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Research article

On second-order differential subordination for certain meromorphically multivalent functions

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Abstract: A new class $\mathcal{R}_n(A, B, \lambda)$ of meromorphically multivalent functions defined by the second-order differential subordination is introduced. Several geometric properties of this new class are studied. The sharp upper bound on $|z| = r < 1$ for the functional $\mathrm{Re}\{(\lambda - 1)z^{p+1} f'(z) + \frac{\lambda}{p+1} z^{p+2} f''(z)\}$ over the class $\mathcal{R}_n(A, B, 0)$ is obtained.

Keywords: meromorphically multivalent function; differential subordination; coefficient estimate; sharp bound
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1. Introduction

Throughout our present investigation, we assume that

$$n,\ p \in \mathbb{N},\ -1 \leq B < 1,\ B < A \ \text{ and } \lambda < 0. \tag{1.1}$$

Let $\Sigma_n(p)$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^{k-p} \tag{1.2}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* = \{z : 0 < |z| < 1\}$ with a pole at $z = 0$. The class $\Sigma_n(p)$ is closed under the Hadamard product (or convolution)

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^{k-p} = (f_2 * f_1)(z),$$

where

$$f_j(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,j} z^{k-p} \in \Sigma_n(p) \quad (j = 1, 2).$$

For functions $f(z)$ and $g(z)$ analytic in $\mathbb{U} = \{z : |z| < 1\}$, we say that $f(z)$ is subordinate to $g(z)$ and write $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$|w(z)| \le |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In this paper we introduce and investigate the following subclass of $\Sigma_n(p)$.

Definition. A function $f(z) \in \Sigma_n(p)$ is said to be in the class $\mathcal{R}_n(A, B, \lambda)$ if it satisfies the second-order differential subordination:

$$(\lambda - 1) z^{p+1} f'(z) + \frac{\lambda}{p+1} z^{p+2} f''(z) \prec p \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \tag{1.3}$$

Recently, several authors (see, e.g., [1–9, 11–15] and the references cited therein) introduced and investigated various subclasses of meromorphically multivalent functions. Some properties such as distortion bounds, inclusion relations and coefficient estimates were given. In this note we obtain inclusion relation and coefficient estimate for functions $f(z)$ in the class $\mathcal{R}_n(A, B, \lambda)$. Furthermore, we investigate a new problem. It is to find

$$\max_{|z|=r<1} \mathrm{Re} \left\{ (\lambda - 1) z^{p+1} f'(z) + \frac{\lambda}{p+1} z^{p+2} f''(z) \right\},$$

where $f(z)$ varies in the class:

$$\mathcal{R}_n(A, B, 0) = \left\{ f(z) \in \Sigma_n(p) : -z^{p+1} f'(z) \prec p \frac{1 + Az}{1 + Bz} \right\}. \tag{1.4}$$

We need the following lemma in order to derive our main results for the class $\mathcal{R}_n(A, B, \lambda)$.

Lemma [10]. Let $g(z)$ be analytic in $\mathbb{U}$ and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z),$$

where $\mathrm{Re}\mu \ge 0$ and $\mu \ne 0$, then $g(z) \prec h(z)$.

2. Geometric properties of functions in class $\mathcal{R}_n(A, B, \lambda)$

Theorem 1. Let $\lambda_2 < \lambda_1 < 0$. Then

$$\mathcal{R}_n(A, B, \lambda_2) \subset \mathcal{R}_n(A, B, \lambda_1).$$

Proof. Suppose that

$$g(z) = -z^{p+1} f'(z) \tag{2.1}$$

for $f(z) \in \mathcal{R}_n(A, B, \lambda_2)$. Then $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = p$. By using (1.3) and (2.1), we have

$$\begin{aligned}(\lambda_2 - 1)z^{p+1} f'(z) + \frac{\lambda_2}{p+1} z^{p+2} f''(z) &= g(z) - \frac{\lambda_2}{p+1} z g'(z) \\ &\prec p\frac{1+Az}{1+Bz}.\end{aligned} \tag{2.2}$$

Hence an application of Lemma with $\mu = -\frac{p+1}{\lambda_2} > 0$ yields

$$g(z) \prec p\frac{1+Az}{1+Bz}. \tag{2.3}$$

Note that $0 < \frac{\lambda_1}{\lambda_2} < 1$ and that the function $\frac{1+Az}{1+Bz}$ is convex univalent in $\mathbb{U}$, then it follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}&(\lambda_1 - 1)z^{p+1} f'(z) + \frac{\lambda_1}{p+1} z^{p+2} f''(z) \\ &= \frac{\lambda_1}{\lambda_2}\left((\lambda_2 - 1)z^{p+1} f'(z) + \frac{\lambda_2}{p+1} z^{p+1} f''(z)\right) + \left(1 - \frac{\lambda_1}{\lambda_2}\right) g(z) \\ &\prec p\frac{1+Az}{1+Bz}.\end{aligned}$$

Thus $f(z) \in \mathcal{R}_n(A, B, \lambda_1)$. The proof of Theorem 1 is completed. □

Theorem 2. Let

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^{k-p} \in \mathcal{R}_n(A, B, \lambda). \tag{2.4}$$

Then

$$|a_k| \leq \frac{p(p+1)(A-B)}{(p+1-\lambda k)|k-p|} \quad (k \geq n \text{ and } k \neq p). \tag{2.5}$$

The result is sharp for each $k \geq n$ $(k \neq p)$.

Proof. It is known that, if

$$\varphi(z) = \sum_{j=1}^{\infty} c_j z^j \prec \psi(z) \quad (z \in \mathbb{U}),$$

where $\varphi(z)$ is analytic in $\mathbb{U}$ and $\psi(z) = z + \cdots$ is analytic and convex univalent in $\mathbb{U}$, then $|c_j| \leq 1$ $(j \in \mathbb{N})$.

By (2.4) we have

$$\begin{aligned}\frac{(\lambda - 1)z^{p+1} f'(z) + \frac{\lambda}{p+1} z^{p+1} f''(z) - p}{p(A-B)} &= \sum_{k=n}^{\infty} \frac{(k-p)(\lambda k - p - 1)}{p(p+1)(A-B)} a_k z^k \\ &\prec \frac{z}{1+Bz} \quad (z \in \mathbb{U}).\end{aligned} \tag{2.6}$$

Because the function $\frac{z}{1+Bz}$ is analytic and convex univalent in $\mathbb{U}$, it follows from (2.6) that

$$\frac{|k-p|(p+1-\lambda k)}{p(p+1)(A-B)}|a_k| \leq 1 \quad (k \geq n \text{ and } k \neq p),$$

which gives (2.5).

Next we consider the function $f_k(z)$ defined by

$$f_k(z) = z^{-p} + p(p+1)(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}z^{km-p}}{(km-p)(\lambda km-p-1)} \quad (z \in \mathbb{U};\ k \geq n,\ k \neq p).$$

Since

$$(\lambda-1)z^{p+1}f_k'(z) + \frac{\lambda}{p+1}z^{p+2}f_k''(z) = p\frac{1+Az^k}{1+Bz^k} \prec p\frac{1+Az}{1+Bz} \quad (z \in \mathbb{U})$$

and

$$f_k(z) = z^{-p} + \frac{p(p+1)(A-B)}{(k-p)(\lambda k-p-1)}z^{k-p} + \cdots$$

for each $k \geq n$ ($k \neq p$), the proof of Theorem 2 is completed. □

Theorem 3. Let $f(z) \in \mathcal{R}_n(A,B,\lambda)$, $g(z) \in \Sigma_n(p)$ and

$$\operatorname{Re}(z^p g(z)) > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{2.7}$$

Then $(f * g)(z) \in \mathcal{R}_n(A,B,\lambda)$.

Proof. For $f(z) \in \mathcal{R}_n(A,B,\lambda)$ and $g(z) \in \Sigma_n(p)$, we have

$$\begin{aligned}&(\lambda-1)z^{p+1}(f*g)'(z) + \frac{\lambda}{p+1}z^{p+2}(f*g)''(z)\\ &= (\lambda-1)\left(z^{p+1}f'(z)\right)*(z^p g(z)) + \frac{\lambda}{p+1}\left(z^{p+2}f''(z)\right)*(z^p g(z))\\ &= h(z)*(z^p g(z)),\end{aligned} \tag{2.8}$$

where

$$h(z) = (\lambda-1)z^{p+1}f'(z) + \frac{\lambda}{p+1}z^{p+2}f''(z) \prec p\frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{2.9}$$

From (2.7), we can see that the function $z^p g(z)$ has Herglotz representation:

$$z^p g(z) = \int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z \in \mathbb{U}), \tag{2.10}$$

where $\mu(x)$ is a probability measure on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$.

Because the function $\frac{1+Az}{1+Bz}$ is convex univalent in $\mathbb{U}$, it follows from (2.8)–(2.10) that

$$\begin{aligned}&(\lambda-1)z^{p+1}(f*g)'(z) + \frac{\lambda}{p+1}z^{p+2}(f*g)''(z)\\ &= \int_{|x|=1}h(xz)d\mu(x) \prec p\frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).\end{aligned}$$

This shows that $(f*g)(z) \in \mathcal{R}_n(A,B,\lambda)$. The proof of Theorem 3 is completed. □

Theorem 4. Let $f(z) \in \mathcal{R}_n(A, B, 0)$. Then for $|z| = r < 1$,

(i) if $M_n(A, B, \lambda, r) \geq 0$, we have

$$\begin{aligned}&\operatorname{Re}\left\{(\lambda-1)z^{p+1}f'(z)+\frac{\lambda}{p+1}z^{p+2}f''(z)\right\}\\&\leq \frac{p[p+1+((p+1)(A+B)-\lambda n(A-B))r^n+(p+1)ABr^{2n}]}{(p+1)(1+Br^n)^2};\end{aligned} \tag{2.11}$$

(ii) if $M_n(A, B, \lambda, r) \leq 0$, we have

$$\operatorname{Re}\left\{(\lambda-1)z^{p+1}f'(z)+\frac{\lambda}{p+1}z^{p+2}f''(z)\right\} \leq \frac{p(4\lambda^2 K_A K_B - L_n^2)}{4\lambda(p+1)(A-B)r^{n-1}(1-r^2)K_B}, \tag{2.12}$$

where

$$\begin{cases} K_A = 1 - A^2 r^{2n} + nAr^{n-1}(1-r^2), \\ K_B = 1 - B^2 r^{2n} + nBr^{n-1}(1-r^2), \\ L_n = 2\lambda(1-ABr^{2n}) + \lambda n(A+B)r^{n-1}(1-r^2) - (p+1)(A-B)r^{n-1}(1-r^2), \\ M_n(A,B,\lambda,r) = 2\lambda K_B(1+Ar^n) - L_n(1+Br^n). \end{cases} \tag{2.13}$$

The results are sharp.

Proof. Equality in (2.11) occurs for $z = 0$. Thus we assume that $0 < |z| = r < 1$.

For $f(z) \in \mathcal{R}_n(A, B, 0)$, we can write

$$-\frac{z^{p+1}f'(z)}{p} = \frac{1+Az^n\varphi(z)}{1+Bz^n\varphi(z)} \quad (z \in \mathbb{U}), \tag{2.14}$$

where $\varphi(z)$ is analytic and $|\varphi(z)| \leq 1$ in $\mathbb{U}$. It follows from (2.14) that

$$\begin{aligned}&(\lambda-1)z^{p+1}f'(z)+\frac{\lambda}{p+1}z^{p+2}f''(z)\\&=-z^{p+1}f'(z)-\frac{\lambda p(A-B)(nz^n\varphi(z)+z^{n+1}\varphi'(z))}{(p+1)(1+Bz^n\varphi(z))^2}\\&=-z^{p+1}f'(z)+\frac{\lambda np}{(p+1)(A-B)}\left(\frac{z^{p+1}f'(z)}{p}+1\right)\left(A+B\frac{z^{p+1}f'(z)}{p}\right)-\frac{\lambda p(A-B)z^{n+1}\varphi'(z)}{(p+1)(1+Bz^n\varphi(z))^2}.\end{aligned} \tag{2.15}$$

Using the Carathéodory inequality:

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-r^2},$$

we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{z^{n+1}\varphi'(z)}{(1+Bz^n\varphi(z))^2}\right\} &\leq \frac{r^{n+1}(1-|\varphi(z)|^2)}{(1-r^2)|1+Bz^n\varphi(z)|^2}\\&\leq \frac{r^{2n}|A+\frac{B}{p}z^{p+1}f'(z)|^2-|\frac{1}{p}z^{p+1}f'(z)+1|^2}{(A-B)^2r^{n-1}(1-r^2)}.\end{aligned} \tag{2.16}$$

Put $-\frac{z^{p+1}f'(z)}{p} = u + iv \quad (u, v \in \mathbb{R})$. Note that $\lambda < 0$, then (2.15) and (2.16) provide

$$\operatorname{Re}\left\{(\lambda-1)z^{p+1}f'(z)+\frac{\lambda}{p+1}z^{p+2}f''(z)\right\} \leq p\left(1-\frac{\lambda n(A+B)}{(p+1)(A-B)}\right)u+\frac{\lambda npA}{(p+1)(A-B)}$$

$$
\begin{aligned}
&+\frac{\lambda npB}{(p+1)(A-B)}(u^2-v^2)-\frac{\lambda p[r^{2n}((A-Bu)^2+(Bv)^2)-((u-1)^2+v^2)]}{(p+1)(A-B)r^{n-1}(1-r^2)}\\
&=p\left(1-\frac{\lambda n(A+B)}{(p+1)(A-B)}\right)u+\frac{\lambda np}{(p+1)(A-B)}(A+Bu^2)-\frac{\lambda p(r^{2n}(A-Bu)^2-(u-1)^2)}{(p+1)(A-B)r^{n-1}(1-r^2)}\\
&+\frac{\lambda p}{(p+1)(A-B)}\left(\frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)}-nB\right)v^2. \qquad (2.17)
\end{aligned}
$$

Further, we can see that

$$
\begin{aligned}
\frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)} \geq \frac{1-r^{2n}}{r^{n-1}(1-r^2)} &= \frac{1}{r^{n-1}}\left(1+r^2+r^4+\cdots+r^{2(n-2)}+r^{2(n-1)}\right)\\
&=\frac{1}{2r^{n-1}}\left[(1+r^{2(n-1)})+(r^2+r^{2(n-2)})+\cdots+(r^{2(n-1)}+1)\right]\\
&\geq n \geq -nB. \qquad (2.18)
\end{aligned}
$$

Combining (2.17) and (2.18), we have

$$
\begin{aligned}
\mathrm{Re}\left\{(\lambda-1)z^{p+1}f'(z)+\frac{\lambda}{p+1}z^{p+2}f''(z)\right\} &\leq p\left(1-\frac{\lambda n(A+B)}{(p+1)(A-B)}\right)u+\frac{\lambda np}{(p+1)(A-B)}(A+Bu^2)\\
&+\frac{\lambda p((u-1)^2-r^{2n}(A-Bu)^2)}{(p+1)(A-B)r^{n-1}(1-r^2)}\\
&=: \psi_n(u). \qquad (2.19)
\end{aligned}
$$

It is known that for $|\xi| \leq \sigma$ $(\sigma < 1)$,

$$
\frac{1-A\sigma}{1-B\sigma} \leq \mathrm{Re}\left(\frac{1+A\xi}{1+B\xi}\right) \leq \frac{1+A\sigma}{1+B\sigma}. \qquad (2.20)
$$

From (2.20) and(2.14) we have

$$
\frac{1-Ar^n}{1-Br^n} \leq u = \mathrm{Re}\left(-\frac{z^{p+1}f'(z)}{p}\right) \leq \frac{1+Ar^n}{1+Br^n}.
$$

Now we calculate the maximal value of $\psi_n(u)$ on the segment $\left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right]$. Obviously,

$$
\psi_n'(u) = p\left(1-\frac{\lambda n(A+B)}{(p+1)(A-B)}\right)+\frac{2\lambda npB}{(p+1)(A-B)}u+\frac{2\lambda p((1-B^2r^{2n})u-(1-ABr^{2n}))}{(p+1)(A-B)r^{n-1}(1-r^2)},
$$

$$
\begin{aligned}
\psi_n''(u) &= \frac{2\lambda p}{(p+1)(A-B)}\left(\frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)}+nB\right)\\
&\leq \frac{2\lambda np(1+B)}{(p+1)(A-B)} \leq 0 \quad \text{(see (2.18) and (1.1))} \qquad (2.21)
\end{aligned}
$$

and $\psi_n'(u) = 0$ if and only if

$$
u = u_n = \frac{2\lambda(1-ABr^{2n})+\lambda n(A+B)r^{n-1}(1-r^2)-(p+1)(A-B)r^{n-1}(1-r^2)}{2\lambda(1-B^2r^{2n}+nBr^{n-1}(1-r^2))}
$$

$$= \frac{L_n}{2\lambda K_B}, \tag{2.22}$$

where L_n and K_B are given by (2.13). From (2.13) and (2.18) one can see that $K_B > 0$ and $L_n < 0$.

Since

$$\begin{aligned}
&2\lambda K_B(1 - Ar^n) - L_n(1 - Br^n) \\
&= 2\lambda\left[(1 - Ar^n)(1 - B^2r^{2n}) - (1 - Br^n)(1 - ABr^{2n})\right] \\
&\quad + \lambda nr^{n-1}(1 - r^2)\left[2B(1 - Ar^n) - (A + B)(1 - Br^n)\right] + (p + 1)(A - B)r^{n-1}(1 - r^2)(1 - Br^n) \\
&= -2\lambda(A - B)r^n(1 - Br^n) - \lambda n(A - B)r^{n-1}(1 - r^2)(1 + Br^n) \\
&\quad + (p + 1)(A - B)r^{n-1}(1 - r^2)(1 - Br^n) \\
&> 0 \quad (\lambda < 0),
\end{aligned}$$

we see that

$$u_n > \frac{1 - Ar^n}{1 - Br^n}. \tag{2.23}$$

But u_n is not always less than $\frac{1+Ar^n}{1+Br^n}$. The following two cases arise.

(i) $u_n \geq \frac{1+Ar^n}{1+Br^n}$, that is, $M_n(A, B, \lambda, r) \geq 0$ (see (2.13)). In view of $\psi_n'(u_n) = 0$ and (2.21), the function $\psi_n(u)$ is increasing on the segment $\left[\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right]$. Thus we deduce from (2.19) that, if $M_n(A, B, \lambda, r) \geq 0$, then

$$\begin{aligned}
&\text{Re}\left\{(\lambda - 1)z^{p+1}f'(z) + \frac{\lambda}{p + 1}z^{p+2}f''(z)\right\} \leq \psi_n\left(\frac{1 + Ar^n}{1 + Br^n}\right) \\
&= p\left(1 - \frac{\lambda n(A + B)}{(p + 1)(A - B)}\right)\left(\frac{1 + Ar^n}{1 + Br^n}\right) + \frac{\lambda np}{(p + 1)(A - B)}\left(A + B\left(\frac{1 + Ar^n}{1 + Br^n}\right)^2\right) \\
&= p\frac{1 + Ar^n}{1 + Br^n} + \frac{\lambda np}{(p + 1)(A - B)}\left(1 - \frac{1 + Ar^n}{1 + Br^n}\right)\left(A - B\frac{1 + Ar^n}{1 + Br^n}\right) \\
&= \frac{p[p + 1 + ((p + 1)(A + B) - \lambda n(A - B))r^n + (p + 1)ABr^{2n}]}{(p + 1)(1 + Br^n)^2}.
\end{aligned}$$

This proves (2.11).

Next we consider the function $f(z) \in \mathcal{R}_n(A, B, 0)$ defined by

$$-\frac{z^{p+1}f'(z)}{p} = \frac{1 + Az^n}{1 + Bz^n}.$$

It is easy to find that

$$(\lambda - 1)r^{p+1}f'(r) + \frac{\lambda}{p + 1}r^{p+2}f''(r) = \frac{p[p + 1 + ((p + 1)(A + B) - \lambda n(A - B))r^n + (p + 1)ABr^{2n}]}{(p + 1)(1 + Br^n)^2},$$

which shows that the inequality (2.11) is sharp.

(ii) $u_n \leq \frac{1+Ar^n}{1+Br^n}$, that is, $M_n(A, B, \lambda, r) \leq 0$. In this case we easily obtain

$$\text{Re}\left\{(\lambda - 1)z^{p+1}f'(z) + \frac{\lambda}{p + 1}z^{p+2}f''(z)\right\} \leq \psi_n(u_n). \tag{2.24}$$

In view of (2.13), $\psi_n(u)$ in (2.19) can be written as

$$\psi_n(u) = \frac{p(\lambda K_B u^2 - L_n u + \lambda K_A)}{(p+1)(A-B)r^{n-1}(1-r^2)}. \tag{2.25}$$

Therefore, if $M_n(A, B, \lambda, r) \leq 0$, then it follows from (2.22), (2.24) and (2.25) that

$$\begin{aligned}\operatorname{Re}\left\{(\lambda-1)z^{p+1}f'(z) + \frac{\lambda}{p+1}z^{p+2}f''(z)\right\} &\leq \frac{p(\lambda K_B u_n^2 - L_n u_n + \lambda K_A)}{(p+1)(A-B)r^{n-1}(1-r^2)} \\ &= \frac{p(4\lambda^2 K_A K_B - L_n^2)}{4\lambda(p+1)(A-B)r^{n-1}(1-r^2)K_B}.\end{aligned}$$

To show that the inequality (2.12) is sharp, we consider the function $f(z)$ defined by

$$-\frac{z^{p+1}f'(z)}{p} = \frac{1+Az^n\varphi(z)}{1+Bz^n\varphi(z)} \quad \text{and} \quad \varphi(z) = \frac{z-c_n}{1-c_n z} \quad (z \in \mathbb{U}),$$

where $c_n \in \mathbb{R}$ is determined by

$$-\frac{r^{p+1}f'(r)}{p} = \frac{1+Ar^n\varphi(r)}{1+Br^n\varphi(r)} = u_n \in \left(\frac{1-Ar^n}{1-Br^n}, \frac{1+Ar^n}{1+Br^n}\right].$$

Clearly, $-1 < \varphi(r) \leq 1$, $-1 \leq c_n < 1$, $|\varphi(z)| \leq 1$ $(z \in \mathbb{U})$, and so $f(z) \in \mathcal{R}_n(A, B, 0)$. Since

$$\varphi'(r) = \frac{1-c_n^2}{(1-c_n r)^2} = \frac{1-|\varphi(r)|^2}{1-r^2},$$

from the above argument we obtain that

$$(\lambda-1)r^{p+1}f'(r) + \frac{\lambda}{p+1}r^{p+2}f''(r) = \psi_n(u_n).$$

Now the proof of Theorem 4 is completed. □

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Conflict of interest

The authors declare no conflicts of interest.

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