{3}{2}\Big)^5\Big(t+k_1+\frac{1}{2}\Big),$$

$$\tilde{q}_2(t)>\int_{k_1}^{k_1+1}\Big(\frac{1}{15}\Big)^5\cdot15^5(t+\xi)d\xi=t+k_1+\frac{1}{2}.$$

It is easy to verify that

$$\int_{t_0}^{\infty}\Big[\frac{1}{r_2(u)}\int_{u}^{\infty}\Big(\frac{1}{r_1(v)}\int_{v}^{\infty}q_1(s)ds\Big)^{\frac{1}{\alpha_1}}dv\Big]^{\frac{1}{\alpha_2}}du$$

$$> \int_{k_1}^{\infty} u^{-\frac{1}{2}} \int_{u}^{\infty} \Big(\int_{v}^{\infty} 32\Big(s + k_1 + \frac{1}{2}\Big)ds\Big)^3 dvdu = \infty,$$

and

$$\int_{t_*}^{t} \Big[\tilde{\delta}_1^{\alpha_1}(s)\tilde{q}_2(s)\tilde{\gamma}(s)\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(s)), t_1) - \Big(\frac{\alpha_1}{\alpha_1+1}\Big)^{\alpha_1+1} \frac{1}{\tilde{\delta}_1(s) r_1^{\frac{1}{\alpha_1}}(s)}\Big]ds$$
$$> \int_{4k_1+3}^{t} \Big[\Big(\frac{1}{2s^2}\Big)^{\frac{1}{3}}\Big(s + k_1 + \frac{1}{2}\Big)\Big((s - 3(k_1+1))^{\frac{1}{2}} - k_1^{\frac{1}{2}}\Big)^{\frac{1}{3}} - \frac{4^{-\frac{4}{3}}}{s}\Big]ds \to \infty,$$

as $t \to \infty$, where we set $t_* = 4k_1 + 3$. Therefore, conditions (H1)–(H5), (1.3), (1.5), (1.6), (2.1), (2.4), (3.1) and (3.2) hold. We choose $\rho(t) = 1$. Applying Theorem 3.1, it remains to check (2.3), and we see that

$$\int_{t_*}^{t} \Big[\Big(\frac{\delta_3(\tau^{-1}(\sigma_2(s)), t_1)}{\delta_2(\tau^{-1}(\sigma_2(s)), t_1)}\Big)^{\alpha_1\alpha_2} \frac{\rho(s)q_2(s)\gamma(\tau^{-1}(\sigma_2(s)))}{r_2^{\alpha_1}(\tau^{-1}(\sigma_2(s)))} - \frac{(\rho'_+(s))^{\alpha_1+1} r_1(\tau^{-1}(\sigma_2(s)))}{(\alpha_1+1)^{\alpha_1+1}\rho^{\alpha_1}(s)}\Big]ds$$
$$> \int_{4k_1+3}^{t} \Big[\Big(\frac{89}{48k_1^2}\Big)^{\frac{1}{3}} \frac{\big(\frac{3}{2}\big)^5\big(s + k_1 + \frac{1}{2}\big)}{(s - 3(k_1+1))^{\frac{1}{6}}}\Big]ds \to \infty,$$

as $t \to \infty$. Hence, every solution of (5.1) is either oscillatory or converges to zero by Theorem 3.1.

Example 4.2 Consider the third-order neutral delay differential equation with distributed deviating arguments

$$[t^2(t^2(x(t) + 2x(t-1))')']' + \int_1^2 12^3 \lambda t \xi x^3\Big(t - 1 + \frac{1}{\xi}\Big)d\xi = 0, \; t \geq 1, \tag{5.2}$$

where λ is a positive constant. Choose $t_0 = t_1 = 1$. Then we get

$$\sigma_2(t) = \sigma(t, 2) = t - \frac{1}{2},$$
$$\delta_1(t, t_1) = \int_1^t s^{-2}ds = 1 - \frac{1}{t}, \;\; \delta_2(t, t_1) = \frac{\delta_1(t, k_1)}{t^2} = \frac{1}{t^2} - \frac{1}{t^3},$$
$$\delta_3(t, t_1) = \int_1^t \Big(\frac{1}{s^2} - \frac{1}{s^3}\Big)ds = \frac{1}{2t^2} - \frac{1}{t} + \frac{1}{2},$$
$$\tilde{\delta}_1(t) = \int_t^{\infty} s^{-2}ds = \frac{1}{t}, \;\; \tilde{\delta}_2(t, t_1) = \int_1^t s^{-2}ds = 1 - \frac{1}{t}.$$

Furthermore, we deduce that

$$p_1(t) = \frac{1}{2}\Big(1 - \frac{1}{2}\Big) = \frac{1}{4} > 0, \;\; p_2(t) = \frac{1}{2}\Big(1 - \frac{1}{2} \cdot \frac{\frac{1}{2(t+2)^2} - \frac{1}{t+2} + \frac{1}{2}}{\frac{1}{2(t+1)^2} - \frac{1}{t+1} + \frac{1}{2}}\Big) > \frac{1}{12} > 0,$$
$$q_1(t) = \int_1^2 \Big(\frac{1}{4}\Big)^3 12^3 \lambda t \xi d\xi = \frac{3^4}{2}\lambda t, \;\; q_2(t) > \int_1^2 \Big(\frac{1}{12}\Big)^3 12^3 \lambda t \xi d\xi = \frac{3}{2}\lambda t,$$
$$\tilde{p}_2(t) = \frac{1}{2}\Big(1 - \frac{1}{2} \cdot \frac{1 - \frac{1}{t+2}}{1 - \frac{1}{t+1}}\Big) > \frac{1}{4} > 0, \;\; \tilde{q}_2(t) > \int_1^2 \Big(\frac{1}{4}\Big)^3 12^3 \lambda t \xi d\xi = \frac{3^4}{2}\lambda t.$$

It is easy to verify that

$$\int_{t_0}^{\infty}\Big[\frac{1}{r_2(u)}\int_u^{\infty}\Big(\frac{1}{r_1(v)}\int_v^{\infty}q_1(s)ds\Big)^{\frac{1}{\alpha_1}}dv\Big]^{\frac{1}{\alpha_2}}du=\int_2^{\infty}u^{-2}\int_u^{\infty}v^{-2}\int_v^{\infty}\frac{3^4}{2}\lambda s ds dv du=\infty,$$

$$\int_{t_*}^{\infty}\Big[\frac{1}{r_2(u)}\int_{t_*}^{u}\Big(\frac{1}{r_1(v)}\int_{t_*}^{v}q_1(s)ds\Big)^{\frac{1}{\alpha_1}}dv\Big]^{\frac{1}{\alpha_2}}du=\int_2^{\infty}u^{-2}\int_2^{u}v^{-2}\int_2^{v}\frac{3^4}{2}\lambda s ds dv du=\infty,$$

and

$$\begin{aligned}&\int_{t_*}^{t}\Big[\tilde{\delta}_1^{\alpha_1}(s)\tilde{q}_2(s)\tilde{\gamma}(\tau^{-1}(\sigma_2(s)))\tilde{\delta}_2^{\alpha_1\alpha_2}(\tau^{-1}(\sigma_2(s)),t_1)-\Big(\frac{\alpha_1}{\alpha_1+1}\Big)^{\alpha_1+1}\frac{1}{\tilde{\delta}_1(s)r_1^{\frac{1}{\alpha_1}}(\tau^{-1}(\sigma_2(s)))}\Big]ds\\ &>\int_2^{t}\Big[\frac{1}{s}\frac{3^4}{2}\lambda s\Big(1-\frac{1}{s+\frac{1}{2}}\Big)-\frac{1}{4}\frac{s}{\Big(s+\frac{1}{2}\Big)^2}\Big]ds\to\infty,\end{aligned}$$

as $t\to\infty$, if $\lambda\geq 1$, where we set $t_*=2$. Therefore, conditions (H1)–(H5), (1.4), (1.5), (1.6), (2.2), (2.4), (3.1), (3.17) and (4.1) hold. We choose $\rho(t)=t^2$. Applying Theorem 4.2, it remans to check (2.25), and we get

$$\begin{aligned}&\int_{t_*}^{t}\Big[\Big(\frac{\delta_3(\tau(s),t_1)}{\delta_2(\tau(s),t_1)}\Big)^{\alpha_1\alpha_2}\frac{\rho(s)q_2(s)\gamma(\tau(s))}{r_2^{\alpha_1}(\tau(s))}-\frac{(\rho'_+(s))^{\alpha_1+1}r_1(\tau(s))}{(\alpha_1+1)^{\alpha_1+1}\rho^{\alpha_1}(s)}\Big]ds\\ &>\int_2^{t}\Big[\frac{\frac{1}{2(s-1)^2}-\frac{1}{s-1}+\frac{1}{2}}{\frac{1}{(s-1)^2}-\frac{1}{(s-1)^3}}\cdot\frac{\frac{3}{2}\lambda s^3}{(s-1)^2}-(s-1)^2\Big]ds\to\infty,\end{aligned}$$

as $t\to\infty$. Hence, every solution of (5.2) is either oscillatory or converges to zero by Theorem 4.2.

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Conflict of interest

The authors declare that they have no conflict of interest.

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