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Research article

Stability and domination exponentially in some graphs

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Abstract: For a graph $G = (V, E)$ and the exponential dominating set $S \subseteq V(G)$ of G such that $\sum_{u \in S} (1/2)^{\overline{d}(u,v)-1} \geq 1$, $\forall v \in V(G)$, where $\overline{d}(u, v)$ is the length of a shortest path in $\langle V(G) - (S - \{u\}) \rangle$ if such a path exists, and ∞ otherwise, the minimum exponential domination number, $\gamma_e(G)$ is the smallest cardinality of S. The minimum exponential domination number can be decreased or increased by removal of some vertices from G. In this paper, we continue to study on exponential domination number and stability of some graphs. We consider γ_e^+ and γ_e^- stability of the lollipop graph $L_{m,n}$, the comet graph $C_{m,n}$, the sunflower graph SF_n, the helm graph H_n, the diamond-necklace graph N_n, the diamond-bracelet graph B_n and the diamond-chain graph L_n to give us an idea about the resistance of these graphs.

Keywords: graph vulnerability; network design and communication; domination; exponential domination number
Mathematics Subject Classification: 05C69, 05C40, 68M10, 68R10

1. Introduction

A graph G, a mathematical modelling in which we show some structures is represented by a set of vertices $V(G)$ and a set of edges $E(G)$. Firstly, we mention some of the definitions related to graph theory in this article. For any vertex $v \in V(G)$, the *open neighbourhood* of v is $N(v) = \{u \in V(G) | uv \in E(G)\}$ and closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. The number of the neighbour vertices of the vertex v is called degree of v and denoted by $deg_G(v)$. A vertex v is said to be pendant vertex if $deg_G(v) = 1$. A vertex u is called support if u is adjacent to a pendant vertex [1]. Throughout this article, the largest integer not greater than x is denoted by $\lfloor x \rfloor$ and the least integer not less than x is denoted by $\lceil x \rceil$. The graph with n vertices labeled $x_1, x_2, ..., x_n$ and edges $x_1x_2, x_2x_3, ..., x_{n-1}x_n$ is called a path of length

$n - 1$, denoted by P_n. The cycle of length n, C_n is the graph with n vertices $x_1, x_2, ..., x_n$ and the edges $x_1x_2, x_2x_3, ..., x_nx_1$ [2]. Paths are trees. A tree is a path if and only if its maximum degree is 2. The wheel with $n + 1$ vertices, W_n, is the graph that consists of an n-cycle and one additional vertex that is adjacent to all the vertices of the cycle. Complete graph K_n is the graph with n vertices, and every vertex is adjacent to every other vertex [2]. A star is a tree consisting of one vertex adjacent to all the others. The $n-$vertex star is the biclique $K_{1,n-1}$ [3]. The complement $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(\overline{G})$ [3].

The domination in graph theory, which has an important role in many fields of study such as optimization, design and analysis of communication networks, social sciences and military surveillance. A *dominating set* in a graph G is a set of vertices of G such that every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [4, 5].

There are many domination parameters that measure the robustness and stability of graphs under any attack or influence. In such cases some vertices in the graph are critical. When we remove a vertex from the graph, resume of the remaining structure gives us information about the importance of the vertex.

Dankelmann et al. [6] recently defined exponential domination. Let G be a graph and $S \subseteq V(G)$. We denote by $\langle S \rangle$ the subgraph of G induced by S. For each vertex $u \in S$ and for each $v \in V(G) - S$, we define $\overline{d}(u, v)$ to be the length of a shortest $u - v$ path in $\langle G - (S - \{u\}) \rangle$ if such a path exists, and ∞ otherwise. If, for each $v \in V(G) - S$ we have $\sum_{u \in S} 1/2^{\overline{d}(u,v)} \geq 1$, then S is an exponential dominating set. The smallest cardinality of an exponential dominating set is the *exponential domination number*, $\gamma_e(G)$. One can think of this in the following way: each vertex dominates its neighbors, 1/2-dominates those at distance 2, and so on. Hence a vertex v can be dominated by a neighbor of v or by a number of vertices that are not too far from v. Such a model could be used, for example, for the analysis of dissemination of information in social networks, where the impact of the information decreases every time it is passed on. The assumption is that gossip heard directly from a source is totally reliable, while gossip passed from person to person loses half its credibility with each individual in the chain. Finding the exponential domination number in this application amounts to determining the minimum number of sources needed so that each person gets fully reliable information.

In this paper, firstly known results are given. Then, some results about the exponential domination number, γ_e^+-stability and γ_e^--stability for some graphs are established. Finally, conclusion section is presented.

2. Known results

Theorem 2.1. [6] $\forall n \in \mathbb{Z}^+$, $\gamma_e(P_n) = \lceil (n + 1)/4 \rceil$.

Theorem 2.2. [7] $\forall n \geq 6 \in \mathbb{Z}^+$,

$$\gamma_e^+(P_n) = \begin{cases} 2, & if\ n \equiv 0 (mod\ 4) \\ 1, & otherwise \end{cases}$$

Theorem 2.3. [7] $\forall n \geq 7 \in \mathbb{Z}^+$,

$$\gamma_e^-(P_n) = \begin{cases} 4, & if\ n \equiv 3(mod\,4) \\ 3, & if\ n \equiv 2(mod\,4) \\ 2, & if\ n \equiv 1(mod\,4) \\ 1, & if\ n \equiv 0(mod\,4) \end{cases}$$

Theorem 2.4. [6] $\forall n \in \mathbb{Z}^+$,

$$\gamma_e(C_n) = \begin{cases} 2, & if\ n = 4 \\ \lceil n/4 \rceil, & if\ n \neq 4 \end{cases}$$

Theorem 2.5. [7] $\forall n \geq 12 \in \mathbb{Z}^+$,

$$\gamma_e^+(C_n) = \begin{cases} 3, & if\ n \equiv 1(mod\,4) \\ 2, & otherwise \end{cases}$$

Theorem 2.6. [7] $\forall n \geq 6 \in \mathbb{Z}^+$,

$$\gamma_e^-(C_n) = \begin{cases} 5, & if\ n \equiv 0(mod\,4) \\ 2, & if\ n \equiv 1(mod\,4) \\ 3, & if\ n \equiv 2(mod\,4) \\ 4, & if\ n \equiv 3(mod\,4) \end{cases}$$

Theorem 2.7. [6] *If G is a connected graph with diametre d, then* $\gamma_e(G) \geq \frac{\lceil d+2 \rceil}{4}$.

Theorem 2.8. [6] *If G is a connected graph with order n, then* $\gamma_e(G) \leq \frac{2}{5}(n+2)$.

Theorem 2.9. [6] *Let G be a connected graph with order n and T be a spanning tree of G. Then,* $\gamma_e(G) \leq \gamma_e(T)$.

Theorem 2.10. [6] *For every graph G,* $\gamma_e(G) \leq \gamma(G)$. *Also,* $\gamma_e(G) = 1$ *if and only if* $\gamma(G) = 1$.

Theorem 2.11. [6] *There is a tree T with order 375 and* $\gamma_e(T) = 144$.

Theorem 2.12. [8] *If G is a connected graph with n vertices and there is a vertex such that* $\exists v \in V(G)$ $deg(v) = n-1$, *then* $\gamma_e(G) = 1$.

Theorem 2.13. [8] *Let G be a connected graph with n vertices. If* $diam(G) = 2$ *and there isn't any vertex such that* $deg(v) = n-1$, *then* $\gamma_e(G) = 2$.

Theorem 2.14. [8] *For binary graph operations join and corona:*
a) For any two graphs G_1 *and* G_2, $\gamma_e(G_1 \circ G_2) \geq \lceil \frac{diam(G_1 \circ G_2)}{2} \rceil$.
b) Let G_1 *and* G_2 *be any two graphs with diameters* d_1 *and* d_2 *respectively. If* $diam(G_1) = d_1 < diam(G_2) = d_2$, *then* $\gamma_e(G_1 + G_2) = \gamma_e(G_1)$.

Theorem 2.15. [1, 3] *If G is a simple graph and* $diam(G) \geq 3 \Rightarrow diam(\overline{G}) \leq 3$.

Corollary 2.1. [1, 3] *If graph G has diameter at least 3, then* $\gamma(\overline{G}) \leq 2$.

Theorem 2.16. [9] *Let G is a connected graph with n vertices. Then the exponential domination number of the complement prism* $G\overline{G}$ *with* $2n$ *vertices is* $\gamma_e(G\overline{G}) = 2$.

Theorem 2.17. *Let G be a graph with* $diam(G) = d$ *and* $|V(G)| = n$, *then* $G^d \cong K_n$.

Theorem 2.18. [7] *Let* $K_{m,n}$ *be a bipartite complete graph with* $m+n$ *vertices* $(m < n)$. *Then,* $\gamma_e^-(K_{m,n}) = m-1$ *and* $\gamma_e^+(K_{m,n}) = m$.

3. Exponential domination and stability of a graph

We partition the vertices of G into three disjoint sets according to how their removal affects $\gamma_e(G)$ [10]. Let $V = V_e^0 \cup V_e^+ \cup V_e^-$ for

$$V_e^0(G) = \{v \in V(G) : \gamma_e(G - v) = \gamma_e(G)\}$$
$$V_e^+(G) = \{v \in V(G) : \gamma_e(G - v) > \gamma_e(G)\}$$
$$V_e^-(G) = \{v \in V(G) : \gamma_e(G - v) < \gamma_e(G)\}$$

Definition 3.1. [11]
i.) *γ_e- stability of graph G is the minimum number of vertices whose removal changes $\gamma_e(G)$.*
ii.) *γ_e^+- stability ($\gamma_e^- -$) stability of a graph G written γ_e^+ (γ_e^-) is the minimum number of vertices whose removal increase (decrease) $\gamma_e(G)$.*

When the graph under consideration is clear from the context we simply write V_e^0, V_e^+, V_e^-. For the graph in Figure 1, $S = \{v_2, v_4, v_7\}$ is an any minimum exponential domination set. So, $\gamma_e(G) = 3$. If we remove $V_e^+(G) = \{v_9, v_6\}$ we have a remaining graph G^* has an any minimum exponential domination set $S^* = \{v_2, v_5, v_7, v_{11}\}$. So, $\gamma_e(G^*) = 4$ and $\gamma_e^+(G) = 2$. If we remove $V_e^-(G) = \{v_1\}$ from G then, we have a remaining graph G^{**} has an any minimum exponential domination set $S^{**} = \{v_3, v_7\}$. So, $\gamma_e(G^{**}) = 2$ and $\gamma_e^-(G) = 1$.

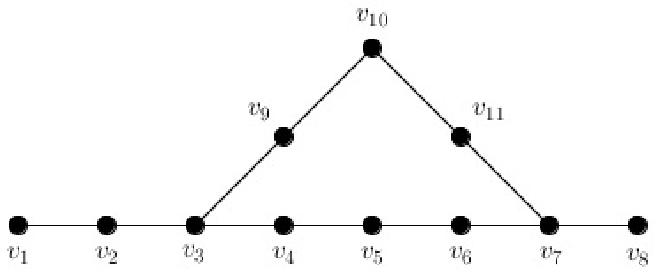

Figure 1. Graph G.

Definition 3.2. [12] *The graph G is said to be a lollipop graph if there exists one edge $e \in E$ such that the removal of the bridge e disconnects G into two graphs K_m and P_n such that K_m is a clique and P_n is a path graph. The lollipop graph $L_{4,5}$ can be depicted as in the following figure:*

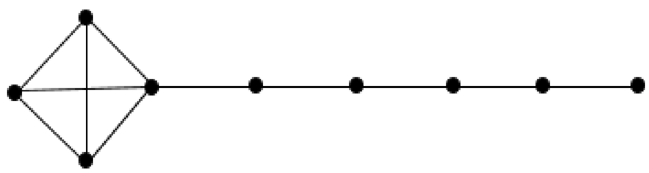

Figure 2. Lollipop Graph $L_{4,5}$.

Theorem 3.1. *Let $L_{m,n}$ be a lollipop graph with $m+n$ vertices. Then,*
a.) $\gamma_e(L_{m,n}) = \lceil \frac{n+3}{4} \rceil$.
b.)

$$\gamma_e^+(L_{m,n}) = \begin{cases} 1, & if\ n \equiv 0,1(mod\, 4) \\ 2, & if\ n \equiv 2,3(mod\, 4) \end{cases}$$

c.)

$$\gamma_e^-(L_{m,n}) = \begin{cases} 3, & if\ n \equiv 0(mod\, 4) \\ 4, & if\ n \equiv 1(mod\, 4) \\ 1, & if\ n \equiv 2(mod\, 4) \\ 2, & if\ n \equiv 3(mod\, 4) \end{cases}$$

Proof. ***a.)*** The graph $L_{m,n}$ consists of a path graph with $n+2$ vertices and a complete graph K_{m-2} with $m-2$ vertices. Let S be an exponential domination set of $L_{m,n}$. We know from the Theorem 2.1 that $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$. The first vertex of the path graph and any vertex of the complete graph are common. So, adding the first vertex of P_{n+2} to the set S, all vertices of K_{m-2} are dominated. So, we have $\gamma_e(L_{m,n}) = \lceil \frac{n+3}{4} \rceil$.
b.) **Case 1.** $n \equiv 0,1(\text{mod } 4)$
In this case, if we remove the vertex $v \in K_m$ with $deg(v) = m$, then we have two graphs with a complete graph K_{m-1} and a path graph P_n. We know $\gamma_e(K_{m-1}) = 1$ and $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$. If $n \equiv 0,1(\text{mod } 4)$, then $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil = \lceil \frac{n+3}{4} \rceil = \gamma_e(L_{m,n})$. Also, we add any vertex from K_{m-1} to S to be dominated the complete graph. Hence, the exponential domination number of $L_{m,n}$ increases.
Case 2. $n \equiv 2,3(\text{mod } 4)$
In this case, removing the vertex $v \in K_m$ with $deg(v) = m$ isn't enough to increase the exponential domination number. Because, for the remaining graphs K_{m-1} and P_n, $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil < \lceil \frac{n+3}{4} \rceil = \gamma_e(L_{m,n})$ and $K_{m-1} = 1$. So, $\gamma_e(P_n) + \gamma_e(K_{m-1}) = \gamma_e(L_{m,n})$. Therefore, we also remove the second vertex of the path graph and we have three remaining graphs that are K_{m-1}, P_1, P_{n-2} which has $\gamma_e(K_{m-1}) = 1$, $\gamma_e(P_1) = 1$ and $\gamma_e(P_{n-2}) = \lceil \frac{n-1}{4} \rceil = \lceil \frac{n+3}{4} \rceil - 1 = L_{m,n} - 1$. So, $\gamma_e(K_{m-1}) + \gamma_e(P_1) + \gamma_e(P_{n-2}) = \gamma_e(L_{m,n}) + 1$.
c.) **Case 1.** $n \equiv 0(\text{mod } 4)$
In this case, when we remove the last three vertices v_{n-2}, v_{n-1}, v_n of P_n, we have two remaining graphs K_m and P_{n-3}. Hence, we have a lollipop graph $L_{m,n-3}$. $\gamma_e(L_{m,n-3}) = \lceil \frac{n}{4} \rceil < \lceil \frac{n+3}{4} \rceil = \gamma_e(L_{m,n})$.
Case 2. $n \equiv 1(\text{mod } 4)$
In this case, when we remove the last four vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n$ of P_n, we have a lollipop graph $L_{m,n-4}$. Hence, $\gamma_e(L_{m,n-4}) = \lceil \frac{n-1}{4} \rceil < \lceil \frac{n+3}{4} \rceil = \gamma_e(L_{m,n})$.
Case 3. $n \equiv 2(\text{mod } 4)$
In this case, when we remove the last vertex v_n of P_n, we have a lollipop graph $L_{m,n-1}$. Hence, $\gamma_e(L_{m,n-1}) = \lceil \frac{n+2}{4} \rceil < \lceil \frac{n+3}{4} \rceil = \gamma_e(L_{m,n})$.
Case 4. $n \equiv 3(\text{mod } 4)$
In this case, when we remove the last two vertices v_{n-1}, v_n of P_n, we have a lollipop graph $L_{m,n-2}$. Hence, $\gamma_e(L_{m,n-2}) = \lceil \frac{n+1}{4} \rceil < \lceil \frac{n+3}{4} \rceil = \gamma_e(L_{m,n})$.
The proof is completed. □

Definition 3.3. [13] *For integer $m \geq 2$ and $n \geq 1$, the comet graph $C_{m,n}$ is defined to be the graph of order $m+n$ obtained from disjoint union of a star $K_{1,m}$ and a path P_n with n vertices by adding an* *edge joining the central vertex of the star with an end-vertex of the path. The Comet graph $C_{7,6}$ can be depicted as in the following figure:*

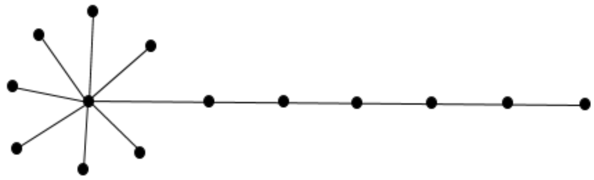

Figure 3. Comet Graph $C_{7,6}$.

Theorem 3.2. *Let $C_{m,n}$ be a comet graph with $m + n$ vertices. Then,*
a.) *[9]* $\gamma_e(C_{m,n}) = \gamma_e(P_{n+2}) = \lceil \frac{n+3}{4} \rceil$
b.) $\gamma_e^+(C_{m,n}) = 1$
c.)

$$\gamma_e^-(C_{m,n}) = \begin{cases} 3, & if\ n \equiv 0(mod\, 4) \\ 4, & if\ n \equiv 1(mod\, 4) \\ 1, & if\ n \equiv 2(mod\, 4) \\ 2, & if\ n \equiv 3(mod\, 4) \end{cases}$$

Proof. ***b.)*** Let S be an exponential domination set of $C_{m,n}$. If we remove the center vertex c of the star graph with $deg(c) = m+1$, then we have a null graph $\overline{K_m}$ and a path graph P_n. We know $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$ and $\gamma_e(\overline{K_m}) = m$. If, $n \equiv 0(\text{mod } 4)$, then $\lceil \frac{n+1}{4} \rceil = \lceil \frac{n}{4} \rceil + 1$, otherwise $\lceil \frac{n+1}{4} \rceil = \lceil \frac{n}{4} \rceil = \gamma_e(C_{m,n}) - 1$. Also, due to $m \geq 2$, if $n \equiv 0(\text{mod } 4)$, then $\gamma_e(\overline{K_m}) + \gamma_e(P_n) = m + \lceil \frac{n+1}{4} \rceil = m + \lceil \frac{n}{4} \rceil + 1 > \gamma_e(C_{m,n})$; otherwise $\gamma_e(\overline{K_m}) + \gamma_e(P_n) = m + \lceil \frac{n+1}{4} \rceil = m + \gamma_e(C_{m,n}) - 1 > \gamma_e(C_{m,n})$. Hence, $\gamma_e^+(C_{m,n}) = 1$.
c.) The proof is similar to the proof of Theorem 3.1.*c.)*
The proof is completed. □

Definition 3.4. [14] *Sunflower graph is a graph obtained by taking a wheel with the central vertex c and the n-cycle $u_1, ..., u_{(n)}$ combined with additional vertices $v_1, ..., v_{(n)}$, where v_i is joined by edges $u_i, u_{(i+1)}$, where $i + 1$ is taken from modulo n. The sunflower graph SF_3 can be depicted as in the following figure:*

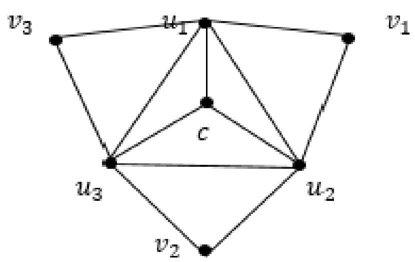

Figure 4. Sunflower Graph SF_3.

Theorem 3.3. *Let SF_n be a sunflower graph and $n > 12$, then*
a.) $\gamma_e(SF_n) = \lceil \frac{n}{5} \rceil + 1$
b.) $\gamma_e^+(SF_n) = 1$,
c.)

$$\gamma_e^-(SF_n) = \begin{cases} 6, & if \ n \equiv 0(mod\,5) \\ n(mod\,5) + 1, & if \ n \equiv otherwise \end{cases}$$

Proof. ***a.)*** We can split the $V(SF_n)$ into three vertex sets that are $V(SF_n) = V_1(SF_n) \cup V_2(SF_n) \cup V_3(SF_n)$ as the following:
$V_1(SF_n) = \{c$: the central vertex with $deg(c) = n\}$
$V_2(SF_n) = \{u_i$: the vertices with $deg(u_i) = 5$ on the cycle graph of $SF_n, i = \overline{(1, n)}\}$
$V_3(SF_n) = \{v_i$: the vertices that are corresponding the edges $(u_i, u_{(i+1)}), i = \overline{(1, n)}\}$
Let S be an exponential dominating set. When we add the vertex c to S, then all vertices of $V_2(SF_n)$ are dominated because of $d(c, u_i) = 1$. If we add the vertices $u_{(i-2)}$ and $u_{(i+4)}$ to S, then the vertex v_i is dominated because of $d(v_i, u_{(i-2)}) = d(v_i, u_{(i+4)}) = 3$ and $d(v_i, c) = 2$. So, we have $w(v_i) = \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2} = 1$. Hence, the distance between the vertices in $V_2(SF_n)$ must be at most 5 for S namely $S = \{u_i, u_{i+5}, u_{i+10}, ...\}$. Therefore, we have $\lceil \frac{n}{5} \rceil$ vertices from $V_2(SF_n)$ for S and considering the central vertex c, we have $\gamma_e(SF_n) = \lceil \frac{n}{5} \rceil + 1$.
b.) If we remove the central vertex c from S, then we need to add the vertices u_{i-1} and u_{i+2} to dominate the vertex v_i since $d(u_{i-1}, v_i) = d(u_{i+2}, v_i) = 2$ for $n > 12$. Hence the distance between the vertices in $V_2(SF_n)$ is at most 3 in S and the exponential domination number increases.
c.) **Case 1.** $n \equiv 0(\text{mod } 5)$
In this case, if we remove the last vertex u_{n-4} from S, then the vertices $v_{n-7}, v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}$ and v_{n-2} can't be dominated exponentially. Therefore, if we omit these vertices from the graph, then we don't need to add the vertex u_{n-4} to S. So, the exponential domination number decreases.
Case 2. $n \equiv 1, 2, 3, 4(\text{mod } 5)$
The distance between the last vertex u_x and the first vertex u_1 that are picked for S in $V_2(SF_n)$ is at 1,2,3,4 in cases $n \equiv 1(\text{mod } 5)$, $n \equiv 2(\text{mod } 5)$, $n \equiv 3(\text{mod } 5)$, $n \equiv 4(\text{mod } 5)$ respectively. If we remove u_x from S, then $v_x, v_{x-1}, v_{x-2}, ..., v_{x-(nmod5)}$ can't be dominated exponentially. Since, the number of these vertices are $n(\text{mod } 5) + 1$, we have $\gamma_e^-(SF_n) = n(\text{mod } 5) + 1$.
The proof is completed. □

Definition 3.5. [15] *A helm H_n is constructed from a wheel W_n by adding n vertices of degree 1, one adjacent to each terminal vertex. It follows that the Helm graph denoted H_n has $2n + 1$ vertices (n vertices of degree 4, n vertices of degree one an one vertex of degree n) and 3n edges. The Helm graph H_n can be depicted as in the following figure:*

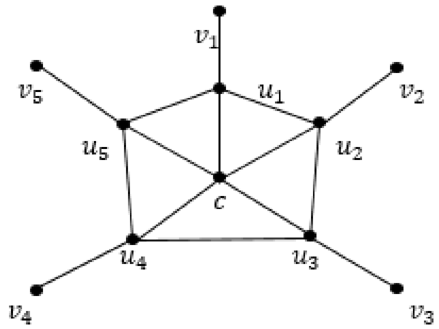

Figure 5. Helm Graph H_5.

Theorem 3.4. *Let H_n be a Helm graph with $2n+1$ vertices and $n > 12$, then*
a.) $\gamma_e(H_n) = \lceil \frac{n}{4} \rceil + 1$
b.) $\gamma_e^+(H_n) = 1$,
c.)

$$\gamma_e^-(H_n) = \begin{cases} 5, & if\ \ n \equiv 0(mod\,4) \\ n(mod\,4)+1, & if\ \ n \equiv otherwise \end{cases}$$

Proof. The proof is similar to the proof of Theorem 3.3. □

Definition 3.6. [16] *For $k \geq 2$ an integer, let N_k be the connected cubic graph constructed as follows. Take k disjoint copies $D_1, D_2, ..., D_k$ of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where a_ib_i is the missing edge in D_i. Let N_k be obtained from the disjoint union of these k diamonds by adding the edges $\{a_ib_{i+1} | i = 1, 2, ..., k-1\}$ and adding the edge a_kb_1. We call N_k a diamond-necklace with k diamonds. A diamond-necklace, N_6, with six diamonds can be depicted as in the following figure:*

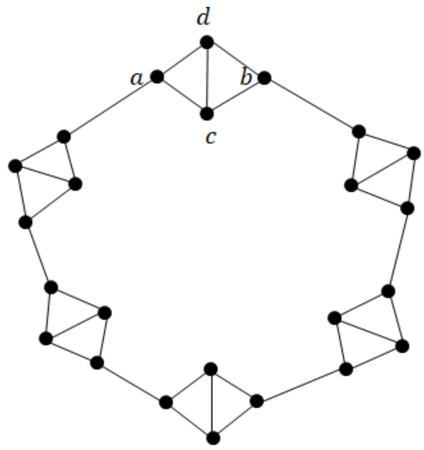

Figure 6. Diamond-Necklace Graph N_6

Theorem 3.5. *Let N_n be a diamond-necklace graph with $4n$ vertices, then*
a.) $\gamma_e(N_n) = \lceil \frac{3n+1}{4} \rceil$
b.)

$$\gamma_e^+(N_n) = \begin{cases} 2, & if\ \gamma_e(N_n) = \gamma_e(N_{n-1}) \\ 4, & otherwise \end{cases}$$

c.)

$$\gamma_e^-(N_n) = \begin{cases} 7, & if\ n \equiv 0(mod\, 4) \\ 5, & if\ n \equiv 1(mod\, 4) \\ 4, & if\ n \equiv 2(mod\, 4) \\ 3, & if\ n \equiv 3(mod\, 4) \end{cases}$$

Proof. ***a.)*** Let S be an exponential dominating set. The vertices c and d are exponentially dominated by the same vertex in S because of $d(c,d) = d(c,a) = d(c,b) = d(d,a) = d(d,b) = 1$ and $d(a,b) = 2$ for every diamond in the graph. Hence, we can regard every diamond as a path P_3. We know $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$ from the Theorem 2.1. There are n diamonds in the diamond-necklace graph N_n. So, $\gamma_e(N_n) = \gamma_e(P_{3n}) = \lceil \frac{3n+1}{4} \rceil$.

b.) If we remove the vertices a_i and b_i from the diamond D_i in N_n, then the graph N_{n-1} and the edge $(c_i d_i)$ remains. To dominate the vertices c_i and d_i, one of them is added to S when $\gamma_e(N_n) = \gamma_e(N_{n-1})$. Hence, the exponential domination number of the graph increases.

But, When $\gamma_e(N_n) \neq \gamma_e(N_{n-1})$ we need to remove the vertices $c_i, d_i \in D_i$ and $c_{i+2}, d_{i+2} \in D_{i+2}$, where $c_{i-1}, c_{i+3} \in S$. Hence, the graph N_{n-3}, one diamond D_{i+1} with two pendant vertices b_i, a_{i+2} and two pendant vertices a_i, b_{i+2} remain. So,

$$\begin{aligned} \gamma_e(N_{n-3}) + \gamma_e(P_5) + 2 &= \lceil \frac{3(n-3)+1}{4} \rceil + \lceil \frac{6}{4} \rceil + 2 \\ &= \lceil \frac{3n-8}{4} \rceil + 2 + 2 \\ &= \lceil \frac{3n}{4} \rceil + 2 \\ &> \lceil \frac{3n+1}{4} \rceil. \\ &= \gamma_e(N_n) \end{aligned}$$

where, $\lceil \frac{3n}{4} \rceil = k$, $k \in \mathbb{Z}$ holds $\forall n \in (\frac{4}{3}k, \frac{4}{3}(k+1))$ and $\lceil \frac{3n}{4} \rceil + 2 = k + 2$. Also, $\lceil \frac{3n-8}{4} \rceil + 4 > \lceil \frac{3n+1}{4} \rceil$ due to $k \leq \lceil \frac{3n+1}{4} \rceil \leq k+1$. Hence, the exponential domination number of the graph is increased by subtracting 4 vertices in total.

c.) We add one vertex from each three consecutive diamonds to S and we don't need to add any vertex from the fourth diamond. Therefore, we have three vertices from every four consecutive diamonds in S. To decrease the exponential domination number by one;

Case 1. $n \equiv 0(\text{mod } 4)$

In this case, six vertices that are four of them at distance 2 and two of them at distance 1 to the vertex $v \in S$ can't be dominated exponentially.

Case 2. $n \equiv 1(\text{mod } 4)$

In this case, four vertices that are three of them at distance 1 and one of them at distance 2 to the vertex $v \in S$ can't be dominated exponentially.

Case 3. $n \equiv 2(\text{mod } 4)$

In this case, three vertices that are two of them at distance 1 and one of them at distance 2 to the vertex $v \in S$ can't be dominated exponentially.

Case 4. $n \equiv 3(\text{mod } 4)$

In this case, two vertices that are at distance 1 to the vertex $v \in S$ can't be dominated exponentially. Hence, the exponential domination number decreases by subtracting these vertices as well as the vertex $v \in S$ from the graph.
The proof is completed. □

Definition 3.7. [16] *For $k \geq 1$, we define a diamond-bracelet B_k with k diamonds as follows. Let B_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, ..., D_{k+1}$ by removing the diamond D_{k+1} and adding a triangle T with $V(T) = \{a, b, c\}$, and adding the edges bb_1 and aa_k. A diamond-bracelet, B_5 with five diamonds can be depicted as in the following figure:*

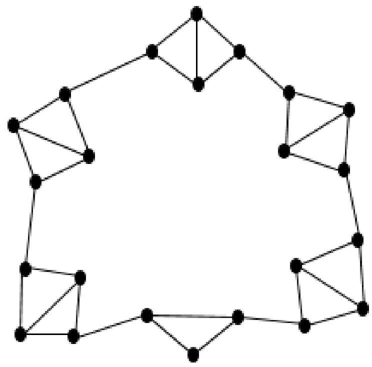

Figure 7. Diamond-bracelet Graph B_5.

Theorem 3.6. *Let B_n for $n \geq 5$ be a diamond-bracelet graph with $4n+3$ vertices, then*
a)

$$\gamma_e(B_n) = \begin{cases} \lceil \frac{3n+1}{4} \rceil, & if\ n \equiv 3, 6, 7 (mod\ 8) \\ \lceil \frac{3n+1}{4} \rceil + 1, & if\ n \equiv 0, 1, 2, 4, 5 (mod\ 8) \end{cases}$$

b)

$$\gamma_e^+(B_n) = \begin{cases} 1, & if\ n = 5, 6, 7 \\ 2, & if\ n = 8 \\ 4, & if\ n \geq 9 \end{cases}$$

c)

$$\gamma_e^-(B_n) = \begin{cases} 4, & if\ n \equiv 0 (mod\ 8) \\ 3, & if\ n \equiv 1 (mod\ 8) \\ 2, & if\ n \equiv 2 (mod\ 8) \\ 6, & if\ n \equiv 3 (mod\ 8) \\ 4, & if\ n \equiv 4 (mod\ 8) \\ 3, & if\ n \equiv 5 (mod\ 8) \\ 7, & if\ n \equiv 6 (mod\ 8) \\ 6, & if\ n \equiv 7 (mod\ 8) \end{cases}$$

Proof. The proof is similar to the proof of Theorem 3.5. □

Definition 3.8. [16] *For $k \geq 1$, we define a diamond-chain L_k with k diamonds as follows. Let L_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, ..., D_{k+1}$ by removing the diamond D_{k+1} and adding two disjoint triangles T_1 and T_2 and adding an edge joining b_1 to a vertex of T_1 and adding an edge joining a_k to a vertex of T_2. A diamond-chain, L_2, with two diamonds can be depicted as in the following figure:*

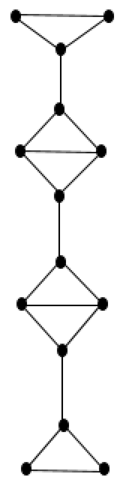

Figure 8. Diamond-chain Graph L_2.

Theorem 3.7. *Let L_n be a diamond-chain graph with $4n+6$ vertices, then*
a.)

$$\gamma_e(L_n) = \begin{cases} 3\lfloor \frac{n}{4} \rfloor + 2, & if\ n \equiv 0, 1 (mod\ 4) \\ 3\lfloor \frac{n}{4} \rfloor + 3, & if\ n \equiv 2 (mod\ 4) \\ 3\lfloor \frac{n}{4} \rfloor + 4, & if\ n \equiv 3 (mod\ 4) \end{cases}$$

b.) $\gamma_e^+(L_n) = 2$
c.)

$$\gamma_e^-(L_n) = \begin{cases} 2, & if\ n \equiv 0 (mod\ 4) \\ 6, & if\ n \equiv 1 (mod\ 4) \\ 4, & if\ n \equiv 2 (mod\ 4) \\ 3, & if\ n \equiv 3 (mod\ 4) \end{cases}$$

Proof. ***a.)*** Let S be an exponential dominating set of L_n. To dominate all vertices in the graph L_n, there must be one vertex from the triangle T_1 and one vertex from the triangle T_2 in S. Also, there must be one vertex from each consecutive three diamonds and there is no need to add any vertex from the fourth diamond for the set S. If we continue in this manner, we have at least $3\lfloor \frac{n}{4} \rfloor + 2$ vertices in S and also;

Case 1. $n \equiv 0, 1(\text{mod } 4)$
In this case, the number of the diamonds is either a multiple of four or one more. So, we don't need to choose another vertex for S.

Case 2. $n \equiv 2(\text{mod } 4)$
In this case, the number of the diamonds is more than two times the quadruple. Hence, to dominate all

vertices exponentially we must also add one vertex from the last diamond to S.
Case 3. $n \equiv 3(\text{mod } 4)$
In this case, the number of the diamonds is more than three times the quadruple. Hence, to dominate all vertices exponentially we must also add two vertices from the last two diamonds to S.
b.) If we remove the vertices a and b from the second diamond in the case $n \equiv 0(\text{mod } 4)$ or from any diamond in other cases, then the exponential domination number increases by one.
c.) To decrease the exponential domination number, it is sufficient to remove the vertex v choosen from the triangle T_2 for S and the vertices that are exponentially dominated by this vertex. So,
Case 1. $n \equiv 0(\text{mod } 4)$
In this case, it is sufficient to remove two vertices with degree 2 in triangle.
Case 2. $n \equiv 1(\text{mod } 4)$
In this case, there is the vertex v with $deg(v) = 3$ from T_2 in S. The vertices except the vertex a in the last diamond before this triangle are exponentially dominated by the vertex v. To decrease the exponential domination number of the graph, we must remove the vertices in T_2 and three vertices except the vertex a from the last diamond .
Case 3. $n \equiv 2(\text{mod } 4)$
In this case, the last two vertices in S are the vertices with degree 2 in T_2 and the other is the vertex a in the last diamond. The vertices c and d are exponentially dominated by the vertex a due to $d(a, c) = d(a, d) = 1$. But, $d(a, b) = 2$. So, we must remove the vertex b and three vertices in T_2 to decrease the exponential domination number.
Case 4. $n \equiv 3(\text{mod } 4)$
In this case, the last two vertices in S are the vertices c or d in the last diamond and the vertex $v \in T_2$. We know $d(c, b) = 1$ and $d(v, c) \geq 2$ $\forall v \in T_2$. So, if we remove the all vertices in the triangle T_2, then the exponential domination number of the graph L_n decreases.
The proof is completed. □

4. Conclusions

If we think of the graph as modelling a communication network, some vertices play a critical role with the deterioration of some centers and connecting lines. Many graph theoretical parameters have been used to describe the stability of communication networks including connectivity, toughness, integrity, scattering number, binding number, domination and its variations [1, 4]. In this paper, we have discussed the graph-theoretic concept of exponential domination number and we investigate the influence of some vertices on this parameter. Analogous work can be carried out for other graph families.

Conflict of interest

The author declares no conflicts of interest in this paper.

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