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Research article

Conformable fractional integral inequalities for *GG*- and *GA*-convex functions

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Abstract: In the article, we present several new Hermite-Hadamard type inequalities for *GG*- and *GA*-convex functions via the conformable fractional integrals. Our results are the generalizations of some previously known results.

Keywords: *GG*-convex function; *GA*-convex function; Hermite-Hadamard inequality; conformable fractional integral
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1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $h : I \to \mathbb{R}$ is said to be convex (concave) on the interval I if the inequality

$$h(t\kappa_1 + (1-t)\kappa_2) \leq (\geq) th(\kappa_1) + (1-t)h(\kappa_2)$$

holds for all $\kappa_1, \kappa_2 \in I$ and $t \in [0, 1]$.

It is well known that convexity (concavity) has wide applications in pure and applied mathematics [1–12]. The well known Hermite-Hadamard inequality [13–20] for the convex (concave) function $h : I \to \mathbb{R}$ can be stated as follows:

$$h\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq (\geq) \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} h(x)dx \leq (\geq) \frac{h(\kappa_1) + h(\kappa_2)}{2}$$

for all $\kappa_1, \kappa_2 \in I$ with $\kappa_1 \neq \kappa_2$.

Recently, many generalizations, invariants and extensions have been made for the convexity, for example, harmonic-convexity [21,22], exponential-convexity [23,24], s-convexity [25,26], Schur-convexity [27–29], strong convexity [30–33], $H_{p,q}$-convexity [34–38], generalized convexity [39], GG- and GA-convexities [40], preinvexity [41] and quasi-convexity [42]. In particular, many remarkable inequalities can be found in the literature [43–58] via the convexity theory.

Niculescu [59,60] defined the GG- and GA-convex functions as follows.

Definition 1.1. (See [59]) A real-valued function $h: I \to [0,\infty)$ is said to be GG-convex on the interval I if the inequality

$$h\left(\kappa_1^t \kappa_2^{1-t}\right) \leq h(\kappa_1)^t h(\kappa_2)^{1-t}$$

holds for all $\kappa_1, \kappa_2 \in I$ and $t \in [0,1]$.

Definition 1.2. (See [60]) A real-valued function $h: I \to [0,\infty)$ is said to be GA-convex if the inequality

$$h\left(\kappa_1^t \kappa_2^{1-t}\right) \leq t h(\kappa_1) + (1-t) h(\kappa_2)$$

holds for all $\kappa_1, \kappa_2 \in I$ and $t \in [0,1]$.

Ardıç et al. [61] established several novel inequalities (Theorem 1.1) involving the GG- and GA-convex functions via an identity (Lemma 1.1) for differentiable functions.

Lemma 1.1. (See [61]) Let $\kappa_1, \kappa_2 \in (0,\infty)$ with $\kappa_1 < \kappa_2$ and $h: [\kappa_1, \kappa_2] \to \mathbb{R}$ be a differentiable function such that $h' \in L([\kappa_1, \kappa_2])$. Then the identity

$$\kappa_2^2 h(\kappa_2) - \kappa_1^2 h(\kappa_1) - 2\int_{\kappa_1}^{\kappa_2} x h(x) dx \tag{1.1}$$

$$= (\log \kappa_2 - \log \eta) \int_0^1 (\kappa_2^t \eta^{1-t})^3 h'(\kappa_2^t \eta^{1-t}) dt + (\log \eta - \log \kappa_1) \int_0^1 (\eta^t \kappa_1^{1-t})^3 h'(\eta^t \kappa_1^{1-t}) dt$$

holds for all $\eta \in [\kappa_1, \kappa_2]$.

Theorem 1.1. (See [61]) Let $\kappa_1, \kappa_2 \in (0,\infty)$ with $\kappa_1 < \kappa_2$ and $h: [\kappa_1, \kappa_2] \to \mathbb{R}$ be a differentiable function such that $h' \in L([\kappa_1, \kappa_2])$. Then the following statements are true:

(1) If $|h'(x)|$ is GG-convex on $[\kappa_1, \kappa_2]$, then the inequality

$$\left| \kappa_2^2 h(\kappa_2) - \kappa_1^2 h(\kappa_1) - 2\int_{\kappa_1}^{\kappa_2} x h(x) dx \right| \tag{1.2}$$

$$\leq (\log \kappa_2 - \log \eta) L(\kappa_2^3 |h'(\kappa_2)|, \eta^3 |h'(\eta)|) + (\log \eta - \log \kappa_1) L(\eta^3 |h'(\eta)|, \kappa_1^3 |h'(\kappa_1)|)$$

holds for all $\eta \in [\kappa_1, \kappa_2]$, where $L(\kappa_1, \kappa_2) = (\kappa_2 - \kappa_1)/(\log \kappa_2 - \log \kappa_1)$ is the logarithmic mean of κ_1 and κ_2.

(2) If $\vartheta, \gamma > 1$ with $1/\vartheta + 1/\gamma = 1$ and $|h'(x)|^\gamma$ is GG-convex on $[\kappa_1, \kappa_2]$, then the inequalities

$$\left| \kappa_2^2 h(\kappa_2) - \kappa_1^2 h(\kappa_1) - 2\int_{\kappa_1}^{\kappa_2} x h(x) dx \right| \tag{1.3}$$

$$\leq (\log \kappa_2 - \log \eta)(L(\kappa_2^{3\vartheta}, \eta^{3\vartheta}))^{\frac{1}{\vartheta}} (L(|h'(\kappa_2)|^\gamma, |h'(\eta)|^\gamma))^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)(L(\eta^{3\vartheta},\kappa_1^{3\vartheta}))^{\frac{1}{\vartheta}}(L(|h\prime(\eta)|^{\gamma},\kappa_1^3|h\prime(\kappa_1)|^{\gamma}))^{\frac{1}{\gamma}},$$

$$\left|\kappa_2^2h(\kappa_2)-\kappa_1^2h(\kappa_1)-2\int_{\kappa_1}^{\kappa_2}xh(x)dx\right| \tag{1.4}$$

$$\leq(\log\kappa_2-\log\eta)(L(\kappa_2^{3\gamma}|h'(\kappa_2)|^{\gamma},\eta^{3\gamma}|h'(\eta)|^{\gamma}))^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)(L(\eta^{3\gamma}|h'(\eta)|^{\gamma},\kappa_1^{3\gamma}|h'(\kappa_1)|^{\gamma}))^{\frac{1}{\gamma}}$$

and

$$\left|\kappa_2^2h(\kappa_2)-\kappa_1^2h(\kappa_1)-2\int_{\kappa_1}^{\kappa_2}xh(x)dx\right| \tag{1.5}$$

$$\leq(\log\kappa_2-\log\eta)(L(\kappa_2^3,\eta^3))^{1-\frac{1}{\gamma}}(L(\kappa_2^3|h'(\kappa_2)|^{\gamma},\eta^3|h\prime(\eta)|^{\gamma}))^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)(L(\eta^3,\kappa_1^3))^{1-\frac{1}{\gamma}}(L(\eta^3|h'(\eta)|^{\gamma},\kappa_1^3|h'(\kappa_1)|^{\gamma}))^{\frac{1}{\gamma}}$$

hold for all $\eta\in[\kappa_1,\kappa_2]$.

(3) If $|h'(x)|$ is GA-convex on $[\kappa_1,\kappa_2]$, then we have

$$\left|\kappa_2^2h(\kappa_2)-\kappa_1^2h(\kappa_1)-2\int_{\kappa_1}^{\kappa_2}xh(x)dx\right| \tag{1.6}$$

$$\leq\frac{|h'(\kappa_2)|}{3}\left[\kappa_2^3-L(\eta^3,\kappa_2^3)\right]+\frac{|h'(\eta)|}{3}[L(\eta^3,\kappa_2^3)-L(\kappa_1^3,\eta^3)]+\frac{|h'(\kappa_1)|}{3}\left[L(\kappa_1^3,\eta^3)-\eta^3\right]$$

for all $\eta\in[\kappa_1,\kappa_2]$.

(4) If $\vartheta,\gamma>1$ with $1/\vartheta+1/\gamma=1$ and $|h'(x)|^{\gamma}$ is GA-convex on $[\kappa_1,\kappa_2]$, then one has

$$\left|\kappa_2^2h(\kappa_2)-\kappa_1^2h(\kappa_1)-2\int_{\kappa_1}^{\kappa_2}xh(x)dx\right| \tag{1.7}$$

$$\leq(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}(L(\kappa_2^3,\eta^3))^{1-\frac{1}{\gamma}}\left(\frac{|h'(\kappa_2)|^{\gamma}\left[\kappa_2^3-L(\eta^3,\kappa_2^3)\right]+|h'(\eta)|^{\gamma}\left[L(\eta^3,\kappa_2^3)-\eta^3\right]}{3}\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}(L(\eta^3,\kappa_1^3))^{1-\frac{1}{\gamma}}\left(\frac{|h'(\eta)|^{\gamma}\left[\eta^3-L(\kappa_1^3,\eta^3)\right]+|h'(\kappa_1)|^{\gamma}\left[L(\kappa_1^3,\eta^3)-\kappa_1^3\right]}{3}\right)^{\frac{1}{\gamma}},$$

$$\left|\kappa_2^2h(\kappa_2)-\kappa_1^2h(\kappa_1)-2\int_{\kappa_1}^{\kappa_2}xh(x)dx\right| \tag{1.8}$$

$$\leq\frac{(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}}{\vartheta^{\frac{1}{\gamma}}}\left(L\left(\kappa_2^{\frac{3(\gamma-\vartheta)}{\gamma-1}},\eta^{\frac{3(\gamma-\vartheta)}{\gamma-1}}\right)\right)^{\frac{\gamma-1}{\gamma}}(A_{\gamma}(\kappa_2,\eta))^{\frac{1}{\gamma}}$$

$$+\frac{(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}}{\vartheta^{\frac{1}{\gamma}}}\left(L\left(\eta^{\frac{3(\gamma-\vartheta)}{\gamma-1}},\kappa_1^{\frac{3(\gamma-\vartheta)}{\gamma-1}}\right)\right)^{\frac{\gamma-1}{\gamma}}(A_{\gamma}(\eta,\kappa_1))^{\frac{1}{\gamma}},$$

$$\left|\kappa_2^2h(\kappa_2)-\kappa_1^2h(\kappa_1)-2\int_{\kappa_1}^{\kappa_2}xh(x)dx\right| \tag{1.9}$$

$$\leq (\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}\left(L\left(\kappa_2^{\frac{3\gamma}{\gamma-1}},\eta^{\frac{3\gamma}{\gamma-1}}\right)\right)^{1-\frac{1}{\gamma}}\left(\frac{|h'(\kappa_2)|^\gamma+|h'(\eta)|^\gamma}{2}\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}\left(L\left(\eta^{\frac{3\gamma}{\gamma-1}},\kappa_1^{\frac{3\gamma}{\gamma-1}}\right)\right)^{1-\frac{1}{\gamma}}\left(\frac{|h'(\eta)|^\gamma+|h'(\kappa_1)|^\gamma}{2}\right)^{\frac{1}{\gamma}},$$

$$\left|\kappa_2^2h(\kappa_2)-\kappa_1^2h(\kappa_1)-2\int_{\kappa_1}^{\kappa_2}xh(x)dx\right| \tag{1.10}$$

$$\leq \frac{(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}}{\gamma^{\frac{1}{\gamma}}}(A_\gamma(\kappa_2,\eta))^{1/\gamma}+\frac{(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}}{\gamma^{\frac{1}{\gamma}}}(A_\gamma(\eta,\kappa_1))^{1/\gamma},$$

where

$$A_\gamma(\kappa_2,\eta)=\frac{|h'(\kappa_2)|^\gamma\left[\kappa_2^{3\gamma}-L(\eta^{3\gamma},\kappa_2^{3\gamma})\right]+|h'(\eta)|^\gamma\left[L(\eta^{3\gamma},\kappa_2^{3\gamma})-\eta^{3\gamma}\right]}{3}$$

and

$$A_\gamma(\eta,\kappa_1)=\frac{|h'(\eta)|^\gamma\left[\eta^{3\gamma}-L(\kappa_1^{3\gamma},\eta^{3\gamma})\right]+|h'(\kappa_1)|^\gamma\left[L(\kappa_1^{3\gamma},\eta^{3\gamma})-\kappa_1^{3\gamma}\right]}{3}.$$

The conformable fractional derivative $D_\alpha(h)(t)$ [62] of order $0<\alpha\leq 1$ at $t>0$ for a function $h:[0,\infty)\to\mathbb{R}$ is defined by

$$D_\alpha(h)(t)=\lim_{\epsilon\to 0}\frac{h(t+\epsilon t^{1-\alpha})-h(t)}{\epsilon},$$

h is said to be α-fractional differentiable if the conformable fractional derivative $D_\alpha(h)(t)$ exists. The conformable fractional derivative at 0 is defined by $h^\alpha(0)=\lim_{t\to 0^+}h^\alpha(t)$. If h_1 and h_2 are α-differentiable at $t>0$, and $\kappa_1,\kappa_2,\lambda,c\in\mathbb{R}$ are constants, then the conformable fractional derivative satisfies the following formulas

$$\frac{d_\alpha}{d_\alpha t}\left(t^\lambda\right)=\lambda t^{\lambda-\alpha},\quad \frac{d_\alpha}{d_\alpha t}(c)=0,$$

$$\frac{d_\alpha}{d_\alpha t}\left(\kappa_1h_1(t)+\kappa_2h_2(t)\right)=\kappa_1\frac{d_\alpha}{d_\alpha t}(h_1(t))+\kappa_2\frac{d_\alpha}{d_\alpha t}(h_2(t)),$$

$$\frac{d_\alpha}{d_\alpha t}\left(h_1(t)h_2(t)\right)=h_1(t)\frac{d_\alpha}{d_\alpha t}(h_2(t))+h_2(t)\frac{d_\alpha}{d_\alpha t}(h_1(t)),$$

$$\frac{d_\alpha}{d_\alpha t}\left(\frac{h_1(t)}{h_2(t)}\right)=\frac{h_2(t)\frac{d_\alpha}{d_\alpha t}(h_1(t))-h_1(t)\frac{d_\alpha}{d_\alpha t}(h_2(t))}{(h_2(t))^2}$$

and

$$\frac{d_\alpha}{d_\alpha t}\left(h_1(h_2(t))\right)=h_1'(h_2(t))\frac{d_\alpha}{d_\alpha t}(h_2(t))$$

if h_1 differentiable at $h_2(t)$. Moreover,

$$\frac{d_\alpha}{d_\alpha t}\left(h_1(t)\right)=t^{1-\alpha}\frac{d}{dt}(h_1(t))$$

if h_1 is differentiable.

Let $\alpha \in (0,1]$ and $0 \leq \kappa_1 < \kappa_2$. Then the function $h : [\kappa_1,\kappa_2] \to \mathbb{R}$ is said to be α-fractional integrable on $[\kappa_1,\kappa_2]$ if the integral

$$\int_{\kappa_1}^{\kappa_2} h(x)d_\alpha x = \int_{\kappa_1}^{\kappa_2} h(x)x^{\alpha-1}dx$$

exists and is finite. All α-fractional integrable functions on $[\kappa_1,\kappa_2]$ is denoted by $L_\alpha([\kappa_1,\kappa_2])$. Note that

$$I_\alpha^{\kappa_1}(h_1)(s) = I_1^{\kappa_1}(s^{\alpha-1}h_1) = \int_{\kappa_1}^{s} \frac{h_1(x)}{x^{1-\alpha}}dx$$

for all $\alpha \in (0,1]$, where the integral is the usual Riemann improper integral.

Recently, the conformable integrals and derivatives have attracted the attention of many researchers. Anderson [63] established the conformable integral version of the Hermite-Hadamard inequality as follows:

$$\frac{\alpha}{\kappa_2^\alpha - \kappa_1^\alpha}\int_{\kappa_1}^{\kappa_2} h(x)d_\alpha x \leq \frac{h(\kappa_1)+h(\kappa_2)}{2}$$

if $\alpha \in (0,1]$ and $h : [\kappa_1,\kappa_2] \to \mathbb{R}$ is an α-fractional differentiable function such that $D_\alpha(h)$ is increasing. Moreover, if function h is decreasing on $[\kappa_1,\kappa_2]$, then

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{\alpha}{\kappa_2^\alpha - \kappa_1^\alpha}\int_{\kappa_1}^{\kappa_2} h(x)d_\alpha x.$$

The main purpose of the article is to establish the conformable fractional integral versions of the Hermite-Hadamard type inequality for *GG*- and *GA*-convex functions.

2. Main results

In order to establish our main results, we need a lemma which we present in this section.

Lemma 2.1. Let $\kappa_1,\kappa_2 \in (0,\infty)$ with $\kappa_1 < \kappa_2$, $\alpha \in (0,1]$ and $h : [\kappa_1,\kappa_2] \to \mathbb{R}$ be an α-fractional differentiable function on (κ_1,κ_2) such that $D_\alpha(h) \in L_\alpha([\kappa_1,\kappa_2])$. Then the identity

$$\kappa_2^{2\alpha}h(\kappa_2) - \kappa_1^{2\alpha}h(\kappa_1) - 2\alpha\int_{\kappa_1}^{\kappa_2} x^\alpha h(x)d_\alpha x \tag{2.1}$$

$$= (\log\kappa_2 - \log\eta)\int_0^1 (\kappa_2^t\eta^{1-t})^{3\alpha}D_\alpha(h)(\kappa_2^t\eta^{1-t})t^{1-\alpha}dt$$

$$+(\log\eta - \log\kappa_1)\int_0^1 (\eta^t\kappa_1^{1-t})^{3\alpha}D_\alpha(h)(\eta^t\kappa_1^{1-t})t^{1-\alpha}dt$$

holds for all $\eta \in [\kappa_1,\kappa_2]$.

Proof. Integration by parts, we get

$$I_1 = \int_0^1 (\kappa_2^t \eta^{1-t})^{3\alpha} D_\alpha(h)(\kappa_2^t \eta^{1-t}) t^{1-\alpha} dt$$

$$= \int_0^1 (\kappa_2^t \eta^{1-t})^{2\alpha+1} h'(\kappa_2^t \eta^{1-t}) dt.$$

Let $x = \kappa_2^t \eta^{1-t}$. Then I_1 can be rewritten as

$$I_1 = \frac{1}{\log \kappa_2 - \log \eta} \int_\eta^{\kappa_2} x^{2\alpha} h'(x) dx$$

$$= \frac{1}{\log \kappa_2 - \log \eta} \left[\kappa_2^\alpha h(\kappa_2) - \eta^\alpha h(\eta) - 2\alpha \int_\eta^{\kappa_2} x^{2\alpha-1} h(x) dx \right]$$

$$= \frac{1}{\log \kappa_2 - \log \eta} \left[\kappa_2^\alpha h(\kappa_2) - \eta^\alpha h(\eta) - 2\alpha \int_\eta^{\kappa_2} x^\alpha h(x) d_\alpha x \right].$$

Similarly, we have

$$I_2 = \int_0^1 (\eta^t \kappa_1^{1-t})^{3\alpha} D_\alpha(h)(\eta^t \kappa_1^{1-t}) t^{1-\alpha} dt$$

$$= \frac{1}{\log \eta - \log \kappa_1} \left[\eta^\alpha h(\eta) - \kappa_1^\alpha h(\kappa_1) - 2\alpha \int_{\kappa_1}^{\eta} x^\alpha h(x) d_\alpha x \right].$$

Multiplying I_1 by $(\log \kappa_2 - \log \eta)$ and I_2 by $(\log \eta - \log \kappa_1)$, then add them we get the desired identity. □

Remark 2.1. Let $\alpha = 1$. Then identity (2.1) reduces to (1.1).

Theorem 2.1. Let $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$, $\alpha \in (0, 1]$, $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an α-fractional differentiable function on (κ_1, κ_2) such that $D_\alpha(h) \in L_\alpha([\kappa_1, \kappa_2])$ and $|h'(x)|$ be a GG-convex function on $[\kappa_1, \kappa_2]$. Then the inequality

$$\left| \kappa_2^{2\alpha} h(\kappa_2) - \kappa_1^{2\alpha} h(\kappa_1) - 2\alpha \int_{\kappa_1}^{\kappa_2} x^\alpha h(x) d_\alpha x \right| \tag{2.2}$$

$$\leq (\log \kappa_2 - \log \eta) L(\kappa_2^{2\alpha+1} |h'(\kappa_2)|, \eta^{2\alpha+1} |h'(\eta)|)$$

$$+ (\log \eta - \log \kappa_1) L(\eta^{2\alpha+1} |h'(\eta)|, \kappa_1^{2\alpha+1} |h'(\kappa_1)|)$$

holds for all $\eta \in [\kappa_1, \kappa_2]$.

Proof. It follows from the GG-convexity of the function $|h'(x)|$ on the interval $[\kappa_1, \kappa_2]$ and Lemma 2.1 that

$$\left| \kappa_2^{2\alpha} h(\kappa_2) - \kappa_1^{2\alpha} h(\kappa_1) - 2\alpha \int_{\kappa_1}^{\kappa_2} x^\alpha h(x) d_\alpha x \right|$$

$$\leq (\log \kappa_2 - \log \eta) \int_0^1 (\kappa_2^t \eta^{1-t})^{2\alpha+1} |h'(\kappa_2^t \eta^{1-t})| dt$$

$$+(\log\eta-\log\kappa_1)\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|dt$$

$$\leq(\log\kappa_2-\log\eta)\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2)|^t|h'(\eta)|^{1-t}dt$$

$$+(\log\eta-\log\kappa_1)\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta)|^t|h'(\kappa_1)|^{1-t}dt$$

$$=(\log\kappa_2-\log\eta)L(\kappa_2^{2\alpha+1}|h'(\kappa_2)|,\eta^{2\alpha+1}|h'(\eta)|)$$

$$+(\log\eta-\log\kappa_1)L(\eta^{2\alpha+1}|h'(\eta)|,\kappa_1^{2\alpha+1}|h'(\kappa_1)|).$$

□

Remark 2.2. Let $\alpha=1$. Then inequality (2.2) reduces to (1.2).

Theorem 2.2. Let $\kappa_1,\kappa_2\in(0,\infty)$ with $\kappa_1<\kappa_2$, $\vartheta,\gamma>1$ with $1/\vartheta+1/\gamma=1$, $\alpha\in(0,1]$, $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an α-fractional differentiable function on (κ_1,κ_2) such that $D_\alpha(h)\in L_\alpha([\kappa_1,\kappa_2])$ and $|h'(x)|^\gamma$ be a GG-convex function on $[\kappa_1,\kappa_2]$. Then the inequality

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right| \tag{2.3}$$

$$\leq(\log\kappa_2-\log\eta)(L(\kappa_2^{(2\alpha+1)\vartheta},\eta^{(2\alpha+1)\vartheta}))^{\frac{1}{\vartheta}}(L(|h'(\kappa_2)|^\gamma,|h'(\eta)|^\gamma))^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)(L(\eta^{(2\alpha+1)\vartheta},\kappa_1^{(2\alpha+1)\vartheta}))^{\frac{1}{\vartheta}}(L(|h'(\eta)|^\gamma,|h'(\kappa_1)|^\gamma))^{\frac{1}{\gamma}}$$

holds for all $\eta\in[\kappa_1,\kappa_2]$.

Proof. From Lemma 2.1, the property of the modulus, GG-convexity of $|h'|^\gamma$ and Hölder inequality we clearly see that

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right|$$

$$\leq(\log\kappa_2-\log\eta)\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|dt$$

$$+(\log\eta-\log\kappa_1)\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|dt$$

$$\leq(\log\kappa_2-\log\eta)\left(\int_0^1(\kappa_2^t\eta^{1-t})^{(2\alpha+1)\vartheta}dt\right)^{\frac{1}{\vartheta}}\left(\int_0^1|h'(\kappa_2^t\eta^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1(\eta^t\kappa_1^{1-t})^{(2\alpha+1)\vartheta}dt\right)^{\frac{1}{\vartheta}}\left(\int_0^1|h'(\eta^t\kappa_1^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$\leq(\log\kappa_2-\log\eta)\left(\int_0^1(\kappa_2^t\eta^{1-t})^{(2\alpha+1)\vartheta}dt\right)^{\frac{1}{\vartheta}}\left(\int_0^1|h'(\kappa_2)|^{\gamma t}|h'(\eta)|^{(1-t)\gamma}dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1(\eta^t\kappa_1^{1-t})^{(2\alpha+1)\vartheta}dt\right)^{\frac{1}{\vartheta}}\left(\int_0^1|h'(\eta)|^{\gamma t}|h'(\kappa_1)|^{(1-t)\gamma}dt\right)^{\frac{1}{\gamma}}$$

$$=(\log\kappa_2-\log\eta)(L(\kappa_2^{(2\alpha+1)\vartheta},\eta^{(2\alpha+1)\vartheta}))^{\frac{1}{\vartheta}}(L(|h'(\kappa_2)|^\gamma,|h'(\eta)|^\gamma))^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)(L(\eta^{(2\alpha+1)\vartheta},\kappa_1^{(2\alpha+1)\vartheta}))^{\frac{1}{\vartheta}}(L(|h'(\eta)|^\gamma,|h'(\kappa_1)|^\gamma))^{\frac{1}{\gamma}}.$$

□

Remark 2.3. Let $\alpha = 1$. Then inequality (2.3) reduces to (1.3).

Theorem 2.3. Let $\kappa_1,\kappa_2\in(0,\infty)$ with $\kappa_1<\kappa_2$, $\gamma>1$, $\alpha\in(0,1]$, $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an α-fractional differentiable function on (κ_1,κ_2) such that $D_\alpha(h)\in L_\alpha([\kappa_1,\kappa_2])$ and $|h'(x)|^\gamma$ be a GG-convex function on $[\kappa_1,\kappa_2]$. Then the inequality

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right| \tag{2.4}$$

$$\leq(\log\kappa_2-\log\eta)(L(\kappa_2^{(2\alpha+1)\gamma}|h'(\kappa_2)|^\gamma,\eta^{(2\alpha+1)\gamma}|h'(\eta)|^\gamma))^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)(L(\eta^{(2\alpha+1)\gamma}|h'(\eta)|^\gamma,\kappa_1^{(2\alpha+1)\gamma}|h'(\kappa_1)|^\gamma))^{\frac{1}{\gamma}}$$

holds for all $\eta\in[\kappa_1,\kappa_2]$.

Proof. It follows from Lemma 2.1 that

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right|$$

$$\leq(\log\kappa_2-\log\eta)\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|dt$$

$$+(\log\eta-\log\kappa_1)\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|dt.$$

Let $\vartheta>1$ such that $\vartheta^{-1}+\gamma^{-1}=1$. Then making use of the Hölder integral inequality and the GG-convexity of $|h'|^\gamma$, we get

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right|$$

$$\leq(\log\kappa_2-\log\eta)\left(\int_0^1dt\right)^{\frac{1}{\vartheta}}\left(\int_0^1(\kappa_2^t\eta^{1-t})^{(2\alpha+1)\gamma}|h'(\kappa_2^t\eta^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1dt\right)^{\frac{1}{\vartheta}}\left(\int_0^1(\eta^t\kappa_1^{1-t})^{(2\alpha+1)\gamma}|h'(\eta^t\kappa_1^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$\leq(\log\kappa_2-\log\eta)\left(\int_0^1dt\right)^{\frac{1}{\vartheta}}\left(\int_0^1(\kappa_2^t\eta^{1-t})^{(2\alpha+1)\gamma}|h'(\kappa_2)|^{\gamma t}|h'(\eta)|^{(1-t)\gamma}dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1 dt\right)^{\frac{1}{\vartheta}}\left(\int_0^1(\eta^t\kappa_1^{1-t})^{(2\alpha+1)\gamma}|h'(\eta)|^{\gamma t}|h'(\kappa_1)|^{(1-t)\gamma}dt\right)^{\frac{1}{\gamma}}$$
$$=(\log\kappa_2-\log\eta)(L(\kappa_2^{(2\alpha+1)\gamma}|h'(\kappa_2)|^{\gamma},\eta^{(2\alpha+1)\gamma}|h'(\eta)|^{\gamma}))^{\frac{1}{\gamma}}$$
$$+(\log\eta-\log\kappa_1)(L(\eta^{(2\alpha+1)\gamma}|h'(\eta)|^{\gamma},\kappa_1^{(2\alpha+1)\gamma}|h'(\kappa_1)|^{\gamma}))^{\frac{1}{\gamma}}.$$

□

Remark 2.4. Let $\alpha=1$. Then inequality (2.4) reduces to (1.4).

Theorem 2.4. Let $\kappa_1,\kappa_2\in(0,\infty)$ with $\kappa_1<\kappa_2$, $\gamma>1$, $\alpha\in(0,1]$, $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an α-fractional differentiable function on (κ_1,κ_2) such that $D_\alpha(h)\in L_\alpha([\kappa_1,\kappa_2])$ and $|h'(x)|^\gamma$ be a GG-convex function on $[\kappa_1,\kappa_2]$. Then the inequality

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right| \tag{2.5}$$
$$\le(\log\kappa_2-\log\eta)(L(\kappa_2^{(2\alpha+1)},\eta^{(2\alpha+1)}))^{1-\frac{1}{\gamma}}(L(\kappa_2^{(2\alpha+1)}|h'(\kappa_2)|^{\gamma},\eta^{(2\alpha+1)}|h'(\eta)|^{\gamma}))^{\frac{1}{\gamma}}$$
$$+(\log\eta-\log\kappa_1)(L(\eta^{(2\alpha+1)},\kappa_1^{(2\alpha+1)}))^{1-\frac{1}{\gamma}}(L(\eta^{(2\alpha+1)}|h'(\eta)|^{\gamma},\kappa_1^{(2\alpha+1)}|h'(\kappa_1)|^{\gamma}))^{\frac{1}{\gamma}}$$

holds whenever $\eta\in[\kappa_1,\kappa_2]$.

Proof. From the GG-convexity of $|h'|^\gamma$, power mean inequality, the property of the modulus and Lemma 2.1 we clearly see that

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right|$$
$$\le(\log\kappa_2-\log\eta)\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|dt$$
$$+(\log\eta-\log\kappa_1)\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|dt$$
$$\le(\log\kappa_2-\log\eta)\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$
$$+(\log\eta-\log\kappa_1)\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$
$$\le(\log\kappa_2-\log\eta)\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2)|^{\gamma t}|h'(\eta)|^{(1-t)\gamma}dt\right)^{\frac{1}{\gamma}}$$
$$+(\log\eta-\log\kappa_1)\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta)|^{\gamma t}|h'(\kappa_1)|^{(1-t)\gamma}dt\right)^{\frac{1}{\gamma}}$$
$$=(\log\kappa_2-\log\eta)(L(\kappa_2^{(2\alpha+1)},\eta^{(2\alpha+1)}))^{1-\frac{1}{\gamma}}(L(\kappa_2^{(2\alpha+1)}|h'(\kappa_2)|^{\gamma},\eta^{(2\alpha+1)}|h'(\eta)|^{\gamma}))^{\frac{1}{\gamma}}$$
$$+(\log\eta-\log\kappa_1)(L(\eta^{(2\alpha+1)},\kappa_1^{(2\alpha+1)}))^{1-\frac{1}{\gamma}}(L(\eta^{(2\alpha+1)}|h'(\eta)|^{\gamma},\kappa_1^{(2\alpha+1)}|h'(\kappa_1)|^{\gamma}))^{\frac{1}{\gamma}}.$$

□

Remark 2.5. Let $\alpha = 1$. Then inequality (2.5) reduces to (1.5).

Theorem 2.5. Let $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$, $\alpha \in (0, 1]$, $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an α-fractional differentiable function on (κ_1, κ_2) such that $D_\alpha(h) \in L_\alpha([\kappa_1, \kappa_2])$ and $|h'(x)|$ be a GA-convex function on $[\kappa_1, \kappa_2]$. Then the inequality

$$\left|\kappa_2^{2\alpha} h(\kappa_2) - \kappa_1^{2\alpha} h(\kappa_1) - 2\alpha \int_{\kappa_1}^{\kappa_2} x^\alpha h(x) d_\alpha x\right| \tag{2.6}$$

$$\leq \frac{|h(\kappa_2)|}{2\alpha + 1}\left[\kappa_2^{2\alpha+1} - L(\eta^{2\alpha+1}, \kappa_2^{2\alpha+1})\right] + \frac{|h'(\eta)|}{2\alpha + 1}\left[L(\eta^{2\alpha+1}, \kappa_2^{2\alpha+1}) - L(\kappa_1^{2\alpha+1}, \eta^{2\alpha+1})\right]$$
$$+ \frac{|h'(\kappa_1)|}{2\alpha + 1}\left[L(\kappa_1^{2\alpha+1}, \eta^{2\alpha+1}) - \eta^{2\alpha+1}\right]$$

holds for each $\eta \in [\kappa_1, \kappa_2]$.

Proof. It follows from the GA-convexity of $|h'(x)|$ and Lemma 2.1 that

$$\left|\kappa_2^{2\alpha} h(\kappa_2) - \kappa_1^{2\alpha} h(\kappa_1) - 2\alpha \int_{\kappa_1}^{\kappa_2} x^\alpha h(x) d_\alpha x\right|$$

$$\leq (\log \kappa_2 - \log \eta) \int_0^1 (\kappa_2^t \eta^{1-t})^{2\alpha+1} |h'(\kappa_2^t \eta^{1-t})| dt$$
$$+ (\log \eta - \log \kappa_1) \int_0^1 (\eta^t \kappa_1^{1-t})^{2\alpha+1} |h'(\eta^t \kappa_1^{1-t})| dt$$
$$\leq (\log \kappa_2 - \log \eta) \int_0^1 (\kappa_2^t \eta^{1-t})^{2\alpha+1} \left[t|h'(\kappa_2)| + (1-t)|h'(\eta)|\right] dt$$
$$+ (\log \eta - \log \kappa_1) \int_0^1 (\eta^t \kappa_1^{1-t})^{2\alpha+1} \left[t|h'(\eta)| + (1-t)|h'(\kappa_1)|\right] dt$$
$$= \frac{|h'(\kappa_2)|}{2\alpha + 1}\left[\kappa_2^{2\alpha+1} - L(\eta^{2\alpha+1}, \kappa_2^{2\alpha+1})\right] + \frac{|h'(\eta)|}{2\alpha + 1}\left[L(\eta^{2\alpha+1}, \kappa_2^{2\alpha+1}) - L(\kappa_1^{2\alpha+1}, \eta^{2\alpha+1})\right]$$
$$+ \frac{|h'(\kappa_1)|}{2\alpha + 1}\left[L(\kappa_1^{2\alpha+1}, \eta^{2\alpha+1}) - \eta^{2\alpha+1}\right].$$

□

Remark 2.6. Let $\alpha = 1$. Then inequality (2.6) becomes (1.6).

Theorem 2.6. Let $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$, $\alpha \in (0, 1]$, $\gamma > 1$, $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an α-fractional differentiable function on (κ_1, κ_2) such that $D_\alpha(h) \in L_\alpha([\kappa_1, \kappa_2])$ and $|h'(x)|^\gamma$ be a GA-convex function on $[\kappa_1, \kappa_2]$. Then the inequality

$$\left|\kappa_2^{2\alpha} h(\kappa_2) - \kappa_1^{2\alpha} h(\kappa_1) - 2\alpha \int_{\kappa_1}^{\kappa_2} x^\alpha h(x) d_\alpha x\right| \tag{2.7}$$

$$\leq (\log \kappa_2 - \log \eta)^{1-\frac{1}{\gamma}} (L(\kappa_2^{(2\alpha+1)}, \eta^{(2\alpha+1)}))^{1-\frac{1}{\gamma}}$$

$$\times\left(\frac{|h'(\kappa_2)|^\gamma\left[\kappa_2^{2\alpha+1}-L(\eta^{2\alpha+1},\kappa_2^{2\alpha+1})\right]+|h'(\eta)|^\gamma\left[L(\eta^{2\alpha+1},\kappa_2^{2\alpha+1})-\eta^{2\alpha+1}\right]}{2\alpha+1}\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}(L(\eta^{(2\alpha+1)},\kappa_1^{(2\alpha+1)}))^{1-\frac{1}{\gamma}}$$

$$\times\left(\frac{|h'(\eta)|^\gamma\left[\eta^{2\alpha+1}-L(\kappa_1^{2\alpha+1}),\eta^{2\alpha+1}\right]+|h'(\kappa_1)|^\gamma\left[L(\kappa_1^{2\alpha+1},\eta^{2\alpha+1})-\kappa_1^{2\alpha+1}\right]}{2\alpha+1}\right)^{\frac{1}{\gamma}}$$

holds for any $\eta\in[\kappa_1,\kappa_2]$.

Proof. From the GA-convexity of $|h'|^\gamma$, power mean inequality, the property of the modulus and Lemma 2.1, one has

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right|$$

$$\leq(\log\kappa_2-\log\eta)\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|dt$$

$$+(\log\eta-\log\kappa_1)\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|dt$$

$$\leq(\log\kappa_2-\log\eta)\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$\leq(\log\kappa_2-\log\eta)\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}\left[t|h'(\kappa_2)|^\gamma+(1-t)|h'(\eta)|^\gamma\right]dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}\left[t|h'(\eta)|^\gamma+(1-t)|h'(\kappa_1)|^\gamma\right]dt\right)^{\frac{1}{\gamma}}$$

$$=(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}(L(\kappa_2^{(2\alpha+1)},\eta^{(2\alpha+1)}))^{1-\frac{1}{\gamma}}$$

$$\times\left(\frac{|h'(\kappa_2)|^\gamma\left[\kappa_2^{2\alpha+1}-L(\eta^{2\alpha+1},\kappa_2^{2\alpha+1})\right]+|h'(\eta)|^\gamma\left[L(\eta^{2\alpha+1},\kappa_2^{2\alpha+1})-\eta^{2\alpha+1}\right]}{2\alpha+1}\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}(L(\eta^{(2\alpha+1)},\kappa_1^{(2\alpha+1)}))^{1-\frac{1}{\gamma}}$$

$$\times\left(\frac{|h'(\eta)|^\gamma\left[\eta^{2\alpha+1}-L(\kappa_1^{2\alpha+1}),\eta^{2\alpha+1}\right]+|h'(\kappa_1)|^\gamma\left[L(\kappa_1^{2\alpha+1},\eta^{2\alpha+1})-\kappa_1^{2\alpha+1}\right]}{2\alpha+1}\right)^{\frac{1}{\gamma}}.$$

□

Remark 2.7. Let $\alpha = 1$. Then inequality (2.7) reduces to (1.7).

Theorem 2.7. Let $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$, $\vartheta, \gamma > 1$ with $1/\vartheta + 1/\gamma = 1$, $\alpha \in (0, 1]$, $h : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an α-fractional differentiable function on (κ_1, κ_2) such that $D_\alpha(h) \in L_\alpha([\kappa_1, \kappa_2])$ and $|h'(x)|^\gamma$ be a GA-convex function on $[\kappa_1, \kappa_2]$. Then the inequality

$$\left|\kappa_2^{2\alpha} h(\kappa_2) - \kappa_1^{2\alpha} h(\kappa_1) - 2\alpha \int_{\kappa_1}^{\kappa_2} x^\alpha h(x) d_\alpha x\right| \tag{2.8}$$

$$\leq \frac{(\log\kappa_2 - \log\eta)^{1-\frac{1}{\gamma}}}{\vartheta^{\frac{1}{\gamma}}} \left(L\left(\kappa_2^{\frac{(\gamma-\vartheta)(2\alpha+1)}{\gamma-1}}, \eta^{\frac{(\gamma-\vartheta)(2\alpha+1)}{\gamma-1}}\right)\right)^{\frac{\gamma-1}{\gamma}} (A_\gamma(\kappa_2, \eta))^{\frac{1}{\gamma}}$$

$$+\frac{(\log\eta - \log\kappa_1)^{1-\frac{1}{\gamma}}}{\vartheta^{\frac{1}{\gamma}}} \left(L\left(\eta^{\frac{(\gamma-\vartheta)(2\alpha+1)}{\gamma-1}}, \kappa_1^{\frac{(\gamma-\vartheta)(2\alpha+1)}{\gamma-1}}\right)\right)^{\frac{\gamma-1}{\gamma}} (A_\gamma(\eta, \kappa_1))^{\frac{1}{\gamma}}$$

holds for any $\eta \in [\kappa_1, \kappa_2]$, where

$$A_\gamma(\kappa_2, \eta) = \frac{|h'(\kappa_2)|^\gamma \left[\kappa_2^{\gamma(2\alpha+1)} - L\left(\eta^{\gamma(2\alpha+1)}, \kappa_2^{\gamma(2\alpha+1)}\right)\right] + |h'(\eta)|^\gamma \left[L\left(\eta^{\gamma(2\alpha+1)}, \kappa_2^{\gamma(2\alpha+1)}\right) - \eta^{\gamma(2\alpha+1)}\right]}{2\alpha+1},$$

$$A_\gamma(\eta, \kappa_1) = \frac{|h'(\eta)|^\gamma \left[\eta^{\gamma(2\alpha+1)} - L\left(\kappa_1^{\gamma(2\alpha+1)}, \eta^{\gamma(2\alpha+1)}\right)\right] + |h'(\kappa_1)|^\gamma \left[L(\kappa_1^{\gamma(2\alpha+1)}, \eta^{\gamma(2\alpha+1)}) - \kappa_1^{\gamma(2\alpha+1)}\right]}{2\alpha+1}.$$

Proof. It follows from Lemma 2.1, the GA-convexity of $|h'|^\gamma$, power mean inequality, Hölder integral inequality and the property of the modulus that

$$\left|\kappa_2^{2\alpha} h(\kappa_2) - \kappa_1^{2\alpha} h(\kappa_1) - 2\alpha \int_{\kappa_1}^{\kappa_2} x^\alpha h(x) d_\alpha x\right|$$

$$\leq (\log\kappa_2 - \log\eta) \int_0^1 (\kappa_2^t \eta^{1-t})^{2\alpha+1} |h'(\kappa_2^t \eta^{1-t})| dt$$

$$+(\log\eta - \log\kappa_1) \int_0^1 (\eta^t \kappa_1^{1-t})^{2\alpha+1} |h'(\eta^t \kappa_1^{1-t})| dt$$

$$\leq (\log\kappa_2 - \log\eta) \left(\int_0^1 (\kappa_2^{(2\alpha+1)t} \eta^{(2\alpha+1)(1-t)})^{\frac{\gamma-\vartheta}{\gamma-1}} dt\right)^{\frac{\gamma-1}{\gamma}}$$

$$\times \left(\int_0^1 \left(\kappa_2^{(2\alpha+1)t} \eta^{(2\alpha+1)(1-t)}\right)^\vartheta |h'(\kappa_2^t \eta^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta - \log\kappa_1) \left(\int_0^1 (\eta^{(2\alpha+1)t} \kappa_1^{(2\alpha+1)(1-t)})^{\frac{\gamma-\vartheta}{\gamma-1}} dt\right)^{\frac{\gamma-1}{\gamma}}$$

$$\times \left(\int_0^1 \left(\eta^{(2\alpha+1)t} \kappa_1^{(2\alpha+1)(1-t)}\right)^\vartheta |h'(\eta^t \kappa_1^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$\leq (\log\kappa_2 - \log\eta) \left(\int_0^1 (\kappa_2^{(2\alpha+1)t} \eta^{(2\alpha+1)(1-t)})^{\frac{\gamma-\vartheta}{\gamma-1}} dt\right)^{\frac{\gamma-1}{\gamma}}$$

$$\times\left(\int_0^1 \left(\kappa_2^{(2\alpha+1)t}\eta^{(2\alpha+1)(1-t)}\right)^{\vartheta}\left[t|h'(\kappa_2)|^{\gamma}+(1-t)|h'(\eta)|^{\gamma}\right]dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1 (\eta^{(2\alpha+1)t}\kappa_1^{(2\alpha+1)(1-t)})^{\frac{\gamma-\vartheta}{\gamma-1}}dt\right)^{\frac{\gamma-1}{\gamma}}$$

$$\times\left(\int_0^1 \left(\eta^{(2\alpha+1)t}\kappa_1^{(2\alpha+1)(1-t)}\right)^{\vartheta}\left[t|h'(\eta)|^{\gamma}+(1-t)|h'(\kappa_1)|^{\gamma}\right]dt\right)^{\frac{1}{\gamma}}$$

$$=\frac{(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}}{\vartheta^{\frac{1}{\gamma}}}\left(L\left(\kappa_2^{\frac{(\gamma-\vartheta)(2\alpha+1)}{\gamma-1}},\eta^{\frac{(\gamma-\vartheta)(2\alpha+1)}{\gamma-1}}\right)\right)^{\frac{\gamma-1}{\gamma}}(A_{\gamma}(\kappa_2,\eta))^{\frac{1}{\gamma}}$$

$$+\frac{(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}}{\vartheta^{\frac{1}{\gamma}}}\left(L\left(\eta^{\frac{(\gamma-\vartheta)(2\alpha+1)}{\gamma-1}},\kappa_1^{\frac{(\gamma-\vartheta)(2\alpha+1)}{\gamma-1}}\right)\right)^{\frac{\gamma-1}{\gamma}}(A_{\gamma}(\eta,\kappa_1))^{\frac{1}{\gamma}}.$$

□

Remark 2.8. Let $\alpha=1$. Then inequality (2.8) becomes (1.8).

Theorem 2.8. Let $\kappa_1,\kappa_2\in(0,\infty)$ with $\kappa_1<\kappa_2$, $\gamma>1$, $\alpha\in(0,1]$, $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an α-fractional differentiable function on (κ_1,κ_2) such that $D_\alpha(h)\in L_\alpha([\kappa_1,\kappa_2])$ and $|h'(x)|^{\gamma}$ be a GA-convex function on $[\kappa_1,\kappa_2]$. Then the inequality

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^{\alpha}h(x)d_\alpha x\right| \tag{2.9}$$

$$\leq(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}\left(L\left(\kappa_2^{\frac{\gamma(2\alpha+1)}{\gamma-1}},\eta^{\frac{\gamma(2\alpha+1)}{\gamma-1}}\right)\right)^{1-\frac{1}{\gamma}}(A(|h'(\kappa_2)|^{\gamma},|h'(\eta)|^{\gamma}))^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}\left(L\left(\eta^{\frac{\gamma(2\alpha+1)}{\gamma-1}},\kappa_1^{\frac{\gamma(2\alpha+1)}{\gamma-1}}\right)\right)^{1-\frac{1}{\gamma}}\left(A\left(|h'(\eta)|^{\gamma},|h'(\kappa_1)|^{\gamma}\right)\right)^{\frac{1}{\gamma}}$$

holds for any $\eta\in[\kappa_1,\kappa_2]$.

Proof. From Lemma 2.1, the GG-convexity of $|h'|^{\gamma}$, Hölder inequality and the property of the modulus, we have

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^{\alpha}h(x)d_\alpha x\right|$$

$$\leq(\log\kappa_2-\log\eta)\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|dt$$

$$+(\log\eta-\log\kappa_1)\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|dt$$

$$\leq(\log\kappa_2-\log\eta)\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1|h'(\kappa_2^t\eta^{1-t})|^{\gamma}dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1|h'(\eta^t\kappa_1^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}$$

$$\leq(\log\kappa_2-\log\eta)\left(\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1\left[t|h'(\kappa_2)|^\gamma+(1-t)|h'(\eta)|^\gamma\right]dt\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)\left(\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1\left[t|h'(\eta)|^\gamma+(1-t)|h'(\kappa_1)|^\gamma\right]dt\right)^{\frac{1}{\gamma}}$$

$$=(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}\left(L\left(\kappa_2^{\frac{\gamma(2\alpha+1)}{\gamma-1}},\eta^{\frac{\gamma(2\alpha+1)}{\gamma-1}}\right)\right)^{1-\frac{1}{\gamma}}\left(A(|h'(\kappa_2)|^\gamma,|h'(\eta)|^\gamma)\right)^{\frac{1}{\gamma}}$$

$$+(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}\left(L\left(\eta^{\frac{\gamma(2\alpha+1)}{\gamma-1}},\kappa_1^{\frac{\gamma(2\alpha+1)}{\gamma-1}}\right)\right)^{1-\frac{1}{\gamma}}\left(A\left(|h'(\eta)|^\gamma,|h'(\kappa_1)|^\gamma\right)\right)^{\frac{1}{\gamma}}.$$

□

Remark 2.9. Let $\alpha=1$. Then inequality (2.9) leads to (1.9).

Theorem 2.9. Let $\kappa_1,\kappa_2\in(0,\infty)$ with $\kappa_1<\kappa_2$, $\gamma>1$, $\alpha\in(0,1]$, $h:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an α-fractional differentiable function on (κ_1,κ_2) such that $D_\alpha(h)\in L_\alpha([\kappa_1,\kappa_2])$ and $|h'(x)|^\gamma$ be a GA-convex function on $[\kappa_1,\kappa_2]$. Then the inequality

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right| \tag{2.10}$$

$$\leq\frac{(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}}{\gamma^{\frac{1}{\gamma}}}B_\gamma(\kappa_2,\eta)+\frac{(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}}{\gamma^{\frac{1}{\gamma}}}B_\gamma(\eta,\kappa_1)$$

holds for any $\eta\in[\kappa_1,\kappa_2]$, where

$$B_\gamma(\kappa_2,\eta)=\left(\frac{|h'(\kappa_2)|^\gamma\left[\kappa_2^{\gamma(2\alpha+1)}-L\left(\eta^{\gamma(2\alpha+1)},\kappa_2^{\gamma(2\alpha+1)}\right)\right]+|h'(\eta)|^\gamma\left[L(\eta^{\gamma(2\alpha+1)},\kappa_2^{\gamma(2\alpha+1)})-\eta^{\gamma(\alpha+1)}\right]}{2\alpha+1}\right)^{\frac{1}{\gamma}},$$

$$B_\gamma(\eta,\kappa_1)=\left(\frac{|h'(\eta)|^\gamma\left[\eta^{\gamma(2\alpha+1)}-L\left(\kappa_1^{\gamma(2\alpha+1)},\eta^{\gamma(2\alpha+1)}\right)\right]+|h'(\kappa_1)|^\gamma\left[L(\kappa_1^{\gamma(2\alpha+1)},\eta^{\gamma(2\alpha+1)})-\kappa_1^{\gamma(\alpha+1)}\right]}{2\alpha+1}\right)^{\frac{1}{\gamma}}.$$

Proof. It follows from Lemma 2.1, the GA-convexity of $|h'|^\gamma$, power mean inequality and property of the modulus that

$$\left|\kappa_2^{2\alpha}h(\kappa_2)-\kappa_1^{2\alpha}h(\kappa_1)-2\alpha\int_{\kappa_1}^{\kappa_2}x^\alpha h(x)d_\alpha x\right|$$

$$\leq(\log\kappa_2-\log\eta)\int_0^1(\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|dt$$

$$+(\log\eta-\log\kappa_1)\int_0^1(\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|dt$$

$$
\begin{aligned}
&\leq (\log\kappa_2-\log\eta)\left(\int_0^1 dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 (\kappa_2^t\eta^{1-t})^{2\alpha+1}|h'(\kappa_2^t\eta^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}\\
&+(\log\eta-\log\kappa_1)\left(\int_0^1 dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 (\eta^t\kappa_1^{1-t})^{2\alpha+1}|h'(\eta^t\kappa_1^{1-t})|^\gamma dt\right)^{\frac{1}{\gamma}}\\
&\leq (\log\kappa_2-\log\eta)\left(\int_0^1 dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 (\kappa_2^t\eta^{1-t})^{2\alpha+1}\left[t|h'(\kappa_2)|^\gamma+(1-t)|h'(\eta)|^\gamma\right]dt\right)^{\frac{1}{\gamma}}\\
&+(\log\eta-\log\kappa_1)\left(\int_0^1 dt\right)^{1-\frac{1}{\gamma}}\left(\int_0^1 (\eta^t\kappa_1^{1-t})^{2\alpha+1}\left[t|h'(\eta)|^\gamma+(1-t)|h'(\kappa_1)|^\gamma\right]dt\right)^{\frac{1}{\gamma}}\\
&=\frac{(\log\kappa_2-\log\eta)^{1-\frac{1}{\gamma}}}{\gamma^{\frac{1}{\gamma}}}B_\gamma(\kappa_2,\eta)+\frac{(\log\eta-\log\kappa_1)^{1-\frac{1}{\gamma}}}{\gamma^{\frac{1}{\gamma}}}B_\gamma(\eta,\kappa_1).
\end{aligned}
$$

□

Remark 2.10. Let $\alpha = 1$. Then inequality (2.10) reduces to (1.10).

3. Conclusion

We have generalized the Hermite-Hadamard type inequalities for GG- and GA-convex functions established by Ardıç, Akdemir and Yıdız in [61] to the conformable fractional integrals. Our ideas and approach may lead to a lot of follow-up research.

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Conflict of interest

The authors declare no conflict of interest.

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