Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4959–4973.
DOI: 10.3934/math.2020317
Received: 21 March 2020
Accepted: 09 May 2020
Published: 05 June 2020

Research article

Approximation properties of modified (p, q)-Szász-Mirakyan-Kantorovich operators

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Abstract: In this paper, we introduce a new kind of modified (p, q)-Szász-Mirakyan-Kantorovich operators based on (p, q)-calculus. Next, the moments computation formulas, the second and fourth order central moments computation formulas and other quantitative properties are investigated. Then, the approximation properties including local approximation, weighted approximation, rate of convergence and Voronovskaja type theorem are obtained. Finally, we generalize the operators by adding a parameter λ.

Keywords: modified (p, q)-Szász-Mirakyan-Kantorovich operators; moduli of continuity; rate of convergence; weighted approximation
Mathematics Subject Classification: 41A10, 41A25, 41A36

1. Introduction

For any fixed $n \in \mathbb{N}$ and $\zeta \in C[0, \infty)$, the classical Szász-Mirakyan operators $S_n\zeta$ are defined by

$$S_n(\zeta; x) = \sum_{k=0}^{\infty} s_{n,k}(x)\zeta\left(\frac{k}{n}\right), \quad x \in [0, \infty) \tag{1.1}$$

where $s_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$ are Szász-Mirakyan (Possion) basic functions. Then the classical Szász-Mirakyan-Kantorovich operators are defined by

$$K_n(\zeta; x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} \zeta(t)\mathrm{d}t. \tag{1.2}$$

The above operators can be rewritten as follows:

$$K_n(\zeta; x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^1 \zeta\left(\frac{k+t}{n}\right) \mathrm{d}t. \tag{1.3}$$

About contributions on Kantorovich type modification of positive linear operators and many modified Szász-Mirakyan operators , we refer to the papers [1–16]. Recently, several researchers have studied the (p, q)-analogues of positive linear operators and discussed many important approximation properties(see [17–34]). First, we recall some useful concepts and notations about (p, q)-calculus, which can be found in [35]. For each real number λ, (p, q)-analogue of λ named $[\lambda]_{p,q}$ is defined by

$$[\lambda]_{p,q} = \frac{p^\lambda - q^\lambda}{p - q},\ p \neq q.$$

And for any nonnegative integer s, the (p, q)-integer $[s]_{p,q}$ and (p, q)-factorial $[s]_{p,q}!$ are defined by

$$[s]_{p,q} = p^{s-1} + p^{s-2}q + p^{s-3}q^2 + \cdots + pq^{s-2} + q^{s-1} = \begin{cases} \frac{p^s - q^s}{p-q}, & p \neq q \neq 1; \\ sp^{s-1}, & p = q \neq 1; \\ [s]_q, & p = 1; \\ s, & p = q = 1 \end{cases}$$

and

$$[s]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \cdots [s]_{p,q}, & s \geq 1; \\ 1, & s = 0. \end{cases}$$

The (p, q)-analogues of the exponential function named $E_{p,q}(x)$ are defined by

$$E_{p,q}(x) = \sum_{s=0}^{\infty} \frac{q^{\frac{s(s-1)}{2}} x^s}{[s]_{p,q}!}.$$

Let ζ be an arbitrary function and $a \in \mathbb{R}$. The (p, q)-Jackson integral [36] was defined by

$$\int_0^a \zeta(x) \mathrm{d}_{p,q}x = (p - q)a \sum_{i=0}^{\infty} \frac{q^i}{p^{i+1}} \zeta\left(\frac{q^i}{p^{i+1}}\right),\ 0 < q < p \leq 1. \tag{1.4}$$

So a definition for (p, q)-Jackson integral on $[a, b]$ is

$$\int_a^b \zeta(x) \mathrm{d}_{p,q}x = \int_0^b \zeta(x) \mathrm{d}_{p,q}x - \int_0^a \zeta(x) \mathrm{d}_{p,q}x.$$

Therefore, we easily know (p, q)-Jackson integral (1.4) is not positive unless it is assumed that ζ is a nondecreasing function. To solve the problem, T. Acar et al. [25] introduced the new (p, q)-Jackson integral

$$\int_a^b \zeta(x) \mathrm{d}_{p,q}x = (p - q)(b - a) \sum_{i=0}^{\infty} \frac{q^i}{p^{i+1}} \zeta\left(a + (b - a)\frac{q^i}{p^{i+1}}\right),\ 0 < q < p \leq 1. \tag{1.5}$$

It is obvious that integral (1.4) and integral (1.5) of ζ on $[0, 1]$ are equivalence.

In [23], T. Acar introduced (p, q)-analogue of Szász-Mirakyan operators (1.1) as follows:

Definition 1.1. *Let* $0 < q < p \leq 1$ *and* $n \in \mathbb{N}$ *and* $\zeta \in C[0, \infty)$*,* (p, q)*-Szász-Mirakyan operators can be defined by*

$$S_n^{p,q}(\zeta; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\zeta\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right)$$

where $s_{n,k}^{p,q}(x) = \frac{1}{E_{p,q}\left([n]_{p,q}x\right)} q^{\frac{k(k-1)}{2}} \frac{[n]_{p,q}^k x^k}{[k]_{p,q}!}$ *named* (p, q)*-Szász-Mirakyan basis functions.*

Meantime, T. Acar established the following Lemma of moments:

Lemma 1.2. *[23, Lemma 1-2] Let* $0 < q < p \leq 1$*,* $m, n \in \mathbb{N}$*,* $x \in [0, \infty)$*, we have :*

(1). $S_n^{p,q}(1; x) = 1$*;*
(2). $S_n^{p,q}(t; x) = qx$*;*
(3). $S_n^{p,q}(t^2; x) = pqx^2 + \frac{q^2 x}{[n]_{p,q}}$*;*
(4). $S_n^{p,q}(t^3; x) = p^3 x^3 + \frac{x^2(p^2 q + 2pq^2)}{[n]_{p,q}} + \frac{q^3 x}{[n]_{p,q}^2}$*;*
(5). $S_n^{p,q}(t^4; x) = \frac{p^6}{q^2} x^4 + \frac{x^3}{[n]_{p,q}} \frac{p^3 q(p^2 + 2pq + 3q^2)}{q^2} + \frac{x^2}{[n]_{p,q}^2} pq(p^2 + 3pq + 3q^2) + \frac{q^4 x}{[n]_{p,q}^3}$*;*
(6). $S_n^{p,q}(t^{m+1}; x) = \sum_{i=0}^{m} \binom{m}{i} \frac{xp^i}{q^{2i-m-1}[n]_{p,q}^{m-i}} S_n^{p,q}(t^m; x)$.

(p, q)-analogue of the operators (1.2) and their modified operators are discussed and studied in [15, 20,21]. However, we can not find (p, q)-analogue of the operators (1.3) until now. All this achievement motivates us to construct (p, q)-analogue of the operators (1.3) named modified (p, q)-Szász-Mirakyan-Kantorovich operators as follows:

Definition 1.3. *Let* $0 < q < p \leq 1$ *and* $n \in \mathbb{N}$ *and* $\zeta \in C[0, \infty)$*, modified* (p, q)*-Szász-Mirakyan-Kantorovich operators can be defined by*

$$\mathcal{K}_n^{p,q}(\zeta; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \int_0^1 \zeta\left(\frac{q^{1-k}[k]_{p,q} + t}{[n]_{p,q}}\right) \mathrm{d}_{p,q}t. \tag{1.6}$$

The aim of the present paper is to construct modified (p, q)-Szász-Mirakyan-Kantorovich operators based on (p, q)-calculus and discuss their approximation properties.

2. Auxiliary results

To prove our main approximation theorems about the operators (1.6), we need the following lemmas and corollaries.

Lemma 2.1. *The following equality holds for all* $0 < q < p \leq 1$*,* $x \in [0, \infty)$ *and* $m, n \in \mathbb{N}$

$$\mathcal{K}_n^{p,q}(t^m; x) = \sum_{i=0}^{m} \binom{m}{i} \frac{1}{[i+1]_{p,q}[n]_{p,q}^i q^{m-i}} S_n^{p,q}\left(t^{m-i}; x\right). \tag{2.1}$$

Proof. The recurrence formula can be derived by direct computation,

$$
\begin{aligned}
\mathcal{K}_n^{p,q}(t^m;x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\int_0^1 \left(\frac{q^{1-k}[k]_{p,q}+t}{[n]_{p,q}}\right)^m \mathrm{d}_{p,q}t \\
&= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\int_0^1 \sum_{i=0}^{m}\binom{m}{i}\frac{(q^{1-k}[k]_{p,q})^{m-i}t^i}{[n]_{p,q}^m}\mathrm{d}_{p,q}t \\
&= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\sum_{i=0}^{m}\binom{m}{i}\frac{(q^{1-k}[k]_{p,q})^{m-i}}{[n]_{p,q}^m[i+1]_{p,q}} \\
&= \sum_{i=0}^{m}\binom{m}{i}\frac{1}{[i+1]_{p,q}[n]_{p,q}^i q^{m-i}}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right)^{m-i} \\
&= \sum_{i=0}^{m}\binom{m}{i}\frac{1}{[i+1]_{p,q}[n]_{p,q}^i q^{m-i}} S_n^{p,q}\left(t^{m-i};x\right).
\end{aligned}
$$

□

The following Lemma is immediate.

Lemma 2.2. *Let* $0 < q < p \leq 1$, $n \in \mathbb{N}$, $x \in (0,\infty)$, *we have:*

(1). $\mathcal{K}_n^{p,q}(1;x) = 1$, $\mathcal{K}_n^{p,q}(t;x) = x + \frac{1}{[n]_{p,q}[2]_{p,q}}$;

(2). $\mathcal{K}_n^{p,q}(t^2;x) = \frac{p}{q}x^2 + \frac{2+[2]_{p,q}}{[2]_{p,q}[n]_{p,q}}x + \frac{1}{[3]_{p,q}[n]_{p,q}^2}$;

(3). $\mathcal{K}_n^{p,q}(t^3;x) = \frac{p^3}{q^3}x^3 + \left(\frac{p^2}{q^2} + \left(2 + \frac{3}{[2]_{p,q}}\right)\frac{p}{q}\right)\frac{x^2}{[n]_{p,q}} + \left(1 + \frac{3}{[2]_{p,q}} + \frac{3}{[3]_{p,q}}\right)\frac{x}{[n]_{p,q}^2} + \frac{1}{[4][n]_{p,q}^3}$;

(4).
$$
\begin{aligned}
\mathcal{K}_n^{p,q}(t^4;x) = {}& \frac{p^6}{q^6}x^4 + \left(\frac{p^5}{q^5} + \frac{2p^4}{q^4} + \left(3 + \frac{4}{[2]_{p,q}}\right)\frac{p^3}{q^3}\right)\frac{x^3}{[n]_{p,q}} \\
&+ \left(\frac{p^3}{q^3} + \left(3 + \frac{4}{[2]_{p,q}}\right)\frac{p^2}{q^2} + \left(3 + \frac{8}{[2]_{p,q}} + \frac{6}{[3]_{p,q}}\right)\frac{p}{q}\right)\frac{x^2}{[n]_{p,q}^2} \\
&+ \left(1 + \frac{4}{[2]_{p,q}} + \frac{6}{[3]_{p,q}} + \frac{4}{[4]_{p,q}}\right)\frac{x}{[n]_{p,q}^3} + \frac{1}{[5]_{p,q}[n]_{p,q}^4}.
\end{aligned}
$$

Corollary 2.3. *Using Lemma 2.2, we can easily obtain the following explicit formulas for the first, second and fourth central moments:*

$$A_1^{p,q}(x) := \mathcal{K}_n^{p,q}(t-x;x) = \frac{1}{[n]_{p,q}[2]_{p,q}}; \tag{2.2}$$

$$A_2^{p,q}(x) := \mathcal{K}_n^{p,q}\left((t-x)^2;x\right) = \left(\frac{p}{q}-1\right)x^2 + \frac{2}{[n]_{p,q}[2]_{p,q}}x + \frac{1}{[3]_{p,q}[n]_{p,q}^2}; \tag{2.3}$$

$$\mathcal{K}_n^{p,q}\left((t-x)^4;x\right) = \left(\frac{p^6}{q^6} - \frac{4p^3}{q^3} + \frac{6p}{q} - 3\right)x^4$$

$$+\left(\frac{p^5}{q^5}+\frac{2p^4}{q^4}+\left(3+\frac{4}{[2]_{p,q}}\right)\frac{p^3}{q^3}-\frac{4p^2}{q^2}-\left(8+\frac{12}{[2]_{p,q}}\right)\frac{p}{q}+\frac{8}{[2]_{p,q}}+6\right)\frac{x^3}{[n]_{p,q}} \tag{2.4}$$
$$+\left(\frac{p^3}{q^3}+\left(3+\frac{4}{[2]_{p,q}}\right)\frac{p^2}{q^2}+\left(3+\frac{8}{[2]_{p,q}}+\frac{6}{[3]_{p,q}}\right)\frac{p}{q}-4-\frac{12}{[2]_{p,q}}-\frac{6}{[3]_{p,q}}\right)\frac{x^2}{[n]_{p,q}^2}$$
$$+\left(1+\frac{4}{[2]_{p,q}}+\frac{6}{[3]_{p,q}}\right)\frac{x}{[n]_{p,q}^3}+\frac{1}{[5]_{p,q}[n]_{p,q}^4}.$$

Corollary 2.4. *The sequences (p_n), (q_n) satisfy $0 < q_n < p_n \leq 1$ such that $p_n \to 1$, $q_n \to 1$ and $p_n^n \to a \in [0,1]$, $q_n^n \to b \in [0,1]$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty}[n]_{p_n,q_n}A_1^{p_n,q_n}(x)=\frac{1}{2}; \tag{2.5}$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}A_2^{p_n,q_n}(x)=(a-b)x^2+x; \tag{2.6}$$
$$\lim_{n\to\infty}[n]_{p_n,q_n}\mathcal{K}_n^{p,q}\left((t-x)^4;x\right)=0. \tag{2.7}$$

Proof. The limit equality (2.5) is obvious. Using $[n]_{p_n,q_n}\left(\frac{p_n}{q_n}-1\right) \sim [n]_{p_n,q_n}(p_n-q_n) \sim p_n^n-q_n^n \sim a-b$, we can easily obtain the limit equality (2.6). Combining with

$$\begin{aligned}[n]_{p_n,q_n}\left(\frac{p_n^6}{q_n^6}-\frac{4p_n^3}{q_n^3}+\frac{6p_n}{q_n}-3\right) &\sim [n]_{p_n,q_n}\left(p_n^6-4p_n^3q_n^3+6p_nq_n^5-3q_n^6\right)\\ &\sim [n]_{p_n,q_n}\left(p_n^6-q_n^6-4q_n^3(p_n^3-q_n^3)+6q_n^5(p_n-q_n)\right)\\ &\sim [n]_{p_n,q_n}(p_n-q_n)([6]_{p_n,q_n}-4q_n^3[3]_{p_n,q_n}+6q_n^5)\\ &\sim (a-b)(6-12+6)=0\end{aligned}$$

and

$$\frac{p_n^5}{q_n^5}+\frac{2p_n^4}{q_n^4}+\left(3+\frac{4}{[2]_{p_n,q_n}}\right)\frac{p_n^3}{q_n^3}-\frac{4p_n^2}{q_n^2}-\left(8+\frac{12}{[2]_{p_n,q_n}}\right)\frac{p_n}{q_n}+\frac{8}{[2]_{p_n,q_n}}+6 \sim 8-18+10=0.$$

We can obtain the limit equality (2.7). □

3. Approximation properties

Let us denote the norm $\|\zeta\| = \sup_{x\in[0,\infty)} |\zeta(x)|$ on $C_B[0,\infty)$, the class of real valued continuous bounded functions. For $\zeta \in C_B[0,\infty)$ and $\delta > 0$, the s-th order modulus of continuity is defined by $\omega_s(\zeta;\delta) = \sup_{0<h\leq\delta}\sup_{t\in[0,\infty)} |\Delta_h^s\zeta(t)|$, where Δ_h is the forward difference and $\Delta_h\left(\Delta_h^{s-1}\right)$ for $s \geq 1$. In case $s = 1$, we mean the usual modulus of continuity denoted by $\omega(\zeta;\delta)$. Also, Peetre's K-functional is defined by $K_2(\zeta;\delta) = \inf_{\phi\in W^2}\{\|\zeta-\phi\|+\delta\|\phi''\|\}$, where $W^2 = \{\phi \in C_B[0,\infty) : \phi', \phi'' \in C_B[0,\infty)\}$. By [37, p.177, Theorem 2.4], there exists an absolute positive C such that $K_2(\zeta;\delta^2) \leq C\omega_2(\zeta;\delta)$.

Theorem 3.1. *Let (p_n), (q_n) be the sequences defined in Corollary 2.4 and $\zeta \in C_B[0,\infty)$. Then for all $n \in \mathbb{N}$, there exists an absolute positive $C_1 = 4C$ such that*

$$\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right| \leq C_1\omega_2\left(\zeta;\ \sqrt{A_2^{p_n,q_n}(x) + (A_1^{p_n,q_n}(x))^2}\right) + \omega\left(\zeta; A_1^{p_n,q_n}(x)\right).$$

Proof. We consider the following operators:

$$\mathcal{H}_n^{p_n,q_n}(\zeta;x) = \mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(A_1^{p_n,q_n}(x) + x) + \zeta(x), \quad x \in [0,\infty).$$

Let $x, t \in [0,\infty)$ and $\phi \in W^2$. By Taylor's expansion formula, we have:

$$\phi(t) = \phi(x) + \phi'(x)(t-x) + \int_x^t \phi''(u)(t-u)\mathrm{d}u.$$

Applying $\mathcal{H}_n^{p_n,q_n}$ and using $\mathcal{H}_n^{p_n,q_n}(t-x;x) = 0$, we can get

$$\mathcal{H}_n^{p_n,q_n}(\phi;x) - \phi(x) = \mathcal{H}_n^{p_n,q_n}\left(\int_x^t \phi''(u)(t-u)\mathrm{d}u; x\right).$$

Hence,

$$\begin{aligned}
&\left|\mathcal{H}_n^{p_n,q_n}(\phi;x) - \phi(x)\right| \\
&\leq \mathcal{K}_n^{p_n,q_n}\left(\left|\int_x^t \phi''(u)(t-u)\mathrm{d}u\right|; x\right) + \left|\int_x^{x+A_1^{p_n,q_n}(x)} \phi''(u)\left(A_1^{p_n,q_n}(x) + x - u\right)\mathrm{d}u\right| \\
&\leq \mathcal{K}_n^{p_n,q_n}\left(\int_x^t |\phi''(u)|(t-u)\mathrm{d}u; x\right) + \int_x^{x+A_1^{p_n,q_n}(x)} |\phi''(u)|\left(A_1^{p_n,q_n}(x) + x - u\right)\mathrm{d}u \\
&\leq \|\phi''\|\mathcal{K}_n^{p_n,q_n}\left((t-x)^2; x\right) + \|\phi''\|\int_x^{x+A_1^{p_n,q_n}(x)} \left(A_1^{p_n,q_n}(x) + x - u\right)\mathrm{d}u \\
&\leq \left(A_2^{p_n,q_n}(x) + (A_1^{p_n,q_n}(x))^2\right)\|\phi''\|.
\end{aligned}$$

By Lemma 2.2, we can easily obtain $|\mathcal{H}_n^{p_n,q_n}(\zeta;x)| \leq \|\zeta\|$ for any $\zeta \in C_B[0,\infty)$. Hence,

$$\begin{aligned}
&\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right| \\
&= \left|\mathcal{H}_n^{p_n,q_n}(\zeta;x) + \zeta(A_1^{p_n,q_n}(x) + x) - 2\zeta(x)\right| \\
&\leq |\mathcal{H}_n^{p_n,q_n}(\zeta-\phi;x) - (\zeta-\phi)(x)| + \left|\mathcal{H}_n^{p_n,q_n}(\phi;x) - \phi(x)\right| + \left|\zeta(A_1^{p_n,q_n}(x) + x) - \zeta(x)\right| \\
&\leq 4\|\zeta - \phi\| + \left(A_2^{p_n,q_n}(x) + (A_1^{p_n,q_n}(x))^2\right)\|\phi''\| + \omega\left(\zeta; A_1^{p_n,q_n}(x)\right).
\end{aligned}$$

Taking infimum on the right hand side over all $\phi \in W^2$ and using the property of K-functional, we can get:

$$\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right| \leq C_1\omega_2\left(\zeta;\ \sqrt{A_2^{p_n,q_n}(x) + (A_1^{p_n,q_n}(x))^2}\right) + \omega\left(\zeta; A_1^{p_n,q_n}(x)\right).$$

This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let (p_n), (q_n) be the sequences defined in Corollary 2.4. Then for any $A > 0$ and $\zeta \in C_B[0,\infty)$, then the sequence $\{\mathcal{K}_n^{p_n,q_n}(\zeta;x)\}$ converges to ζ uniformly on $[0,A]$.*

Theorem 3.3. *Let $\gamma \in (0,1]$ and D be any subset of the interval $[0,\infty)$, if $\zeta \in C_B[0,\infty)$ is locally on* $\mathrm{Lip}(\gamma)$*, i.e. the condition*

$$|\zeta(t) - \zeta(x)| \leq C_{\zeta,\gamma}|t-x|^{\gamma}, \quad t \in D \quad and \quad x \in [0,\infty) \tag{3.1}$$

holds, then for each $x \in [0,\infty)$, we can obtain:

$$\left|\mathcal{K}_n^{p,q}(\zeta;x) - \zeta(x)\right| \leq C_{\zeta,\gamma}\left(\left(A_2^{p,q}(x)\right)^{\frac{\gamma}{2}} + 2\mathrm{d}^{\gamma}(x;D)\right)$$

where $C_{\zeta,\gamma}$ is a positive constant depending only on γ and ζ and $\mathrm{d}(x;D)$ denotes the distance between x and D defined by

$$\mathrm{d}(x;D) = \inf\{|t-x| : t \in D\}.$$

Proof. Let $\overline{D}$ be the closure of D. Using the properties of infimum, there is at least a point $t_0 \in \overline{D}$ such that $\mathrm{d}(x;D) = |x - t_0|$. By the triangle inequality

$$|\zeta(t) - \zeta(x)| \leq |\zeta(t) - \zeta(t_0)| + |\zeta(x) - \zeta(t_0)|,$$

we immediately have by (3.1) that

$$\begin{aligned}
\left|\mathcal{K}_n^{p,q}(\zeta;x) - \zeta(x)\right| &\leq \mathcal{K}_n^{p,q}(|\zeta(t) - \zeta(t_0)|;x) + \mathcal{K}_n^{p,q}(|\zeta(x) - \zeta(t_0)|;x) \\
&\leq C_{\zeta,\gamma}\{\mathcal{K}_n^{p,q}(|t-t_0|^{\gamma};x) + |x-t_0|^{\gamma}\} \\
&\leq C_{\zeta,\gamma}\{\mathcal{K}_n^{p,q}(|t-x|^{\gamma} + |x-t_0|^{\gamma};x) + |x-t_0|^{\gamma}\} \\
&= C_{\zeta,\gamma}\{\mathcal{K}_n^{p,q}(|t-x|^{\gamma};x) + 2|x-t_0|^{\gamma}\}.
\end{aligned}$$

Choosing $a_1 = \frac{2}{\gamma}$ and $a_2 = \frac{2}{2-\gamma}$ and using the well-known Hölder inequality, we have:

$$\begin{aligned}
\left|\mathcal{K}_n^{p,q}(\zeta;x) - \zeta(x)\right| &\leq C_{\zeta,\gamma}\{(\mathcal{K}_n^{p,q}(|t-x|^{a_1\gamma};x))^{\frac{1}{a_1}}(\mathcal{K}_n^{p,q}(1^{a_2};x))^{\frac{1}{a_2}} + 2\mathrm{d}^{\gamma}(x;D)\} \\
&\leq C_{\zeta,\gamma}\{\left(\mathcal{K}_n^{p,q}\left((t-x)^2;x\right)\right)^{\frac{\gamma}{2}} + 2\mathrm{d}^{\gamma}(x;D)\} \\
&\leq C_{\zeta,\gamma}\left(\left(A_2^{p,q}(x)\right)^{\frac{\gamma}{2}} + 2\mathrm{d}^{\gamma}(x;D)\right).
\end{aligned}$$

□

Next, we obtain the local direct estimate of the operators $\mathcal{K}_n^{p,q}$, using the Lipcshitz type maximal function of the order γ introduced by Lenze [38] as:

$$\tilde{\omega}_{\gamma}(\zeta;x) = \sup_{x,t\in[0,\infty),x\neq t}\frac{|\zeta(t)-\zeta(x)|}{|t-x|^{\gamma}}, \quad x \in [0,\infty) \quad and \quad \gamma \in (0,1]. \tag{3.2}$$

Theorem 3.4. *Let $\zeta \in C_B[0,\infty)$ and $\gamma \in (0,1]$. Then, for all $x \in [0,\infty)$, we have*

$$\left|\mathcal{K}_n^{p,q}(\zeta;x) - \zeta(x)\right| \leq \tilde{\omega}_{\gamma}(\zeta;x)\left(A_2^{p,q}(x)\right)^{\frac{\gamma}{2}}.$$

Proof. From the equation (3.2), we have

$$\left|\mathcal{K}_n^{p,q}(\zeta;x)-\zeta(x)\right| \leq \tilde{\omega}_\gamma(\zeta;x)\mathcal{K}_n^{p,q}(|t-x|^\gamma;x).$$

Applying the well-known Hölder inequality with $a_1=\frac{2}{\gamma}$ and $a_2=\frac{2}{2-\gamma}$, we can get:

$$\begin{aligned}\left|\mathcal{K}_n^{p,q}(\zeta;x)-\zeta(x)\right| &\leq \tilde{\omega}_\gamma(\zeta;x)\left(\mathcal{K}_n^{p,q}\left((t-x)^2;x\right)\right)^{\frac{\gamma}{2}}\\ &\leq \tilde{\omega}_\gamma(\zeta;x)\left(A_2^{p,q}(x)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

□

We consider the following three weighted spaces of functions which are defined on $[0,\infty)$. Let $w(x)$ be the weighted function and C_ζ be a positive constant depending only on the function ζ, we define the weighted space of functions and weighted modulus of continuity as

1. $B_w[0,\infty)$ be the space of function ζ defined on $[0,\infty)$ satisfying $\zeta(x)\leq M_\zeta w(x)$ for all $x\in[0,\infty)$.
2. $C_w[0,\infty)$ be the subspace of all continuous functions in $B_w[0,\infty)$.
3. $C_w^0[0,\infty)$ be the subspace of functions $\zeta\in C_w[0,\infty)$, where $\lim\limits_{x\to\infty}\dfrac{|\zeta(x)|}{w(x)}$ is finite and endowed with the norm $\|\zeta\|_{w(x)}=\sup\limits_{x\in[0,\infty)}\dfrac{|\zeta(x)|}{w(x)}$.
4. While $w(x)=1+x^2$, we define the weighted modulus of continuity as:

$$\Omega(\zeta;\delta)=\sup_{0<t\leq\delta,x\geq 0}\frac{|\zeta(x+t)-\zeta(x)|}{w(x)w(t)} \tag{3.3}$$

and have the inequality

$$\Omega(\zeta;\alpha\delta)\leq 2(1+\alpha)(1+\delta^2)\Omega(\zeta;\delta) \tag{3.4}$$

hold, where $\alpha>0$ and $\zeta\in C_w^0[0,\infty)$(see [39]). Meantime, for $\kappa>0$, the modulus of continuity of ζ on $[0,\kappa]$ is defined by $\omega_\kappa(\zeta;\delta)=\sup\limits_{|t-x|<\delta}\ \sup\limits_{0\leq x,t\leq\kappa}|\zeta(t)-\zeta(x)|$.

Theorem 3.5. *Let $\zeta\in C_w[0,\infty)$, $0<q<p\leq 1$ and $\omega_{\kappa+1}(\zeta;\delta)$ be its modulus of continuity on the finite interval $[0,\kappa]\subset[0,\infty)$, where $\kappa>0$. Then, for every $n\in\mathbb{N}$, we have:*

$$\|\mathcal{K}_n^{p,q}(\zeta;x)-\zeta(x)\|_{C[0,\kappa]}\leq C_\zeta(3+2\kappa^2)A_2^{p,q}(\kappa)+2\omega_{\kappa+1}\left(\zeta;\sqrt{A_2^{p,q}(\kappa)}\right).$$

Proof. For any $x\in[0,\kappa]$ and $t>\kappa+1$, we easily have $1\leq(t-\kappa)^2\leq(t-x)^2$. Thus,

$$\begin{aligned}|\zeta(t)-\zeta(x)| &\leq |\zeta(t)|+|\zeta(x)|\leq C_\zeta(2+t^2+x^2)\\ &= C_\zeta\left(2+x^2+(t-x+x)^2\right)\leq C_\zeta\left(2+2x^2+(t-x)^2\right)\\ &\leq C_\zeta(3+2x^2)(t-x)^2\leq C_\zeta(3+2\kappa^2)(t-x)^2,\end{aligned} \tag{3.5}$$

and for any $x\in[0,\kappa]$, $t\in[0,\kappa+1]$ and $\delta>0$, we have:

$$|\zeta(t)-\zeta(x)|\leq\omega_{\kappa+1}(|t-x|;x)\leq\left(1+\frac{|t-x|}{\delta}\right)\omega_\kappa(\zeta;\delta). \tag{3.6}$$

From (3.5) and (3.6), we can get

$$|\zeta(t)-\zeta(x)| \leq C_{\zeta}(3+2\kappa^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{\kappa}(\zeta;\delta).$$

By Schwarz's inequality and Corollary 2.4, for any $x \in [0,\kappa]$, we can get

$$\begin{aligned}
|\mathcal{K}_n^{p,q}(\zeta;x)-\zeta(x)| &\leq \mathcal{K}_n^{p,q}(|\zeta(t)-\zeta(x)|;x) \\
&\leq C_{\zeta}(3+2\kappa^2)\mathcal{K}_n^{p,q}((t-x)^2;x) + \mathcal{K}_n^{p,q}\left(\left(1+\frac{|t-x|}{\delta}\right);x\right)\omega_{\kappa+1}(\zeta;\delta) \\
&\leq C_{\zeta}(3+2\kappa^2)\mathcal{K}_n^{p,q}((t-x)^2;x) + \omega_{\kappa+1}(\zeta;\delta)\left(1+\frac{1}{\delta}\sqrt{\mathcal{K}_n^{p,q}((t-x)^2;x)}\right) \\
&\leq C_{\zeta}(3+2\kappa^2)A_2^{p,q}(x) + \omega_{\kappa+1}(\zeta;\delta)\left(1+\frac{1}{\delta}\sqrt{A_2^{p,q}(x)}\right) \\
&\leq C_{\zeta}(3+2\kappa^2)A_2^{p,q}(\kappa) + \omega_{\kappa+1}(\zeta;\delta)\left(1+\frac{1}{\delta}\sqrt{A_2^{p,q}(\kappa)}\right)
\end{aligned}$$

By taking $\delta = \sqrt{A_2^{p,q}(\kappa)}$ and supremum over all $x \in [0,\kappa]$, we accomplish the proof of Theorem 3.5. □

Theorem 3.6. *Let $\zeta \in C_w^0[0,\infty)$ and the sequences (p_n), (q_n) satisfy $0 < q_n < p_n \leq 1$ such that $p_n \to 1$, $q_n \to 1$ and $p_n^n \to 1$, $q_n^n \to 1$, $[n]_{p_n,q_n} \to \infty$ as $n \to \infty$, then there exists $N_1 \in \mathbb{N}_+$ such that for any $n > N_1$ and $\nu > 0$, the inequality*

$$\sup_{x\in[0,\infty)} \frac{\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x)-\zeta(x)\right|}{(1+x^2)^{\frac{5}{2}+\nu}} \leq 48\Omega\left(\zeta;\frac{1}{\sqrt{[n]_{p_n,q_n}}}\right)$$

holds.

Proof. For $\zeta \in C_w^0[0,\infty)$, from (3.3) and (3.4), we can write:

$$\begin{aligned}
|\zeta(t)-\zeta(x)| &\leq \left(1+(t-x)^2\right)(1+x^2)\Omega(\zeta;|t-x|) \\
&\leq 2\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(\zeta;\delta)\left(1+(t-x)^2\right)(1+x^2) \\
&\leq \begin{cases} 4(1+\delta^2)^2(1+x^2)\Omega(\zeta;\delta), & |t-x|\leq\delta, \\ 4(1+\delta^2)(1+x^2)\Omega(\zeta;\delta)\dfrac{|t-x|+|t-x|^3}{\delta}, & |t-x|>\delta. \end{cases}
\end{aligned} \tag{3.7}$$

Set $\delta \in (0,1)$, for all $x,t \in [0,\infty)$, (3.7) can be rewritten

$$|\zeta(t)-\zeta(x)| \leq 16(1+x^2)\Omega(\zeta;\delta)\left(1+\frac{|t-x|+|t-x|^3}{\delta}\right). \tag{3.8}$$

Using (2.6) and (2.7), there exists $N_1 \in \mathbb{N}_+$ such that for any $n > N_1$,

$$\mathcal{K}_n^{p_n,q_n}((t-x)^2;x) \leq \frac{1}{[n]_{p_n,q_n}}(1+x^2), \quad \mathcal{K}_n^{p_n,q_n}((t-x)^4;x) \leq (1+x^2)^2.$$

By Schwarz's inequality, we can obtain

$$\mathcal{K}_n^{p_n,q_n}(|t-x|;x) \leq \sqrt{\mathcal{K}_n^{p_n,q_n}((t-x)^2;x)} \leq \frac{\sqrt{1+x^2}}{\sqrt{[n]_{p_n,q_n}}} \tag{3.9}$$

and

$$\mathcal{K}_n^{p_n,q_n}(|t-x|^3;x) \leq \sqrt{\mathcal{K}_n^{p_n,q_n}((t-x)^2;x)}\sqrt{\mathcal{K}_n^{p_n,q_n}((t-x)^4;x)} \leq \frac{\sqrt{(1+x^2)^3}}{\sqrt{[n]_{p_n,q_n}}}. \tag{3.10}$$

Since $\mathcal{K}_n^{p_n,q_n}$ is linear and positive, using (3.8–3.10), we can obtain

$$\begin{aligned}\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x)-\zeta(x)\right| &\leq 16(1+x^2)\Omega(\zeta;\delta)\left(1+\frac{\mathcal{K}_n^{p_n,q_n}\left(|t-x|+|t-x|^3;x\right)}{\delta}\right)\\ &\leq 16(1+x^2)\left(1+\frac{2\sqrt{(1+x^2)^3}}{\delta\sqrt{[n]_{p_n,q_n}}}\right)\Omega(\zeta;\delta).\end{aligned}$$

Choosing $\delta = \frac{1}{\sqrt{[n]_{p_n,q_n}}}$, the conclusion holds. □

Theorem 3.7. *Let (p_n), (q_n) be the sequences defined in Theorem 3.6. Then, for any $\zeta \in C_w^0[0,\infty)$, we have:*

$$\lim_{n\to\infty}\left\|\mathcal{K}_n^{p_n,q_n}(\zeta;x)-\zeta\right\|_{w(x)} = 0.$$

Proof. By weighted Korovkin theorem in [40], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\left\|\mathcal{K}_n^{p_n,q_n}(t^k;x)-x^k\right\|_{w(x)} = 0, \quad k=0,1,2. \tag{3.11}$$

Since $\mathcal{K}_n^{p_n,q_n}(1;x)=1$, then (3.11) holds true for $k=0$. By Lemma 2.2, we can obtain:

$$\begin{aligned}\left\|\mathcal{K}_n^{p_n,q_n}(t;x)-x\right\|_{w(x)} &= \sup_{x\in[0,\infty)}\left|\mathcal{K}_n^{p_n,q_n}(t;x)-x\right|\\ &= \frac{1}{[n]_{p_n,q_n}[2]_{p_n,q_n}} \to 0, \quad n\to\infty,\end{aligned}$$

and

$$\begin{aligned}&\left\|\mathcal{K}_n^{p_n,q_n}(t^2;x)-x^2\right\|_{w(x)}\\ &\leq \sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\left|\frac{p_n}{q_n}-1\right| + \sup_{x\in[0,\infty)}\frac{x}{1+x^2}\frac{2+[2]_{p_n,q_n}}{[2]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{1}{[3]_{p_n,q_n}[n]_{p_n,q_n}^2}\\ &= \left|\frac{p_n}{q_n}-1\right| + \frac{2+[2]_{p_n,q_n}}{2[2]_{p_n,q_n}[n]_{p_n,q_n}} + \frac{1}{[3]_{p_n,q_n}[n]_{p_n,q_n}^2}\\ &\to 0, \quad n\to\infty.\end{aligned}$$

Thus, the proof of Theorem 3.7 is completed. □

Now, we present a weighted approximation theorem for functions in $C_w^0[0,\infty)$.

Theorem 3.8. *Let (p_n), (q_n) be the sequences defined in Theorem 3.6. Then, for any $\zeta \in C_w[0,\infty)$ and $\rho > 0$, we have:*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right|}{(1+x^2)^{1+\rho}} = 0.$$

Proof. Let $x_0 \in (0,\infty)$ be arbitrary but fixed. Then,

$$\begin{aligned}
&\sup_{x\in[0,\infty)} \frac{\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right|}{(1+x^2)^{1+\rho}} \\
&\leq \sup_{x\in[0,x_0]} \frac{\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right|}{(1+x^2)^{1+\rho}} + \sup_{x\in(x_0,\infty)} \frac{\left|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right|}{(1+x^2)^{1+\rho}} \\
&\leq \|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta\|_{C[0,x_0]} + \|\zeta\|_{w(x)} \sup_{x\in(x_0,\infty)} \frac{\left|\mathcal{K}_n^{p_n,q_n}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\rho}} \\
&+ \sup_{x\in(x_0,\infty)} \frac{|\zeta(x)|}{(1+x^2)^{1+\rho}}.
\end{aligned}$$

Since $|\zeta(x)| \leq C_\zeta(1+x^2)$, we have $\sup_{x\in(x_0,\infty)} \frac{|\zeta(x)|}{(1+x^2)^{1+\rho}} \leq \frac{C_\zeta\|\zeta\|_{w(x)}}{(1+x_0^2)^\rho}$. Let $\epsilon > 0$ be arbitrary, we can choose x_0 to be sufficiently large that

$$\frac{C_\zeta\|\zeta\|_{w(x)}}{(1+x_0^2)^\rho} < \frac{\epsilon}{3}. \tag{3.12}$$

In view of Lemma 2.2, while $x \in (x_0,\infty)$, we can obtain:

$$\|\zeta\|_{w(x)} \lim_{n\to\infty} \frac{\left|\mathcal{K}_n^{p_n,q_n}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\rho}} \to 0.$$

Hence, we can choose N sufficiently large such that for any $n > N$ the inequality

$$\sup_{x\in[x_0,\infty)} \|\zeta\|_{w(x)} \frac{\left|\mathcal{K}_n^{p_n,q_n}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\rho}} < \frac{\epsilon}{3}. \tag{3.13}$$

holds. Also, the first term of the above inequality tends to zero by Corollary 3.2, that is

$$\|\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta\|_{C[0,x_0]} < \frac{\epsilon}{3}. \tag{3.14}$$

Thus, combining (3.12)–(3.14), we obtain the desired result. □

The last result is a Voronovskaja-type asymptotic formula for the operators $\mathcal{K}_n^{p_n,q_n}(\zeta;x)$.

Theorem 3.9. *Let (p_n), (q_n) be the sequences defined in Corollary 2.4 and $\zeta \in C_B[0,\infty)$. Supposing that $\zeta''(x)$ exists at a point $x \in [0,\infty)$, then we can obtain:*

$$\lim_{n\to\infty} [n]_{p_n,q_n} \left(\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right) = \frac{1}{2}\zeta'(x) + \frac{1}{2}\left((a-b)x^2 + x\right)\zeta''(x).$$

Proof. Using Taylor's expansion formula, we can obtain:

$$\zeta(t) = \zeta(x) + \zeta'(x)(t-x) + \frac{1}{2}\zeta''(x)(t-x)^2 + \Theta(t,x)(t-x)^2, \tag{3.15}$$

where $\Theta(t,x)$ is the Peano form of the remainder and $\lim_{t\to x}\Theta(t,x) = 0$. Applying $\mathcal{K}_n^{p_n,q_n}$ to the both sides of (3.15), we have:

$$[n]_{p_n,q_n}\left(\mathcal{K}_n^{p_n,q_n}(\zeta;x) - \zeta(x)\right) = [n]_{p_n,q_n}\zeta'(x)A_1^{p_n,q_n}(x)$$
$$+\frac{1}{2}[n]_{p_n,q_n}\zeta''(x)A_2^{p_n,q_n}(x) + [n]_{p_n,q_n}\mathcal{K}_n^{p_n,q_n}\left(\Theta(t,x)(t-x)^2;x\right).$$

By the Schwarz inequality, we have:

$$\mathcal{K}_n^{p_n,q_n}\left(\Theta(t,x)(t-x)^2;x\right) \le \sqrt{\mathcal{K}_n^{p_n,q_n}\left(\Theta^2(t,x);x\right)}\sqrt{\mathcal{K}_n^{p_n,q_n}\left((t-x)^4;x\right)}. \tag{3.16}$$

We observe that $\Theta^2(x,x) = 0$ and $\Theta^2(\cdot,x) \in C_B[0,\infty)$. Then, it follows from Corollary 3.2 that

$$\lim_{n\to\infty}\mathcal{K}_n^{p_n,q_n}\left(\Theta^2(t,x);x\right) = \Theta^2(x,x) = 0. \tag{3.17}$$

Hence, from (2.7), (3.16), (3.17), we can obtain:

$$\lim_{n\to\infty}[n]_{p_n,q_n}\mathcal{K}_n^{p_n,q_n}\left(\Theta(t,x)(t-x)^2;x\right) = 0. \tag{3.18}$$

Combining (2.5), (2.6), (3.18), we obtain the required result. □

4. Better estimates

Motivated by the references [41–43], we will construct a generalization of modified (p,q)-Szász-Mirakyan-Kantorovich operators by adding a parameter. Let $\lambda > 0$, λ-generalization of the operators given in (1.6) is defined by

$$\mathcal{K}_{n,\lambda}^{p,q}(\zeta;x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\int_0^1 \zeta\left(\frac{q^{1-k}[k]_{p,q} + t^\lambda}{[n]_{p,q}}\right)\mathrm{d}_{p,q}t.$$

Then, we have

$$\mathcal{K}_{n,\lambda}^{p,q}(1;x) = 1;\ \mathcal{K}_{n,\lambda}^{p,q}(t;x) = x + \frac{1}{[n]_{p,q}[\lambda+1]_{p,q}};$$
$$\mathcal{K}_{n,\lambda}^{p,q}(t^2;x) = \frac{p}{q}x^2 + \left(1 + \frac{2}{[\lambda+1]_{p,q}}\right)\frac{x}{[n]_{p,q}} + \frac{1}{[2\lambda+1]_{p,q}[n]_{p,q}^2}.$$

Now, Theorem 3.1 and Theorem 3.9 can be modified as following:

Theorem 4.1. *Let (p_n), (q_n) be the sequences defined in Corollary 2.4 and $\zeta \in C_B[0,\infty)$. Then for all $n \in \mathbb{N}$, there exists an absolute positive $C_1 = 4C$ such that*

$$\left|\mathcal{K}_{n,\lambda}^{p_n,q_n}(\zeta;x) - \zeta(x)\right| \le C_1\omega_2\left(\zeta;\sqrt{\left(A_\lambda^{p_n,q_n}(x)\right)^2 + B_\lambda^{p_n,q_n}(x)}\right) + \omega\left(\zeta;A_\lambda^{p_n,q_n}(x)\right),$$

where $A_\lambda^{p_n,q_n}(x) := \mathcal{K}_{n,\lambda}^{p_n,q_n}(t-x;x) = \frac{1}{[n]_{p_n,q_n}[\lambda+1]_{p_n,q_n}}$, $B_\lambda^{p_n,q_n}(x) := \mathcal{K}_{n,\lambda}^{p_n,q_n}\left((t-x)^2;x\right) = \left(\frac{p_n}{q_n}-1\right)x^2 + \frac{x}{[n]_{p_n,q_n}} + \frac{1}{[2\lambda+1]_{p_n,q_n}[n]_{p_n,q_n}^2}$.

Theorem 4.2. *Let (p_n), (q_n) be the sequences defined in Corollary 2.4, $\lambda > 0$ and $\zeta \in C_B[0,\infty)$. Supposing that $\zeta''(x)$ exists at a point $x \in [0,\infty)$, then we can obtain:*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(\mathcal{K}_{n,\lambda}^{p_n,q_n}(\zeta;x)-\zeta(x)\right)=\frac{1}{\lambda+1}\zeta'(x)+\frac{1}{2}\left((a-b)x^2+x\right)\zeta''(x).$$

The proof of Theorem 4.1 and Theorem 4.2 is on similar lines, thus we omit the details.

5. Conclusions

In this paper, we have constructed modified (p,q)-Szász-Mirakyan-Kantorovich operators via new method and investigated their approximation properties. We have obtained a rate of convergence, weighted approximation and Voronovskaya-type theorem for our new operators. Finally, we generalize the operators by adding a parameter λ.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No. 11626031), the Key Natural Science Research Project in Universities of Anhui Province (Grant No. KJ2019A0572), the Philosophy and Social Sciences General Planning Project of Anhui Province of China (Grant No. AHSKYG2017D153) and the Natural Science Foundation of Anhui Province of China (Grant No. 1908085QA29).

Conflict of interest

The authors declare no conflict of interest in this paper.

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