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Research article

A certain new Gauss sum and its fourth power mean

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Abstract: The main purpose of this paper is using the elementary methods and the properties of the Legendre symbol to study the computational problem of the fourth power mean of a certain generalized quadratic Gauss sum, and give two exact calculating formulae for it.

Keywords: congruence equation; certain generalized quadratic Gauss sums; the fourth power mean; calculating formula
Mathematics Subject Classification: 11L03, 11L07

1. Introduction

Let $q \geq 3$ is a positive integer. For any integral coefficient polynomial $f(x)$ and any Dirichlet character χ mod q, we define a generalized Gauss sums $G(f,\chi;q)$ as

$$G(f,\chi;q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{f(a)}{q}\right),$$

where as usual, $e(y) = e^{2\pi i y}$.

If $f(x) = x$, then $G(f,\chi;q) = \tau(\chi)$ is the classical Gauss sums. That is,

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right).$$

A very important property of $\tau(\chi)$ is that if χ is a primitive character mod q, then we have the identity $|\tau(\chi)| = \sqrt{q}$ (see Theorem 8.15 in [1]).

If $f(x) = ax^2 + bx$, then $G(f,\chi;q)$ becomes the generalized quadratic Gauss sums. Many authors have studied its properties and obtained interesting results. For example, if $q = p$ is an odd prime, then from A. Weil's important work [2] we can get the estimate

$$|G(f,\chi;p)| \leq 2\cdot\sqrt{p}$$

for all integers a and b with $(a,b,p)=1$.

Let p be an odd prime, α be a positive integer $\alpha\geq 2$, and λ be a primitive character modulo p^α. Zhang Wenpeng and Lin Xin [3] proved that for any integer n with $(n,p)=1$, we have

$$\sideset{}{'}\sum_{m=1}^{p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^2+na}{p^\alpha}\right)\right|^4 = p^{2\alpha}\phi\left(p^\alpha\right)\left(\alpha+1-\frac{5}{p-1}\right);$$

If λ is any non-primitive character modulo p^α, then we have the identity

$$\sideset{}{'}\sum_{m=1}^{p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^2+na}{p^\alpha}\right)\right|^4 = p^{2\alpha}\phi\left(p^\alpha\right),$$

where $\phi(n)$ denotes the Euler function.

Li Xiaoxue and Xu Zhefeng [4] also studied the special case $q=p$, an odd prime, and obtained several interesting identities.

On the other hand, Lv Xingxing and Zhang Wenpeng [5] introduced a new sum analogous to Kloosterman sum as follows:

$$K(m,n,r,\chi;q)=\sum_{a=1}^{q}\chi\left(ma+n\overline{a}\right)e\left(\frac{ra}{q}\right), \tag{1.1}$$

where m, n and r are integers, $\overline{a}$ denotes $a\cdot\overline{a}\equiv 1 \bmod q$.

Using properties of the character modulo p and analytic methods, they proved the following results: For any odd prime p with $p\equiv 3 \bmod 4$ and integers m and n with $(mn,p)=1$, one has the identity

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi\left(b+\overline{b}\right)\right|^2=(p-1)\left(3p^2-6p-1\right)$$

and

$$\begin{aligned}&\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi\left(b+\overline{b}\right)e\left(\frac{nb}{p}\right)\right|^2\\ =\ &(p-1)\left(p^2-2p-1\right)+p(p-1)\left(\sum_{b=2}^{p-2}e\left(\frac{n\left(b+\overline{b}\right)}{p}\right)+\sum_{b=2}^{p-2}e\left(\frac{n\left(b-\overline{b}\right)}{p}\right)\right).\end{aligned}$$

From the second formula and the estimate for Kloosterman sums, Lv Xingxing and Zhang Wenpeng [5] deduced the asymptotic formula

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi\left(b+\overline{b}\right)e\left(\frac{nb}{p}\right)\right|^2=p^3+O\left(p^{\frac{5}{2}}\right).$$

Shane Chern [6] also studied the properties of $K(m, n, r, \chi; q)$, and obtained the identity

$$\sum_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \chi(ma + n\overline{a}) e\left(\frac{ka}{p}\right) \right|^2 \right|^2 = (p-1)\left(p^4 - 7p^3 + 17p^2 - 5p - 25\right),$$

where p is an odd prime, n and k are integers with $(nk, p) = 1$.

Some other papers related to Gauss sums and two-term exponential sums can also be found in [5–13], but we will not go into further detail here.

Inspired by [5] and [6], it is natural to study the $2k$-th power means

$$\sum_{m=0}^{p-1} \sum_{\chi \bmod p} \left| \sum_{a=0}^{p-1} \chi\left(a^2 + 1\right) e\left(\frac{ma}{p}\right) \right|^{2k}, \quad k \geq 2. \tag{1.2}$$

To the best knowledge of the authors, sums of the type (1.2) have not been previously investigated but the authors think that this problem is meaningful, and can be viewed as a new combination of additive and multiplicative functions.

In this paper, we will use the number of the solutions of some congruence equations mod p and the properties of the Legendre symbol to study the problem of calculating (1.2) with $k = 2$, and give two exact calculating formulae according to $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$. That is, we will prove the following results:

Theorem 1. Let p be an odd prime with $p \equiv 3 \bmod 4$, then we have

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} \chi\left(a^2 + 1\right) e\left(\frac{ma}{p}\right) \right|^4 = p(p-1)\left(3p^2 - 4p + 4\right).$$

Theorem 2. Let p be an odd prime with $p \equiv 1 \bmod 4$, then we have

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} \chi\left(a^2 + 1\right) e\left(\frac{ma}{p}\right) \right|^4 = 3p(p-1)\left(p^2 - 6p + 10\right).$$

Some notes: For $k \geq 3$, whether there exists an exact calculating formula for (1.2) is an open problem. It is also interesting to ask whether a general formula analogous to Theorems 1 and 2 can be obtained for all integers $q \geq 3$ and $k = 2$.

2. Some basic lemmas

In this section, we first give four basic lemmas, which are necessary in the proof of our theorems. Of course, in order to prove these lemmas, we need some knowledge of elementary and analytic number theory. They can be found in reference [1], and we do not repeat them all here. First we have the following:

Lemma 1. For any odd prime p, we have the identity

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{\substack{d=0 \\ ab \equiv cd \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} 1 = 2p^2 - p.$$

Proof. Note that if $a,\ b,\ c$ pass through a reduced residue system mod p, then da, db and cd also pass through a reduced residue system mod p, providing $(d,p)=1$. So from these properties we have

$$\begin{aligned}
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{\substack{d=0\\ ab\equiv cd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1 &= 4(p-1)+1+\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ ab\equiv cd \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1\\
&= 4p-3+\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ ab\equiv c \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1 = 4p-3+(p-1)\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ a+b\equiv ab+1 \bmod p}}^{p-1} 1\\
&= 4p-3+(p-1)\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ (a-1)(b-1)\equiv 0 \bmod p}}^{p-1} 1 = 4p-3+(p-1)\left(2\sum_{a=1}^{p-1}1-1\right)\\
&= 4p-3+(p-1)(2p-3)=2p^2-p.
\end{aligned}$$

This proves Lemma 1.

Lemma 2. If p is an odd prime, then we have the identity

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{\substack{d=0\\ ab+cd\equiv 2 \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1 = p^2+\left(\frac{-1}{p}\right)\cdot p.$$

Proof. It is clear that if d passes through a reduced residue system mod p, then $\overline{d}$ also passes through a reduced residue system mod p. From the properties of the reduced residue system and quadratic residue mod p we have

$$\begin{aligned}
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{\substack{d=0\\ ab+cd\equiv 2 \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1 &= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{\substack{d=1\\ ab+cd\equiv 2 \bmod p\\ a+b\equiv c+d \bmod p}}^{p-1} 1+\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ ab\equiv 2 \bmod p\\ a+b\equiv c \bmod p}}^{p-1} 1\\
&= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{\substack{d=1\\ ab+c\equiv 2\overline{d}^2 \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1+\sum_{a=0}^{p-1}\sum_{\substack{b=0\\ ab\equiv 2 \bmod p}}^{p-1} 1 = \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{\substack{d=1\\ ab+c\equiv 2d^2 \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1+(p-1)\\
&= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{\substack{d=0\\ ab+c\equiv 2d \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} \left(1+\left(\frac{d}{p}\right)\right)-\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ ab+c\equiv 0 \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} 1+(p-1)\\
&= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{\substack{c=0\\ a+b\equiv c+1 \bmod p}}^{p-1} 1+\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{\substack{d=0\\ ab+c\equiv 2d \bmod p\\ a+b\equiv c+1 \bmod p}}^{p-1} \left(\frac{d}{p}\right)-\sum_{a=0}^{p-1}\sum_{\substack{b=0\\ ab+a+b\equiv 1 \bmod p}}^{p-1} 1+(p-1)
\end{aligned}$$

$$= p^2 + \sum_{\substack{a=0 \\ ab+a+b\equiv 1+2d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{d=0}^{p-1} \left(\frac{d}{p}\right) - \sum_{\substack{a=1 \\ ab\equiv 2 \bmod p}}^{p-1} \sum_{b=1}^{p-1} 1 + (p-1). \tag{2.1}$$

It is clear that if a passes through a complete residue system mod p, then $a+1$ also passes through a complete residue system mod p. So we make the change of variables a to $a-1$ and b to $b-1$ modulo p to change the congruence from $ab+a+b \equiv c \bmod p$ to $ab \equiv c+1 \bmod p$. Then from (2.1) and the orthogonality property for the Dirichlet character modulo p we have

$$\sum_{\substack{a=0 \\ ab+cd\equiv 2 \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 = p^2 + \sum_{\substack{a=1 \\ ab\equiv 2+2d \bmod p}}^{p} \sum_{b=1}^{p} \sum_{d=0}^{p-1} \left(\frac{d}{p}\right) = p^2 + \left(\frac{2}{p}\right) \sum_{a=1}^{p} \sum_{b=1}^{p} \left(\frac{ab-2}{p}\right)$$

$$= p^2 + \left(\frac{2}{p}\right) \sum_{a=1}^{p-1} \sum_{b=1}^{p} \left(\frac{ab-2}{p}\right) + \left(\frac{2}{p}\right) \sum_{b=1}^{p} \left(\frac{-2}{p}\right)$$

$$= p^2 + \left(\frac{2}{p}\right) \sum_{a=1}^{p-1} \sum_{b=1}^{p} \left(\frac{b-2}{p}\right) + \left(\frac{-1}{p}\right) \cdot p = p^2 + \left(\frac{-1}{p}\right) \cdot p.$$

This proves Lemma 2.

Lemma 3. If p is an odd prime, then we have the identity

$$\sum_{\substack{a=0 \\ ab+cd\equiv 2 \bmod p \\ ab\equiv cd \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 = 2p-4.$$

Proof. It is clear that the conditions $ab \equiv cd \bmod p$ and $ab+cd \equiv 2 \bmod p$ are equivalent to $ab \equiv cd \equiv 1 \bmod p$. So we have

$$\sum_{\substack{a=0 \\ ab+cd\equiv 2 \bmod p \\ ab\equiv cd \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 = \sum_{\substack{a=1 \\ ab\equiv cd\equiv 1 \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a+\overline{a}\equiv c+\overline{c} \bmod p}}^{p-1} \sum_{c=1}^{p-1} 1$$

$$= \sum_{\substack{a=1 \\ (a-c)(ac-1)\equiv 0 \bmod p}}^{p-1} \sum_{c=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a\equiv c \bmod p}}^{p-1} \sum_{c=1}^{p-1} 1 + \sum_{\substack{a=1 \\ ac\equiv 1 \bmod p}}^{p-1} \sum_{c=1}^{p-1} 1 - \sum_{\substack{a=1 \\ a\equiv c \bmod p \\ ac\equiv 1 \bmod p}}^{p-1} \sum_{c=1}^{p-1} 1$$

$$= p-1+p-1-2 = 2p-4.$$

This proves Lemma 3.

Lemma 4. If p is an odd prime with $p \equiv 1 \bmod 4$, then we have the identity

$$\sum_{\substack{a=0 \\ (a^2+1)(b^2+1)\equiv (c^2+1)(d^2+1)\equiv 0 \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 = 16p-26.$$

Proof. Since $p \equiv 1 \bmod 4$, there exists an integer k satisfying the congruence $(\pm k)^2 \equiv -1 \bmod p$. In fact, these are all such solutions modulo p. Choosing $0 \leq h \leq p-1$ for which $a+b \equiv c+d \equiv h \bmod p$, we have

$$\begin{aligned}
\sum_{\substack{a=0 \\ (a^2+1)(b^2+1)\equiv(c^2+1)(d^2+1)\equiv 0 \bmod p \\ a+b\equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 &= \sum_{h=0}^{p-1} \left(\sum_{\substack{a=0 \\ (a^2+1)(b^2+1)\equiv 0 \bmod p \\ a+b\equiv h \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 \right)^2 \\
&= \sum_{h=0}^{p-1} \left(\sum_{\substack{a=0 \\ a^2+1\equiv 0 \bmod p \\ a+b\equiv h \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 + \sum_{\substack{a=0 \\ b^2+1\equiv 0 \bmod p \\ a+b\equiv h \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 - \sum_{\substack{a=0 \\ a^2+1\equiv b^2+1\equiv 0 \bmod p \\ a+b\equiv h \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 \right)^2 \\
&= \sum_{h=0}^{p-1} \left(2 \sum_{\substack{b=0 \\ k+b\equiv h \bmod p}}^{p-1} 1 + 2 \sum_{\substack{b=0 \\ -k+b\equiv h \bmod p}}^{p-1} 1 - \sum_{\substack{a=0 \\ a^2+1\equiv b^2+1\equiv 0 \bmod p \\ a+b\equiv h \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 \right)^2 \\
&= \sum_{\substack{h=0 \\ h\neq -2k,2k,0}}^{p-1} (2+2-0)^2 + \left(4 - \sum_{\substack{a=0 \\ a^2+1\equiv b^2+1\equiv 0 \bmod p \\ a+b\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 \right)^2 \\
&\quad + \left(4 - \sum_{\substack{a=0 \\ a^2+1\equiv b^2+1\equiv 0 \bmod p \\ a+b\equiv 2k \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 \right)^2 + \left(4 - \sum_{\substack{a=0 \\ a^2+1\equiv b^2+1\equiv 0 \bmod p \\ a+b\equiv -2k \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 \right)^2 \\
&= 16(p-3) + (4-2)^2 + (4-1)^2 + (4-1)^2 = 16p - 26.
\end{aligned}$$

3. Proof of the theorems

In this section, we use the four basic lemmas of the previous section to prove our theorems. First we prove Theorem 1. If $p \equiv 3 \bmod 4$, then $\left(\frac{-1}{p}\right) = -1$ and $p \nmid (a^2+1)$ for all $0 \leq a \leq p-1$. Then from Lemma 1, Lemma 2, Lemma 3, the trigonometric identity

$$\sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) = \begin{cases} p, & \text{if } p \mid m; \\ 0, & \text{if } p \nmid m \end{cases}$$

and the orthogonality of characters mod p

$$\sum_{\chi \bmod p} \chi(a) = \begin{cases} p-1, & \text{if } a \equiv 1 \bmod p; \\ 0, & \text{otherwise} \end{cases}$$

we have

$$\begin{aligned}
& \frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} \chi\left(a^2+1\right) e\left(\frac{ma}{p}\right) \right|^4 \\
= & \frac{1}{p} \sum_{\substack{a=0 \\ \left(a^2+1\right)\left(b^2+1\right) \equiv \left(c^2+1\right)\left(d^2+1\right) \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} \sum_{m=0}^{p-1} e\left(\frac{m(a+b-c-d)}{p}\right) \\
= & \sum_{\substack{a=0 \\ a^2b^2+a^2+b^2+1 \equiv c^2d^2+c^2+d^2+1 \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 = \sum_{\substack{a=0 \\ (ab-1)^2+(a+b)^2 \equiv (cd-1)^2+(c+d)^2 \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 \\
= & \sum_{\substack{a=0 \\ (ab-1)^2 \equiv (cd-1)^2 \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 = \sum_{\substack{a=0 \\ (ab-cd)(ab+cd-2) \equiv 0 \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 \\
= & \sum_{\substack{a=0 \\ ab \equiv cd \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 + \sum_{\substack{a=0 \\ ab+cd \equiv 2 \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 - \sum_{\substack{a=0 \\ ab-cd \equiv ab+cd-2 \equiv 0 \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 \\
= & 2p^2 - p + p^2 - p - (2p-4) = 3p^2 - 4p + 4.
\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. Note that if $p \equiv 1 \bmod 4$, then $\left(\frac{-1}{p}\right) = 1$. If $a^2 + 1 \equiv 0 \bmod p$, then for any character $\chi \bmod p$, we have $\chi\left(a^2+1\right) = 0$. So from Lemma 4 and the methods of proving Theorem 1 we have

$$\begin{aligned}
& \frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=0}^{p-1} \chi\left(a^2+1\right) e\left(\frac{ma}{p}\right) \right|^4 \\
= & \sum_{\substack{a=0 \\ \left(a^2+1\right)\left(b^2+1\right) \equiv \left(c^2+1\right)\left(d^2+1\right) \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 - \sum_{\substack{a=0 \\ \left(a^2+1\right)\left(b^2+1\right) \equiv \left(c^2+1\right)\left(d^2+1\right) \equiv 0 \bmod p \\ a+b \equiv c+d \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{d=0}^{p-1} 1 \\
= & 2p^2 - p + p^2 + p - (2p-4) - (16p-26) = 3\left(p^2 - 6p + 10\right).
\end{aligned}$$

This completes the proof of all of our results.

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Conflict of interest

The authors declare no conflict of interest.

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