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Research article

The new reflected power function distribution: Theory, simulation & application

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Abstract: The aim of the paper is to propose a new Reflected Power function distribution (RPFD). We provide the various properties of the new model in detail such as moments, vitality function and order statistics. We characterize the RPFD based on conditional moments (Right and Left Truncated mean) and doubly truncated mean. We also study the shape of the new distribution to be applicable in many real life situations. We estimate the parameters for the proposed RPFD by using different methods such as maximum likelihood method, modified maximum likelihood method, percentile estimator and modified percentile estimator. The aim of the study is to increase the application of the Power function distribution (PFD). Using two different data sets from real life, we conclude that the RPFD perform better as compare to different competitor models already exist in the literature. We hope that the findings of this paper will be useful for researchers in different field of applied sciences.

Keywords: characterization of truncated distribution; percentile estimator; power function distribution; reflected power function distribution
Mathematics Subject Classification: 60E05

1. Introduction

In the field of reliability and engineering sciences, the researchers mostly prefer to use simple models to obtain failure rates over mathematically complex models. The inverse of Pareto distribution was given by Dallas [1] and named as Power function distribution (PFD). Afterwards Meniconi and Barry [2] preferred to use PFD on Exponential, Lognormal and Weibull distributions as a better fit for the failure rate data.

In recent years, the generalization of the probability distribution has gained great attention. For example Gupta et al. [3], Gupta and Kundu [4], Nadarajah et al. [5] gave the generalization of many of the distributions from literature to exponentiated type distributions for being more flexible to fit to many data sets. Cordeiro et al. [6] introduced exponentiated generalized distributions which were then used by many authors for generalizing different distributions. To get more insight to generalized distributions, interested readers are advised to study Marshall and Olkin [7], Eugene et al. [8]. Shaw and Buckley [9], Silva et al. [10], Zografos and Balakrishnan [11], Cordeiro and de Castro [12], Alexander et al. [13], Zea et al. [14], Alzaatreh et al. [15,16], Cordeiro, Ortega, Popović, and Pescim [17], Nadarajah, Cordeiro and Ortega [18], Aryal and Elbatal [19], Cakmakyapan and Ozel [20], Haghbin et al. [21], Iqbal et al. [22], Karishna et al. [23], Lemonte et al. [24], Rodrigues et al. [25] and Ozel et al. [26]. Alizadeh et al. [27], Cordeiro et al. [28], Bhatti et al. [29] and Haq et al. [30].

A lot of work is available in literature on the generalization of PFD for example Tahir et al. [31], Shahzad and Asghar [32], Hassan and Assar [33], Ibrahim [34], Usman et al. [35], Haq et al. [36] and Zaka et al. [37].

In this paper, we suggest a new probability distribution which reflects the PFD by using Cohen [38]. The literature, we have studied up till now provide us the modifications of probability distributions by using some generators which include more complicated mathematical expressions. The idea behind the current work is to provide a simple probability distribution which provide more real life application by inducting only one new parameter which is called reflecting parameter. The detail is discussed under the following sections. We have derived some of the main structural properties and characterizations of this distribution. The application of this distribution has also been demonstrated with the help of a real life situation where this distribution may best perform.

2. Materials and methods

2.1. Model identification

The probability density function (PDF) of PFD is given as follows

$$g(y)=\frac{\gamma y^{\gamma-1}}{\beta^{\gamma}},\quad 0<y<\beta,\ \text{and}\ \gamma,\beta>0, \tag{1}$$

The cumulative density function (CDF) of PFD is

$$G(y)=\left(\frac{y}{\beta}\right)^{\gamma}. \tag{2}$$

Where $\gamma,\ \beta$ are respectively the shape and scale parameters.

Therefore, RPFD with the help of Cohen [38] technique by reflecting the classical PFD about variate axis at $y=\theta-x$ is,

$$F(x)=1-\frac{(\theta-x)^{\gamma}}{\beta^{\gamma}}, \tag{3}$$

And

$$f(x) = \frac{\gamma(\theta-x)^{\gamma-1}}{\beta^{\gamma}}, \theta-\beta < x < \theta, \text{and } \beta,\theta,\gamma > 0. \tag{4}$$

Where "θ" is the reflecting parameter that will reflect the distribution towards positive skewed to negative skewed or negative skewed to positive skewed. Also γ, β are the shape and scale parameters.

The survival function, hazard rate function (HRF) and cumulative hazard rate function of RPFD are written as

$$S(x) = \frac{(\theta-x)^{\gamma}}{\beta^{\gamma}}, \tag{5}$$

$$H(x) = \frac{\gamma}{\theta-x}, \tag{6}$$

$$CH(x) = log\,\beta^{\gamma} - \gamma\, log(\theta - x).$$

2.2. Asymptotic behavior

We see the asymptotic behavior of the PDF, CDF, hazard and survival functions of RPFD as $x \to 0$ and $x \to \infty$.

i. $lim_{x\to 0}\, f(x) = 0\,, if\ \theta = 0, \gamma = 0\ and\ \beta > 0.$

ii. $lim_{x\to \infty}\, f(x) = \infty\,, \forall\ possible\ values\ of\ \theta, \gamma\ and\ \beta.$

iii. $lim_{x\to 0}\, F(x) = 0\,, if\ \theta = 1, \gamma = 1\ and\ \beta = 1.$

iv. $lim_{x\to 0}\, F(x) = 1, if\ \theta = 0, \gamma = 0\ and\ \beta = 1.$

v. $lim_{x\to \infty}\, F(x) = 1, if\ 0 \le \theta \le \infty, \gamma = 0\ and\ \beta \ge 1.$

vi. $lim_{x\to 0}\, S(x) = 0,\ if\ \theta = 0, \gamma > 0\ and\ \beta > 0.$

vii. $lim_{x\to 0}\, S(x) = \infty,\ if\ \theta \ge 0, \gamma > 0\ and\ \beta = 0.$

viii. $lim_{x\to 0}\, S(x) = 1, if\ \theta \ge 0, \gamma = 0\ and\ \beta \ge 1.$

ix. $lim_{x\to \infty}\, S(x) = \infty, if\ \theta \ge 0, \gamma > 0\ and\ \beta \ge 1.$

x. $lim_{x\to \infty}\, S(x) = 1, if\ \theta \ge 0, \gamma = 0\ and\ \beta \ge 1.$

xi. $lim_{x\to 0}\, H(x) = 0, if\ \theta \ge 1, \gamma = 0.$

xii. $lim_{x\to \infty}\, H(x) = 0, if\ \theta \ge 0, \gamma \ge 1.$

2.3. Characteristics of hazard function using Glaser method

We use the conditions defined by Glaser [39] as

$$\eta(x) = -\frac{\acute{f}(x)}{f(x)},$$

$$\eta(x) = \frac{(\gamma - 1)}{(\theta - x)},$$

$$\acute{\eta}(x) = \frac{(\gamma - 1)}{(\theta - x)^2}.$$

If $x > 0$, then $\acute{\eta}(x) > 0$ under the following conditions

i. If $\gamma \geq 2$, then $\acute{\eta}(x) > 0$.

ii. If $\gamma = 1$, then $\acute{\eta}(x) = 0$.

iii. If $\gamma < 1 \; or \; \gamma = 0$, then $\acute{\eta}(x) < 0$.

The above conditions show that the hazard rate function of RFPD is increasing but if $\gamma < 1 \; or \; \gamma = 0$, then it will be decreasing function.

2.4. Shapes

The RPFD can be negative-skewed, positive-skewed, whereas the HRF can be J-shape, monotonically increasing and decreasing shapes. (See Figures 1–3).

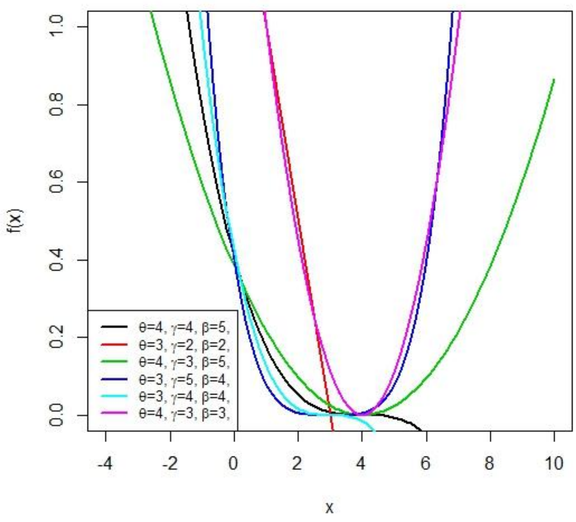

Figure 1. PDF plots of RPFD.

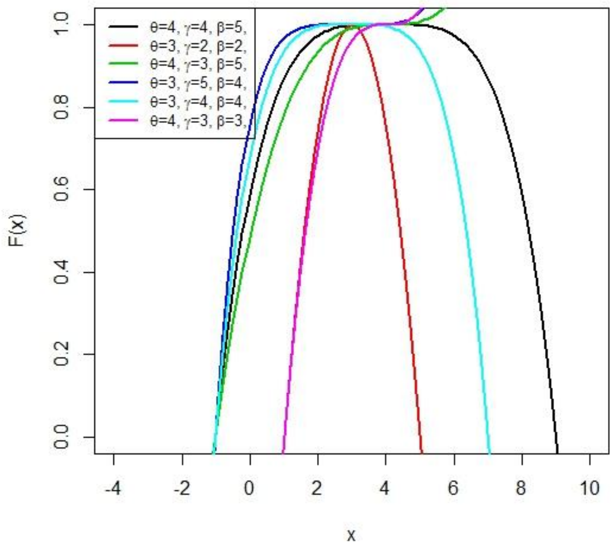

Figure 2. CDF plots of RPFD.

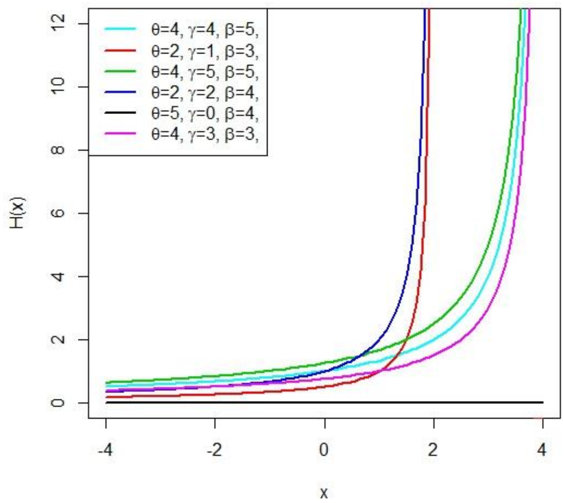

Figure 3. HRF plots of RPFD.

2.5. Moments about Zero

The rth moments about zero of any distribution is described below

$$\mu_r' = \int_{\theta-\beta}^{\theta} x^r \frac{\gamma(\theta - x)^{\gamma-1}}{\beta^\gamma} dx \,,$$

By solving we get

$$\mu_r' = \sum_{j=0}^{\infty} \frac{\gamma(\theta)^{\gamma-j-1}}{\beta^{\gamma}} \frac{(-1)^j \Gamma(\gamma)}{\Gamma(\gamma-j)j!} \left\{\frac{\theta^{r+j+1}-(\theta-\beta)^{r+j+1}}{r+j+1}\right\}, \tag{7}$$

Figure 4 shows the behavior of moments under different parametric values for RPFD.

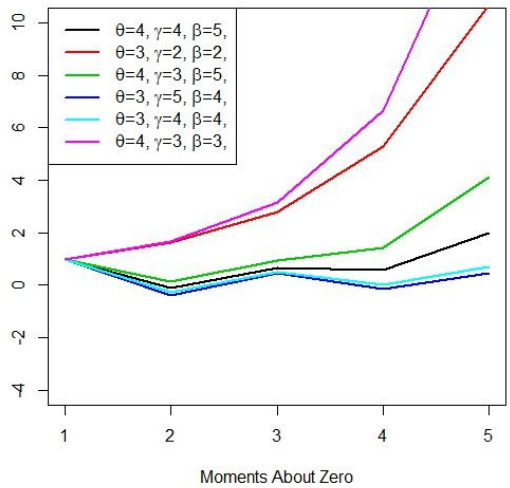

Figure 4. Plots of moments under different parametric values of RPFD.

2.6. Moment generating function

The moment generating function define the characteristic of a random variable. The moment generating function is defined as the linear combination of exponential generalized univariate distributions as

$$M_o(t) = \int_{\theta-\beta}^{\theta} e^{tx} \frac{\gamma(\theta-x)^{\gamma-1}}{\beta^{\gamma}} dx,$$

If “X” follows RPFD, the moment generating function is derived as,

$$M_o(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \sum_{j=0}^{\infty} \frac{\gamma\theta^{\gamma-j-1}}{\beta^{\gamma}} \frac{(-1)^j \Gamma(\gamma)}{\Gamma(\gamma-j)j!} \left(\frac{\theta^{r+j+1}-(\theta-\beta)^{r+j+1}}{r+j+1}\right), \tag{8}$$

2.7. Random number generator

The random number are obtained from

$$R = 1 - \frac{(\theta-x)^{\gamma}}{\beta^{\gamma}}, \tag{9}$$

Where “R" is the random numbers generated from Uniform distribution [0 1].

After simplifying (9) for RPFD we get,

$$x = \theta - \beta(1 - R)^{\frac{1}{\gamma}}.$$

2.8. Inverse moments

The inverse moments are obtained as

$$\mu'_{-r} = \int_{\theta-\beta}^{\theta} x^{-r} \frac{\gamma(\theta - x)^{\gamma-1}}{\beta^{\gamma}} dx \,,$$

We get inverse moments for RPFD as

$$\mu'_{-r} = \sum_{j=0}^{\infty} \frac{\gamma(\theta)^{\gamma-j-1}}{\beta^{\gamma}} \frac{(-1)^{j}\Gamma(\gamma)}{\Gamma(\gamma-j)\, j!} \left(\frac{\theta^{-r+j+1} - (\theta-\beta)^{-r+j+1}}{-r+j+1}\right). \quad (10)$$

2.9. Mean residual function

By definition, the mean residual function is given as

$$e(x) = \int_{\theta-\beta}^{\theta} \frac{S(t)}{S(x)}\, dt,$$

For RPFD, we get mean residual function as

$$e(x) = \frac{\theta - x}{\gamma + 1}. \quad (11)$$

2.10. Vitality function

The vitality function is obtained for RPFD as

$$V(x) = \frac{1}{S(x)} \int_{x}^{\theta} x\, f(x)\, dx,$$

That is obtained as

$$V(x) = \frac{\sum_{j=0}^{\infty} \frac{(-1)^{j}\Gamma(\gamma)\gamma\theta^{\gamma-1-j}}{\Gamma(\gamma-j)\, j!} \left(\frac{\theta^{j+2} - x^{j+2}}{j+2}\right)}{(\theta - x)^{\gamma}}. \quad (12)$$

2.11. Incomplete moments

The incomplete moments are given as

$$\mu_{X|(\alpha,\beta,\gamma)} = \int_{\theta-\beta}^{P} x^r f(x)dx,$$

By simplifying for RPFD we get

$$\mu_{X|(\alpha,\beta,\gamma)} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma)\gamma\theta^{\gamma-j-1}}{\beta^\gamma \Gamma(\gamma-j)j!} \left\{\frac{p^{r+j+1}-(\theta-\beta)^{r+j+1}}{r+j+1}\right\}. \tag{13}$$

2.12. Conditional moments

The conditional moments are given as

$$E[X^r|x>t] = \frac{1}{\bar{F}(t)} \int_t^\theta x^r \sum_{j=0}^{\infty} t_j h_{j+1}(x)\,dx,$$

The conditional moments for RPFD are obtained by using above expression as

$$E[X^r|x>t] = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\gamma)\gamma\theta^{\gamma-j-1}}{\bar{F}(t)\beta^\gamma \Gamma(\gamma-j)j!} \left\{\frac{\theta^{r+j+1}-t^{r+j+1}}{r+j+1}\right\}. \tag{14}$$

2.13. Characterization

2.13.1. Characterization based on conditional moment (Left Truncated Mean)

Let "*X*" be Reflected Power function Variable with Probability density function

$$f(x) = \frac{\gamma(\theta-x)^{\gamma-1}}{\beta^\gamma}, \theta-\beta < x < \theta.$$

And let $\bar{F}(x)$ be the survival function respectively. Then the random variable "*X*" has RPFD if and only if

$$E(X|x \le t) = \frac{1}{F(t)\beta^\gamma}\left[-t\{\theta-t\}^\gamma + (\theta-\beta)\beta^\gamma - \frac{\{\theta-t\}^{\gamma+1}}{\gamma+1} + \frac{\beta^{\gamma+1}}{\gamma+1}\right].$$

where $E(X|x \le t)$Conditional(Left Truncated) mean.

Proof:

Necessary part:

$$E(X^r|x \le t) = \frac{1}{F(t)} \int_{\theta-\beta}^{t} x\, \frac{\gamma(\theta-x)^{\gamma-1}}{\beta^\gamma} dx,$$

$$E(X^r|x \le t) = \frac{1}{F(t)\beta^\gamma}\left[-t\{\theta-t\}^\gamma + (\theta-\beta)\beta^\gamma - \frac{(\theta-t)^{\gamma+1}}{\gamma+1} + \frac{\beta^{\gamma+1}}{\gamma+1}\right], \tag{15}$$

Also Sufficient part

$$E(X^r|x \le t) = \frac{1}{F(t)}\int_{\theta-\beta}^{t} x\ f(x)dx,$$

$$E(X|x \le t) = t - \int_0^t \frac{F(x)}{F(t)}dx, \tag{16}$$

Equate (15) and (16), we get

$$tF(t) - \int_0^t F(x)dx = \frac{1}{\beta^\gamma}\left[-t(\theta-t)^\gamma + (\theta-\beta)\beta^\gamma - \frac{(\theta-t)^{\gamma+1}}{\gamma+1} + \frac{\beta^{\gamma+1}}{\gamma+1}\right],$$

$$tf(t) + F(t) - F(t) = \frac{1}{\beta^\gamma}[t\gamma(\theta-t)^{\gamma-1} - (\theta-t)^\gamma + (\theta-t)^\gamma],$$

$$f(t) = \frac{\gamma(\theta-t)^{\gamma-1}}{\beta^\gamma}, \qquad \theta-\beta < t < \theta, and\ \beta,\gamma,\theta > 0.$$

2.13.2. Characterization based on conditional moment (Right Truncated)

Let “X” be Reflected Power function Variable with Probability density function

$$f(x) = \frac{\gamma(\theta-x)^{\gamma-1}}{\beta^\gamma}, \theta-\beta < x < \theta,$$

And let $\bar{F}(x)$ be the survival function respectively. Then the random variable *“X”* has RPFD if and only if

$$E(X|x \ge t) = \frac{1}{\bar{F}(t)\beta^\gamma}\left[t(\theta-t)^\gamma + \frac{(\theta-t)^{\gamma+1}}{\gamma+1}\right].$$

where $E(X|x \ge t)$ conditional (Right Truncated) mean.

Proof:

$$E(X|x \ge t) = \frac{1}{\bar{F}(t)}\int_t^\theta x\ \frac{\gamma(\theta-X)^{\gamma-1}}{\beta^\gamma}dx,$$

$$E(X|x \ge t) = \frac{1}{\bar{F}(t)\beta^\gamma}\left[t(\theta-t)^\gamma + \frac{(\theta-t)^{\gamma+1}}{\gamma+1}\right], \tag{17}$$

Now sufficient part

$$E(X|x \ge t) = -t - \int_t^\theta \frac{\bar{F}(x)}{\bar{F}(t)}dx, \tag{18}$$

Equate (17) and (18), we get

$$-t\bar{F}(t)-\int_t^\theta \bar{F}(x)dx=\frac{1}{\beta^\gamma}\left[t(\theta-t)^\gamma+\frac{(\theta-t)^{\gamma+1}}{\gamma+1}\right],$$

$$-\Big(t\big(-f(t)\big)+\bar{F}(t)\Big)+\bar{F}(t)=-\frac{1}{\beta^\gamma}[t\gamma(\theta-t)^{\gamma-1}(-1)+(\theta-t)^\gamma+(\theta-t)^\gamma(-1)],$$

$$tf(t)=\frac{t\gamma(\theta-t)^{\gamma-1}}{\beta^\gamma},$$

$$f(t)=\frac{\gamma(\theta-t)^{\gamma-1}}{\beta^\gamma},\theta-\beta<t<\theta, and\ \beta,\theta>0.$$

2.13.3. Characterization based on conditional moment (Doubly Truncated Mean)

Let "*X*" be Reflected Power function Variable with Probability density function

$$f(x)=\frac{\gamma(\theta-x)^{\gamma-1}}{\beta^\gamma},\theta-\beta<x<\theta.$$

And let $\bar{F}(x)$ be the survival function respectively. Then the random variable "*X*" has RPFD if and only if

$$E(X|x<X<y)=\frac{1}{\beta^\gamma\{F(y)-F(x)\}}\left[-y(\theta-y)^\gamma+x(\theta-x)^\gamma-\frac{(\theta-y)^{\gamma+1}}{\gamma+1}+\frac{(\theta-x)^{\gamma+1}}{\gamma+1}\right].$$

where $E(X|x\le X\le y)$: Doubly Truncated Mean .

Proof:

Necessary part:

$$E(X|x\le X\le y)=\frac{1}{F(y)-F(x)}\int_x^y x\frac{\gamma(\theta-x)^{\gamma-1}}{\beta^\gamma}dx,$$

$$E(X|x\le X\le y)=\frac{1}{\beta^\gamma\{F(y)-F(x)\}}\left[-y(\theta-y)^\gamma+x(\theta-x)^\gamma-\frac{(\theta-y)^{\gamma+1}}{\gamma+1}+\frac{(\theta-x)^{\gamma+1}}{\gamma+1}\right] \quad (19)$$

Now Sufficient Part:

$$E(X|x\le X\le y)=\frac{1}{\{F(y)-F(x)\}}\int_x^y x\ f(x)dx,$$

$$E(X|x\le X\le y)=\frac{yF(y)-xF(x)-\int_x^y F(x)dx}{F(y)-F(x)}, \quad (20)$$

Equate (19) and (20), we get

$$\frac{yF(y) - xF(x) - \int_x^y F(x)\,dx}{F(y) - F(x)}$$

$$= \frac{1}{\beta^\gamma\{F(y) - F(x)\}}\left[-y(\theta - y)^\gamma + x(\theta - x)^\gamma - \frac{(\theta - y)^{\gamma+1}}{\gamma + 1} + \frac{(\theta - x)^{\gamma+1}}{\gamma + 1}\right].$$

After differentiating the above equation

$$yf(y) + F(y) - F(y) = \frac{1}{\beta^\gamma}[y\gamma(\theta - y)^{\gamma-1} - (\theta - y)^\gamma + (\theta - y)^\gamma],$$

$$f(y) = \frac{\gamma(\theta - y)^{\gamma-1}}{\beta^\gamma}, \theta - \beta < y < \theta, and\ \beta, \gamma, \theta > 0.$$

3. Estimation of the parameters for RPFD

3.1. Maximum Likelihood Method (MLM)

Let x_1, x_2 ,..., x_n be a random sample of size n from the RPFD. The log-likelihood function for the RPFD is given by

$$L(\gamma, \beta) = nln(\gamma) + (\gamma - 1)\sum_{i=1}^{n} ln(\theta - x_i) - n\gamma\ ln(\beta).$$

The score vector is

$$U_\beta(\gamma, \beta) = \frac{n\gamma}{\beta}, \tag{21}$$

$$U_\gamma(\gamma, \beta) = \frac{n}{\gamma} + \sum_{i=1}^{n} ln(\theta - x_i) - n\,ln(\beta), \tag{22}$$

The parameters of RPFD can be obtained by solving the above equations resulting from setting the two partial derivatives of $L(\gamma, \beta)$ to zero. Since β does not exist, the likelihood function can be maximized by taking

$$\hat{\beta} = x_n, \tag{23}$$

where x_n is the maximum value in the data.

$$\hat{\gamma} = \left(\frac{n}{(n\,ln(\beta) - \sum_{i=1}^{n} ln(\theta - x_i))}\right).$$

3.2. Modified Maximum Likelihood Method (MMLM)

In this modification of the MLM, the Eq (21) is replaced by the median of RPFD.

$$\tilde{x} = \theta - \beta(0.5)^{\frac{1}{\gamma}},$$

By solving the above expression, we get

$$\hat{\beta} = \frac{\theta - \tilde{x}}{(0.5)^{1/\gamma}},$$

$$\frac{n}{\gamma} + \sum_{i=1}^{n} ln(\theta - x_i) - n\, ln\left(\frac{\theta - \tilde{x}}{(0.5)^{\frac{1}{\gamma}}}\right) = 0,$$

$$\hat{\gamma} = \left(\frac{n(1 + ln(0.5))}{(n\, ln(\theta - \tilde{x}) - \sum_{i=1}^{n} ln(\theta - x_i))}\right).$$

3.3. Estimation of RPFD Parameters from "Common Percentiles" (P.E)

Dubey [40] proposed a percentile estimator of the shape parameter, based on any two sample percentiles. Marks [41] also estimated the parameters of Weibull distribution with the help of percentiles and named it as Common Percentile Method.

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n drawn from Probability density function of Reflected Power function distribution. The cumulative distribution function of a Reflected Power function distribution with shape, scale and reflected parameters $(\beta, \gamma\ and\ \theta)$, respectively

$$F(x) = 1 - \frac{(\theta - x)^{\gamma}}{\beta^{\gamma}},$$

By solving we get

$$x = \theta - \beta(1 - R)^{1/\gamma}. \tag{24}$$

Where R = F(x),

Let P75 and P25 are the 75th and 25th Percentiles, therefore (24) becomes

$$P_{75} = \theta - \beta(1 - .75)^{1/\gamma}, \tag{25}$$

$$P_{25} = \theta - \beta(1 - .25)^{1/\gamma}. \tag{26}$$

Solving the above equations, we get

$$\hat{\gamma} = \frac{ln\left(\frac{1 - .75}{1 - .25}\right)}{ln\left(\frac{\theta - P_{75}}{\theta - P_{25}}\right)},$$

$$\text{and} \qquad \hat{\beta} = \frac{(\theta - P_{75})}{(1-.75)^{1/\hat{\gamma}}}.$$

$$\text{generally} \qquad \hat{\gamma} = \frac{ln\left(\frac{1-H}{1-L}\right)}{ln\left(\frac{\theta - P_H}{\theta - P_L}\right)},$$

$$\text{and} \qquad \hat{\beta} = \frac{(\theta - P_H)}{(1-H)^{1/\hat{\gamma}}}.$$

Where H= Maximum Percentage, L= Minimum Percentage and P = Percentile.

3.4. Modified Percentile Estimator (M.P.E)

In this modification of the percentile estimators, (25) is replaced by the Median of Reflected Power function distribution.

$$\tilde{x} = \theta - \beta(0.5)^{\frac{1}{\gamma}},$$

$$\hat{\beta} = \frac{\theta - \tilde{x}}{(0.5)^{1/\gamma}} , \tag{27}$$

Also $\hat{\beta} = \frac{(\theta - P_H)}{(1-H)^{1/\gamma}}$,

Therefore

$$\frac{(\theta - P_H)}{(1-H)^{1/\gamma}} = \frac{\theta - \tilde{x}}{(0.5)^{1/\gamma}},$$

$$\hat{\gamma} = \frac{ln\left(\frac{0.5}{1-H}\right)}{ln\left(\frac{\theta - \tilde{x}}{\theta - P_H}\right)},$$

$$\hat{\beta} = \frac{\theta - \tilde{x}}{(0.5)^{1/\hat{\gamma}}}.$$

Where H= Maximum Percentage and P = Percentile.

A simulation study is used in order to compare the performance of the proposed estimation methods. We carry out this comparison taking the samples of sizes as n = 40 and 100 with pairs of (β, γ) = {(1, 2), (2, 1) and (1.5, 1.5)}. We have generated random samples (using Monte Carlo Simulation) of different sizes by observing that if R_i is random number taking (0, 1), then $x_i = \theta - \beta(1 - R_i)^{1/\gamma}$ is the random number generator from RPFD with $(\gamma, \beta$ and $\theta)$ parameters. All results are based on 5000 replications. Such generated data have been used to obtain estimates of the unknown parameters. The results obtained from parameters estimation of the 3-parameters of

RPFD using different sample sizes and different values of parameters with mean square error MSE.

$$M.S.E\left(\hat{\beta}\right) = E\left[\left(\hat{\beta} - \beta\right)^2\right], M.S.E\left(\hat{\gamma}\right) = E\left[\left(\hat{\gamma} - \gamma\right)^2\right].$$

If we study the results of the Tables (1–4), in which sample sizes are (40 and 100) and the combinations of the values of (β, γ) = {(1, 2), (2, 1) and (1.5, 1.5)}. Then we get the results that P.E is the best for the estimation of β and γ. After P.E, the M.P.E and MMLM are best for the estimation of scale and shape parameters of the Reflected Power function distribution.

Table 1. Estimates for the parameters of RPFD with different estimation methods under the sample size 40 and $\theta = 2$.

Methods	True Values		Estimated Values		M.S.E	
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$
MLM	1	2	1.8605	0.9009	0.745594	1.21558
	2	1	1.9515	1.060192	0.004528007	0.04585
	1.5	1.5	1.887127	1.143243	0.1558585	0.16375
MMLM	1	2	1.006088	2.93567	0.03329484	295.313
	2	1	2.038784	1.222506	0.5796605	27.00332
	1.5	1.5	1.498582	2.008228	0.1275258	80.45536
P.E	1	2	0.9875404	2.178675	0.003350733	0.3289041
	2	1	1.950996	1.09437	0.05049464	0.07982446
	1.5	1.5	1.476918	1.625356	0.0125727	0.1674232
M.P.E	1	2	0.9888006	2.301893	0.01736244	0.7273792
	2	1	1.999309	1.150733	0.3069483	0.1901581
	1.5	1.5	1.492606	1.716947	0.07259521	0.41003

Table 2. Estimates for the parameters of RPFD with different estimation methods under the sample size 100 and $\theta = 2$.

Methods	True Values		Estimated Values		M.S.E	
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$
MLM	1	2	1.911707	0.8737091	0.8332793	1.271054
	2	1	1.980318	1.021556	0.0007741486	0.01242771
	1.5	1.5	1.937788	1.090384	0.193427	0.1768533
MMLM	1	2	0.9994136	2.189761	0.01358523	0.5419498
	2	1	2.021388	1.097434	0.2220572	0.1394513
	1.5	1.5	1.502983	1.64057	0.05312658	0.4238924
P.E	1	2	0.9942939	2.064027	0.001284654	0.09483498
	2	1	1.97935	1.038889	0.02085466	0.02730554
	1.5	1.5	1.491543	1.546539	0.005078406	0.0575562
M.P.E	1	2	0.9947556	2.124453	0.007408561	0.2137276
	2	1	2.008241	1.053431	0.1246356	0.05205558
	1.5	1.5	1.993461	1.586269	0.05174126	0.1175413

Table 3. Estimates for the parameters of Reflected Power function distribution with different estimation methods under the sample size 40 and $\theta = 3$.

Methods	True Values		Estimated Values		M.S.E	
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$
MLM	1	2	2.860517	0.6471663	3.466684	1.831741
	2	1	2.951451	0.7297511	0.7297511	0.08111053
	1.5	1.5	2.884669	0.7645335	1.923084	0.5459775
MMLM	1	2	0.9959269	2.965069	0.03265827	537.9536
	2	1	2.050816	1.334634	0.5815614	75.24864
	1.5	1.5	1.509755	1.938258	0.1358153	32.50174
P.E	1	2	0.9885461	2.173664	0.003262685	0.3074944
	2	1	1.958414	1.086032	0.05149091	0.07893016
	1.5	1.5	1.473992	1.639258	0.01313542	0.1854989
M.P.E	1	2	0.9900613	2.311168	0.01792263	0.7873923
	2	1	1.998593	1.147792	0.3185264	0.1839935
	1.5	1.5	1.486727	1.732373	0.07012046	0.4244317

Table 4. Estimates for the parameters of RPFD with different estimation methods under the sample size 100 and $\theta = 3$.

Methods	True Values		Estimated Values		M.S.E	
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$
MLM	1	2	2.912034	0.6382958	3.658023	1.854838
	2	1	2.980348	0.7178598	0.9614713	0.08242887
	1.5	1.5	2.9371	0.7496401	2.067051	0.5648177
MMLM	1	2	0.9977133	2.197954	0.01377068	0.5290005
	2	1	2.023109	1.096638	0.2262647	0.1675765
	1.5	1.5	1.501211	1.648838	0.054733	0.3114004
P.E	1	2	0.9946443	2.075874	0.001316138	0.1044555
	2	1	1.979712	1.039048	0.02069052	0.02699896
	1.5	1.5	1.490577	1.55466	0.005306528	0.05965253
M.P.E	1	2	0.9953407	2.123411	0.007335557	0.2220309
	2	1	2.004361	1.055047	0.1219972	0.05295546
	1.5	1.5	1.495992	1.582559	0.02916469	0.1177344

4. Application examples

We have taken two different situations from real life and showed the performance of our proposed probability distribution over other already existing probability distributions. The comparison of the probability distributions has been made in both data sets on the basis of Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC).

Finally, using the above mentioned criteria's, we have showed that the proposed RPFD perform better in both data as compared to different competitor models.

4.1. Acute Myelogenous Data

Feigl and Zelen [42] analyzed the survival times (in weeks), of 33 patients suffering from a disease named as Acute Myelogenous Leukaemia. The survival time (in weeks) is given as; 65,156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43. We have compared our proposed distribution with Beta Modified Weibull (BMW) by Silva et al. [10], Exponentiated Generalized Modified Weibull (EGMW) by Aryal and Elbatal [19], Weibull Power function (WPF) by Tahir et al. [31], Transmuated Power Function Distribution (TPFD) by Shahzad and Asghar [32], Exponentiated Weibull Power Function Distribution (EWPFD) by Hassan and Assar [33], Kumaraswamy Power function distribution (KPFD) by Ibrahim [34], and Power function distribution (PFD).

The TTT-plot is displayed in Figure 5, which indicates that the HRF associated with the data set has a decreasing shape, since the plot shows a first convex curvature. So, we can easily fit RPFD on the Acute Myelogenous Data.

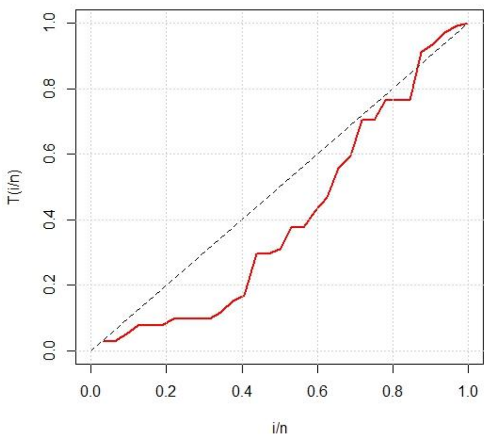

Figure 5. TTT Plot for Acute Myelogenous Data.

From Table 5 and Figure 6, it is clear that the proposed model RPFD is showing better results as compare to the other competitive models by providing smallest AIC, BIC, CAIC and HQIC for the given data.

Table 5. Statistics for Acute Myelogenous Data.

Models	AIC	BIC	CAIC	HQIC
RPFD	304.367	305.8328	304.5004	304.852
EWPFD	305.852	313.335	308.074	308.374
WPF	307.804	313.79	309.232	309.818
EGMW	317.303	324.786	319.525	318.821
BMW	318.967	326.449	321.189	321.484
KPFD	329.734	335.72	331.162	331.748
TPFD	335.131	339.62	335.959	336.642
PFD	965.418	968.411	965.818	966.425

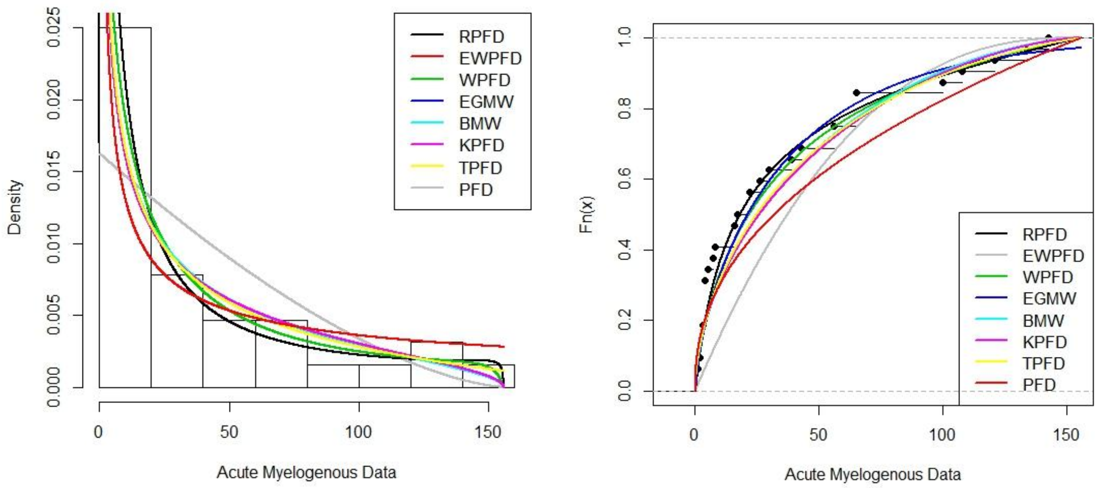

Figure 6. Estimated PDF and CDF curves for Acute Myelogenous Data.

4.2. Bladder cancer patients data

We have adopted the data set consisting the remission time of 128 bladder cancer patients to demonstrate the performance of our proposed Reflected power function distribution. These data were also studied by Zea et al. [14], Lee and Wang [43]. The remission times in months are given: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12 ,79.05.

We have compared our proposed Reflected power function distribution with the Beta Exponentiated Pareto distribution (BEPD) by Zea et al. [14], Marshall-Olkin Power Lomax Distribution (MOPLx) by Haq et al. [30] Kumaraswamy Power function distribution (KPFD) by Ibrahim [34], McDonald`s Power function distribution (McPFD) by Haq et al. [36], and Power function distribution (PFD).

The TTT-plot of the remission time(in months) for bladder cancer patients is exhibited in Figure 7, we may see that the Hazard rate function has little bit bathtub shape, So, we may easily fit RPFD on the bladder cancer data.

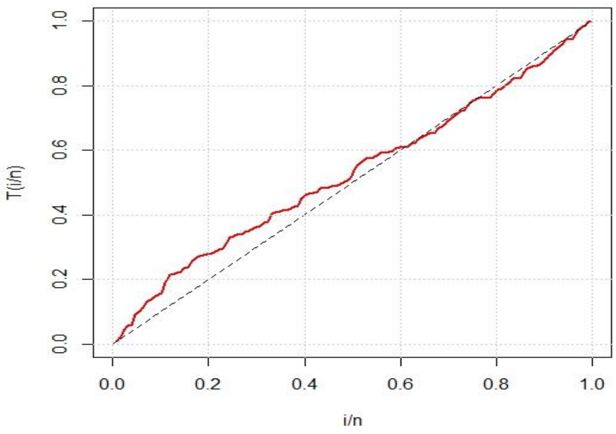

Figure 7. TTT Plot for Bladder Cancer Data.

From Table 6 and Figure 8, we see that the RPFD provides better fit for the above data set as it provides minimum AIC, BIC, CAIC, HQIC.

Table 6. Statistics For Bladder Cancer Data.

Models	AIC	BIC	CAIC	HQIC
RPFD	810.3251	813.1693	810.3571	811.4807
McPFD	811.5785	821.9553	811.9064	816.2008
KPFD	814.0711	822.6037	814.2662	817.5378
MOPLx	827.075	832.483	825.5162	847.3287
BEPD	826.1318	837.5085	826.4596	830.7540
PFD	942.4546	945.2988	942.4866	943.6102

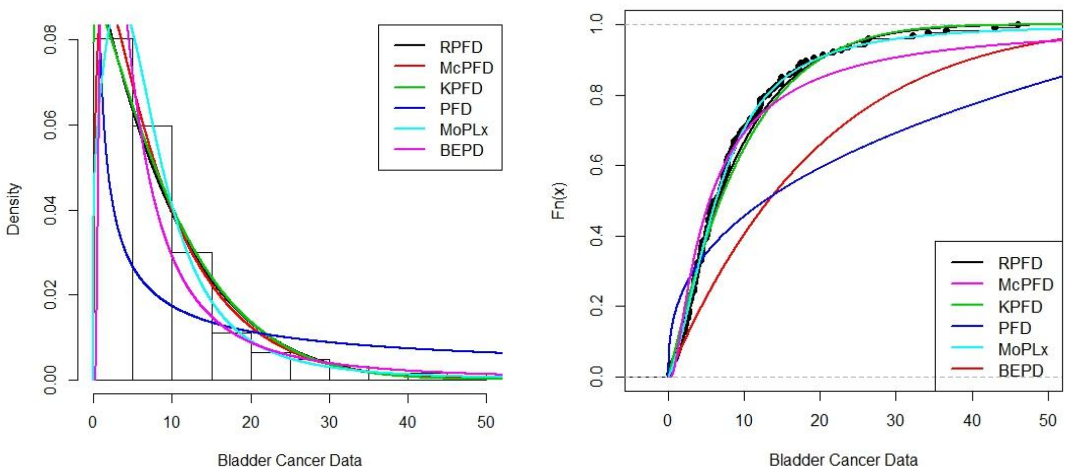

Figure 8. Estimated PDF and CDF curves for Bladder Cancer Data.

5. Conclusions

We propose and study the different properties of RPFD. This distribution has applications in many fields such as reliability, economics, actuaries and survival analysis. We study the several properties of the distribution as moments, survival function, hazard function, inverse moments, shanon entropy, conditional moments, Lorenz curve, incomplete moments and order statistics. We have also characterized the distribution by conditional moments (Right and Left Truncated mean) and doubly truncated mean (DTM). Different estimation methods have been used to estimate the parameters of RPFD including MLM, MMLM, P.E and M.P.E. We have used two real life data sets in order to show the performance of the proposed model over the already available probability models. It is hoped that the findings of this paper will be useful for researchers in different field of applied sciences.

Appendix

A. Simulation Code for MLM

```
phh=c();pll=c();vhat=c(); dsv=c();bhat=c();dsbhat=c()
n=40 #sample size
for(i in 1:5000){
r<-runif(n)
b<-1.5 #scale parameter
v<-1.5 #shape parameter
theta<-3 #reflecting parameter
x<-theta-(b*((1-r)^(1/v)))
vhat[i]<-(n/((n*(log(max(x))))-(sum(log(theta-x)))))
dsv[i]<-((vhat[i]-v)^2)
bhat[i]<-max(x)
dsbhat[i]<-((bhat[i]-b)^2)
        }
estv<-mean(vhat)
estb<-mean(bhat)
msev<-mean(dsv)
mseb<-mean(dsbhat)
```

B. Simulation Code for MMLM

```
phh=c();pll=c();vhat=c(); dsv=c();bhat=c();dsbhat=c()
n=40 #sample size
for(i in 1:5000){
r<-runif(n)
b<-1.5 #scale parameter
v<-1.5 #shape parameter
theta<-3 #reflecting parameter
```

```
x<-theta-(b*((1-r)^(1/v)))
vhat[i]<-(n*(1+log(0.5)))/((n*(log(theta-median(x))))-sum(log(theta-x)))
dsv[i]<-((vhat[i]-v)^2)
bhat[i]<-(theta-median(x))/(0.5^(1/(vhat[i])))
dsbhat[i]<-((bhat[i]-b)^2)
}
estb<-mean(bhat)
estv<-mean(vhat)
msev<-mean(dsv)
mseb<-mean(dsbhat)
```

C. **Simulation Code for P.E**

```
phh=c();pll=c();vhat=c(); dsv=c(); bhat=c(); dsb=c()
n=40 #sample size
h<-0.75 #maximum percentage
l<-0.25 #minimum percentage
for(i in 1:5000){
r<-runif(n)
b<-1.5 # scale parameter
v<-1.5 # shape parameter
theta<-3 #reflecting parameter
x<-theta-(b*((1-r)^(1/v)))
phh[i]<-(quantile(x)[4])
pll[i]<-(quantile(x)[2])
vhat[i]<-((log((1-h)/(1-l)))/(log((theta-phh[i])/(theta-pll[i]))))
dsv[i]<-((vhat[i]-v)^2)
bhat[i]<-(theta-(phh[i]))/((1-h)^(1/((vhat[i]))))
dsb[i]<-((bhat[i]-b)^2)
}
estv<-mean(vhat)
msev<-mean(dsv)
estb<-mean(bhat)
mseb<-mean(dsb)
```

D. **Simulation Code for M.P.E**

```
phh=c();pll=c();vhat=c(); dsv=c(); bhat=c(); dsb=c()
n=40 #sample size
h<-0.75 #maximum percentage
l<-0.25 #minimum percentage
for(i in 1:5000){
r<-runif(n)
```

```
b<-1.5 #scale parameter
v<-1.5 #shape parameter
theta<-3 #reflecting parameter
x<-theta-(b*((1-r)^(1/v)))
phh[i]<-(quantile(x)[4])
pll[i]<-(quantile(x)[2])
vhat[i]<-((log((0.5)/(1-h)))/(log((theta-median(x))/(theta-phh[i]))))
dsv[i]<-((vhat[i]-v)^2)
bhat[i]<-(theta-(median(x)))/((0.5)^(1/((vhat[i]))))
dsb[i]<-((bhat[i]-b)^2)
}
estv<-mean(vhat)
msev<-mean(dsv)
estb<-mean(bhat)
mseb<-mean(dsb)
```

E. **Simulation Code for Application**

```
Acute Myelogenous Data<-c(65, 100, 134, 16, 108, 121, 4, 39, 143,
56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4,
2, 3, 8, 4, 3, 30, 4, 43)
PDF_RPFD<- function(par,x){
v<-par[1] #shape parameter
theta<-156 #reflecting parameter
b<-156 # scale parameter
(theta - x)^(v - 1) * v/(b^v)
}
CDF_RPFD<- function(par,x){
v<-par[1]
theta<-156
b<-156
(1-(((theta-x)^v)/(b^v)))
}
goodness.fit(pdf=PDF_RPFD, cdf=CDF_RPFD,
starts = c(1), data=Acute Myelogenous Data,
method="CG", domain=c(0,156),mle=NULL)
```

Conflicts of interest

All the authors declare no conflict of interest.

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