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Research article

Ulam stability of two fuzzy number-valued functional equations

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Abstract: In this paper, the Ulam stability of two fuzzy number-valued functional equations in Banach spaces is investigated by using the metric defined on a fuzzy number space. Under some suitable conditions, some properties of the solutions for these equations such as existence and uniqueness are discussed.

Keywords: Ulam stability; functional equations; fuzzy number-valued mapping; Banach space
Mathematics Subject Classification: 39B82, 03E72, 39B72

1. Introduction

The study of the stability of functional equations originated from a question by Ulam [1] concerning the stability of group homomorphisms. Since then, Ulam-type stability problems for different types of functional equations in various abstract spaces have been widely and extensively studied (see [2–9] and the refs. contained therein). Meanwhile, it has been successfully implemented in optimization theory (see, e.g., [10]) and economics (see [11]).

Recently, some fuzzy versions of Ulam stability have begun to emerge; however, most of the results were obtained in fuzzy normed spaces (see [12–15]). By contrast, in a Banach space, the Ulam stability of a fuzzy number-valued functional equation was first discussed in the authors' previous work [16], in which it was demonstrated that, under some suitable conditions, the following can be approximated by additive mappings:

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x) \quad \text{and} \quad f(ax+by)=rf(x)+sf(y),$$

where f is a fuzzy number-valued mapping.

The present paper will continue the research originating from the authors' previous work [16]. More precisely, the Ulam stability of the following fuzzy number-valued functional Eqs (1) and (2) are respectively investigated:

$$rf\left(\frac{x+y}{r}\right)+sg\left(\frac{x-y}{s}\right)=2h(x), \tag{1}$$

$$f(x+y+z)=2f\left(\frac{x+y}{2}\right)+f(z). \tag{2}$$

2. Preliminaries

Throughout this paper, R denotes the set of all real numbers, $R_+=(0,\infty)$, X and Y are Banach spaces, $P_{kc}(X)$ denotes the set of all non-empty compact convex subsets of X, and B is a subspace of Y.

Some necessary notions and fundamental results that are used in this paper are herein recalled. The reader is referred to the work by Refs. [17–19] for more information and details.

If a function $u:X\to[0,1]$ satisfies the following conditions:

(i) $[u]^\alpha=\{x\in X:u(x)\geq\alpha\}\in P_{kc}(X)$, $\forall\alpha\in(0,1]$;

(ii) the support set of u: $[u]^0=\text{supp}(u)=\text{cl}\{x:u(x)>0\}$ is a compact set, where the notation "cl" denotes the closure operation, then u is called a fuzzy number on X. The set of all fuzzy numbers on X is denoted by X_F.

For $u,v\in X_F$, $\lambda\in R$, the following properties regarding addition $u+v$ and scalar multiplication $\lambda\cdot u$ can be proven via the Zadeh extension principle (see [17]):

$$[u+v]^\alpha=[u]^\alpha+[v]^\alpha \text{ and } [\lambda\cdot u]^\alpha=\lambda[u]^\alpha.$$

The mapping D: $X_F\times X_F\to R_+\cup\{0\}$ is defined by

$$D(u,v)=\sup_{\alpha\in I}d_H\left([u]^\alpha,[v]^\alpha\right),$$

where d_H is the Hausdorff metric. Then, (X_F,D) is a complete metric space, and D satisfies the following properties: for all $\lambda\in R$ and $u,v,w,e\in X_F$,

(P1) $D(\lambda u,\lambda v)=|\lambda|\cdot D(u,v)$,

(P2) $D(u+w,v+w)=D(u,v)$,

(P3) $D(u+v,w+e)\leq D(u,w)+D(v,e)$.

3. Main results

In this section, the Ulam stability of Eqs (1) and (2) is established, in which f indicates a fuzzy number-valued mapping. The Ulam stability of Eq (1) was investigated in the work by Ebadian et al. [20], in which f was a single value function. Additionally, the Ulam stability of Eq (2) was explored in the work by Lu and Park [21], in which f was a set-valued function. Therefore, the results obtained in the present study are generalizations of the corresponding results in previous works [20, 21].

Theorem 1. If fuzzy number-valued mappings $f, g, h: B \to X_F$ satisfy the inequality

$$D\left(rf\left(\frac{x+y}{r}\right)+sg\left(\frac{x-y}{s}\right),2h(x)\right)<\varepsilon \tag{3}$$

for all $x, y \in B$, where $\varepsilon > 0$ and $r, s \in R\setminus\{0\}$, then there exists a unique additive mapping $T: B \to X_F$ such that $D(T(x),h(x)) \le \frac{3\varepsilon+\delta}{2}$ for all $x \in B$, where $\delta = D(\theta, sg(0)+rf(0))$ and θ is the zero element in X_F.

Moreover, if $h(tx): R \to (X_F, D)$ is continuous for each given $x \in B$, then T is linear on B. Meanwhile, we obtain

$$D\left(T(x), f(x)+\frac{s}{r}g(0)\right)<\frac{4\varepsilon+\delta}{|r|} \quad \text{and} \quad D\left(T(x),\frac{r}{s}f(0)+g(x)\right)<\frac{4\varepsilon+\delta}{|s|}.$$

Proof. In inequality (3), let $y=0$, $y=x$, and $y=-x$, respectively; the following is then obtained:

$$D\left(r f\left(\frac{x}{r}\right)+sg\left(\frac{x}{s}\right),2h(x)\right)<\varepsilon; \tag{4}$$

$$D\left(rf\left(\frac{2x}{r}\right)+sg(0),2h(x)\right)<\varepsilon; \tag{5}$$

$$D\left(rf(0)+sg\left(\frac{2x}{s}\right),2h(x)\right)<\varepsilon. \tag{6}$$

Then,

$$\begin{aligned}
&D\left(\frac{1}{2}h(2x),h(x)\right)\\
&=\frac{1}{4}D(2h(2x),4h(x))\\
&\le\frac{1}{4}D\left(r f\left(\frac{2x}{r}\right)+sg\left(\frac{2x}{s}\right),2h(2x)\right)\\
&+\frac{1}{4}D\left(r f\left(\frac{2x}{r}\right)+sg\left(\frac{2x}{s}\right), r f\left(\frac{2x}{r}\right)+sg(0)+rf(0)+sg\left(\frac{2x}{s}\right)\right)\\
&+\frac{1}{4}D\left(r f\left(\frac{2x}{r}\right)+sg(0)+rf(0)+sg\left(\frac{2x}{s}\right), 4h(x)\right)\\
&\le\frac{\varepsilon}{4}+\delta+\frac{1}{4}D\left(rf\left(\frac{2x}{r}\right)+sg(0),2h(x)\right)\\
&+\frac{1}{4}D\left(rf(0)+sg\left(\frac{2x}{s}\right),2h(x)\right)\\
&<\frac{3\varepsilon+\delta}{4}.
\end{aligned}$$

Next, $h_0(x)=h(x)$ and $h_n(x)=\frac{1}{2^n}h(2^n x)\ (n\in N)$ are set. The following is then obtained:

$$D\left(h_n(x),h_{n-1}(x)\right)=\frac{1}{2^{n-1}}D\left(\frac{1}{2}h\left(2^n x\right),h\left(2^{n-1}x\right)\right)<\frac{3\varepsilon+\delta}{2^{n+1}}. \tag{7}$$

It is then known that $\{f_n(x)\}$ is a Cauchy sequence in X_F. From the completeness of the metric space (X_F,D), there exists a mapping $T:B\to X_F$ such that $T(x)=\lim_{n\to\infty}h_n(x)$ for each $x\in B$.

Next, the additivity of T is proven. From Eqs (3)–(6), it can be concluded that

$$\begin{aligned}
&D\left(h\left(2^n x+2^n y\right)+h\left(2^n x-2^n y\right),h\left(2^{n+1}x\right)\right)\\
&\le D\left(h\left(2^n(x+y)\right)+h\left(2^n(x-y)\right),\ \frac{r}{2}f\left(\frac{2^{n+1}(x+y)}{r}\right)+\frac{s}{2}g(0)+\frac{r}{2}f(0)+\frac{s}{2}g\left(\frac{2^{n+1}(x-y)}{s}\right)\right)\\
&+D\left(\frac{r}{2}f\left(\frac{2^{n+1}(x+y)}{r}\right)+\frac{s}{2}g(0)+\frac{r}{2}f(0)+\frac{s}{2}g\left(\frac{2^{n+1}(x-y)}{s}\right),\right.\\
&\left.\frac{r}{2}f\left(\frac{2^{n+1}(x+y)}{r}\right)+\frac{s}{2}g\left(\frac{2^{n+1}(x-y)}{s}\right)\right)\\
&+D\left(\frac{r}{2}f\left(\frac{2^{n+1}(x+y)}{r}\right)+\frac{s}{2}g\left(\frac{2^{n+1}(x-y)}{s}\right),h\left(2^{n+1}x\right)\right)\\
&<\frac{1}{2}D\left(2h\left(2^n(x+y)\right),rf\left(\frac{2^{n+1}(x+y)}{r}\right)+sg(0)\right)+\frac{1}{2}D\left(2h\left(2^n(x-y)\right),rf(0)+sg\left(\frac{2^{n+1}(x-y)}{s}\right)\right)\\
&+D\left(\frac{s}{2}g(0)+\frac{r}{2}f(0),\theta\right)+\frac{\varepsilon}{2}\\
&<\frac{3\varepsilon+\delta}{2}.
\end{aligned}$$

Therefore,

$$D\left(T(x+y)+T(x-y),T(2x)\right)=\lim_{n\to\infty}\frac{1}{2^n}\left(h\left(2^n(x+y)\right)+h\left(2^n(x-y)\right)\right)=0.$$

Thus, $T(x+y)+T(x-y)=T(2x)$. As a result, $T(x+y)=T(x)+T(y)$ for all $x,y\in B$, i.e., T is additive.

Via inequality (7), the following is obtained:

$$D\left(h(x),T(x)\right)=\lim_{n\to\infty}D\left(h(x),h_n(x)\right)\le\lim_{n\to\infty}\sum_{i=1}^{n}D\left(h_{i-1}(x),h_i(x)\right)\ \le\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3\varepsilon+\delta}{2^{i+1}}=\frac{3\varepsilon+\delta}{2}.$$

Moreover, if $h(tx):R\to(X_F,D)$ is continuous for each given $x\in B$, then

$$\lim_{a\to a_0} T(ax) = \lim_{a\to a_0}\lim_{n\to\infty}\frac{1}{2^n}h\left(2^n ax\right) = \lim_{n\to\infty}\lim_{a\to a_0}\frac{1}{2^n}h\left(2^n ax\right) = \lim_{n\to\infty}\frac{1}{2^n}h\left(2^n a_0 x\right) = T(a_0 x) \qquad (8)$$

for each $a_0 \in R$ and $x \in B$. Recalling that T is additive, $T(cx) = cT(x)$ for each rational number $c \in R$ and $x \in B$. This fact, together with (8), ensures that $T(cx) = cT(x)$ for each $c \in R$ and $x \in B$. As a result, T is linear on B.

Hence, from the linearity of T and the inequality (5), the following is obtained:

$$D\left(f(x) + \frac{s}{r} g(0), T(x) \right)$$

$$\le D\left(f(x) + \frac{s}{r} g(0), \frac{2}{r} h\left(\frac{rx}{2}\right) \right) + D\left(\frac{2}{r} h\left(\frac{rx}{2}\right), \frac{2}{r} T\left(\frac{rx}{2}\right) \right) + D\left(\frac{2}{r} T\left(\frac{rx}{2}\right), T(x) \right)$$

$$= \frac{1}{|r|} D\left(rf(x) + sg(0), 2h\left(\frac{rx}{2}\right) \right) + \frac{2}{|r|} D\left(h\left(\frac{rx}{2}\right), T\left(\frac{rx}{2}\right) \right) + D\left(T(x), T(x)\right).$$

Similarly, the linearity of T and the inequality (6) imply that

$$D\left(g(x) + \frac{r}{s} f(0), T(x) \right) < \frac{4\varepsilon + \delta}{|s|}.$$

Finally, the uniqueness of T is proven. Suppose that there are two additive mappings $T_1, T_2 : B \to X_F$ satisfying $D\left(T_i(x), h(x)\right) \le \frac{3\varepsilon + \delta}{2}$ $\left(i = 1, 2, x \in B\right)$.

Then, as $n \to \infty$,

$$D\left(T_1(x), T_2(x)\right) = \frac{1}{n} D\left(nT_1(x), nT_2(x)\right) \le \frac{1}{n}\left(D\left(T_1(nx), h(nx)\right) + D\left(h(nx), T_2(nx)\right)\right)$$

$$\le \frac{3\varepsilon + \delta}{n} \to 0.$$

Thus, $T_1(x) = T_2(x)$ for all $x \in B$.

Theorem 2. If a fuzzy number-valued mapping $f : B \to X_F$ satisfies the inequality

$$D\left(f(x+y+z), 2f\left(\frac{x+y}{2}\right) + f(z) \right) < \varepsilon \qquad (9)$$

for all $x, y, z \in B$, where $\varepsilon > 0$, then there exists a unique additive mapping $T : B \to X_F$ such that $D\left(T(x), f(x)\right) \le \frac{\varepsilon}{2}$ for all $x \in B$.

Proof. In inequality (9), let $x = y = z$; the following is then obtained:

$$D(f(3x), 3f(x)) < \varepsilon. \qquad (10)$$

Replacing x with $3^n x$ ($n \in N$) in (10), the following is obtained:

$$D\left(f(3^{n+1}x),3f(3^{n}x)\right)<\varepsilon \text{ or } D\left(\frac{f(3^{n+1}x)}{3^{n+1}},\frac{f(3^{n}x)}{3^{n}}\right)<\frac{\varepsilon}{3^{n+1}}.$$

Denoting $f_0(x)=f(x)$, $f_n(x)=\dfrac{f(3^n x)}{3^n}$, then $D\left(f_n(x),f_{n-1}(x)\right)<\dfrac{\varepsilon}{3^n}$ $(n\in N)$. From the completeness of the metric space (X_F,D), a mapping $T:B\to X_F$ with $T(x)=\lim_{n\to\infty}f_n(x)$ is obtained. Moreover, noting that

$$D\left(f_n(x),f(x)\right)\le\sum_{n=1}^{\infty}D\left(f_n(x),f_{n-1}(x)\right)\le\sum_{n=1}^{\infty}\frac{\varepsilon}{3^n}=\frac{\varepsilon}{2},$$

it is known that $D\left(T(x),f(x)\right)\le\dfrac{\varepsilon}{2}$ for all $x\in B$.

It is now demonstrated that T is additive. Via (9), the following is obtained:

$$D\left(T(x+y+z),2T\left(\frac{x+y}{2}\right)+T(z)\right)$$

$$=\lim_{n\to\infty}D\left(f_n(x+y+z),2f_n\left(\frac{x+y}{2}\right)+f_n(z)\right)$$

$$=\lim_{n\to\infty}\frac{1}{3^n}D\left(f\left(3^n(x+y+z)\right),2f\left(\frac{3^n(x+y)}{2}\right)+f(3^n z)\right)\le\lim_{n\to\infty}\frac{\varepsilon}{3^n}=0.$$

Thus,

$$T(x+y+z)=2T\left(\frac{x+y}{2}\right)+T(z),\ \forall x,y,z\in B. \tag{11}$$

Following from Eq (11), it is known that $T(0)=0$ and $2T\left(\dfrac{x}{2}\right)=T(x)$ for all $x\in B$. Consequently,

$$2T\left(\frac{x+y}{2}\right)=T(x+y)=2T\left(\frac{x}{2}\right)+T(y),\quad 2T\left(\frac{y}{2}\right)=T(y).$$

Hence,

$$D\left(T(x+y+z),T(x)+T(y)+T(z)\right)$$

$$=D\left(2T\left(\frac{x+y}{2}\right)+T(z),2T\left(\frac{x}{2}\right)+2T\left(\frac{y}{2}\right)+T(z)\right)=D\left(2T\left(\frac{x+y}{2}\right),2T\left(\frac{x}{2}\right)+2T\left(\frac{y}{2}\right)\right)$$

$$=D\left(2T\left(\frac{x}{2}\right)+T(y),2T\left(\frac{x}{2}\right)+2T\left(\frac{y}{2}\right)\right)=D\left(T(y),2T\left(\frac{y}{2}\right)\right)=0.$$

Thus, T is additive.

Noting that $D\left(T(x), f(x)\right) \leq \frac{\varepsilon}{2}$ for all $x \in B$, the uniqueness of T can be proven by using a similar approach as that in the proof of Theorem 1.

4. Conclusions

The main objective of this paper was to discuss the Ulam stability of two fuzzy number-valued functional equations in Banach spaces via the metric defined on a fuzzy number space. The results made a new a connection between the Ulam stability and fuzzy number-valued functional equations, which together with the authors' previous work. In addition, we will work on different type of fuzzy equations including fuzzy differential equations and higher dimensional fuzzy equations. The work on the Ulam stability of fuzzy differential equations is now in progress.

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Conflict of interest

The authors declare that they have no conflict of interest.

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