AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4931–4945.
DOI:10.3934/math.2020315
Received: 29 March 2020
Accepted: 27 May 2020
Published: 04 June 2020

Research article

Refinements of Jensen's and McShane's inequalities with applications

Muhammad Adil Khan[1], **Josip Pečarić**[2] **and Yu-Ming Chu**[3,4,*]

[1] Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
[2] Russian Peoples' Friendship University, 117198 Moscow, Russia
[3] Department of Mathematics, Huzhou University, Huzhou 313000, P. R. China
[4] Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha 410114, P. R. China

* **Correspondence:** Email: chuyuming2005@126.com; Tel: +865722322189; Fax: +865722321163.

Abstract: In this article, we consider the generalized forms of the Jensen's inequality given by Jessen and McShane for isotonic linear functionals, derive several refinements for the Jessen's and McShane's inequalities connected to certain functions from the linear space, generalize the Jessen's and McShane's inequalities pertaining n certain functions. As applications, we provide some improvements for the generalized means, Hölder and generalized Beck's inequalities.

Keywords: Jensen's inequality; convex function; linear functional; McShane's inequality; Hölder inequality; Beck's inequality
Mathematics Subject Classification: 26B25, 26D15

1. Introduction

It is well-known that the Jensen inequality [1–3] for convex function is one of the most famous and important inequalities in the whole theory of inequalities. Recently, the generalizations and improvements for the Jensen inequality have been the subject of much research, it has been generalized to the s-convex [4], co-ordinate convex [5], ϕ-convex [6], $\alpha(x)$-convex [7] and strongly convex functions [8]. It is worth noting that it is closely related to many other important inequalities such as Cauchy-Schwarz inequalities [9], Ostrowski inequalities [10], Minkowski inequalities [11], Hermite-Hadamard inequalities [12–17], Bessel function inequalities [18], Petrović inequalities [19], Pólya-Szegö inequalities [20], exponentially convex inequalities [21], integral inequalities [22–27], mean value inequalities [28–32], gamma function inequalities [33], generalized convex functions inequalities [34], generalized trigonometric functions inequalities [35] and so on. The Jensen inequality can be stated as follows.

The inequality

$$\psi\left(\frac{\sum_{j=1}^{n}\zeta_j y_j}{\sum_{j=1}^{n}\zeta_j}\right) \le \frac{\sum_{j=1}^{n}\zeta_j\psi(y_j)}{\sum_{j=1}^{n}\zeta_j} \tag{1.1}$$

holds for all $y_j \in I$ and $\zeta_j > 0$ ($j = 1, 2, \ldots, n$) if $\psi : I \to \mathbb{R}$ is a convex function.

The Jensen inequality (1.1) has wild applications in many fields of natural sciences, for example, in optimization theory, statistics, information theory and financial economics [36–38].

The main focus of this article is to present several refinements of the generalized Jensen's inequality for the isotonic linear functionals. Before giving Jessen's and McShane's results, we consider the following hypothesis and recall a definition.

Hypothesis **H**: Suppose that $\mathcal{M}$ is a non empty set and $\mathcal{L}$ is the class of real-valued functions $f : \mathcal{M} \to \mathbb{R}$ such that

(i) $\alpha_1 g_1 + \alpha_2 g_2 \in \mathcal{L}$ if $g_1, g_2 \in \mathcal{L}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$;
(ii) $g \in \mathcal{L}$ if $g(z) = 1$ for all $z \in \mathcal{M}$.

Definition 1.1. The functional $\mathcal{G} : \mathcal{L} \to \mathbb{R}$ is said to be an isotonic linear functional if it satisfies the following two conditions:

(i) $\mathcal{G}(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1\mathcal{G}(g_1) + \alpha_2\mathcal{G}(g_2)$ for $g_1, g_2 \in \mathcal{L}, \alpha_1, \alpha_2 \in \mathbb{R}$;
(ii) $g \in \mathcal{L}, g(t) \ge 0$ on $\mathcal{M} \Rightarrow \mathcal{G}(g) \ge 0$.

In 1931, Jessen [39] constructed the functional version of the Jensen's inequality for convex functions with one variable. In the following Theorem 1.1, we present an weighted version of the Jessen's inequality.

Theorem 1.1. Let the hypothesis **H** be true, $\mathcal{G} : \mathcal{L} \to \mathbb{R}$ be an isotonic linear functional and $\psi : [a, b] \to \mathbb{R}$ be a continuous convex function. Then for all $f, \vartheta \in \mathcal{L}$ such that $\vartheta\psi(f), \vartheta f \in \mathcal{L}$ and $\mathcal{G}(\vartheta) > 0$, we have $\frac{\mathcal{G}(\vartheta f)}{\mathcal{G}(\vartheta)} \in [a, b]$ and

$$\psi\left(\frac{\mathcal{G}(\vartheta f)}{\mathcal{G}(\vartheta)}\right) \le \frac{\mathcal{G}(\vartheta\psi(f))}{\mathcal{G}(\vartheta)}. \tag{1.2}$$

Inequality (1.2) has been applied to obtain the monotonicity of generalized means. We present those results for means as follows.

Let $r \in \mathbb{R}$. Then the generalization of the classical mean $M_r(\vartheta, f; \mathcal{G})$ for isotonic functional $\mathcal{G}$ is defined by

$$M_r(\vartheta, f; \mathcal{G}) = \begin{cases} \left(\frac{\mathcal{G}(\vartheta f^r)}{\mathcal{G}(\vartheta)}\right)^{\frac{1}{r}} & r \ne 0, \\ \exp\left(\frac{\mathcal{G}(\vartheta \log f)}{\mathcal{G}(\vartheta)}\right) & r = 0, \end{cases} \tag{1.3}$$

where $f(x) > 0$ for $x \in \mathcal{M}$, $\vartheta, \vartheta f^r \in \mathcal{L}$ for $r \in \mathbb{R}$, $\vartheta \log f \in \mathcal{L}$ and $\mathcal{G}(\vartheta) > 0$.

The following Theorem 1.2 for the monotonicity of the above generalized mean can be found in the literature [40].

Theorem 1.2. Let the hypothesis **H** be true, f, ϑ be the functions defined on $\mathcal{M}$ such that $f, \vartheta, f^r, \vartheta f^r \in \mathcal{L}$ ($r \in \mathbb{R}$) and $f(x) > 0$ for $x \in \mathcal{M}$, and $\mathcal{G}$ be an isotonic linear functional defined on $\mathcal{L}$ such that $\mathcal{G}(\vartheta) > 0$. Then the inequality

$$M_l(\vartheta, f; \mathcal{G}) \geq M_p(\vartheta, f; \mathcal{G}) \tag{1.4}$$

holds for all $p \leq l$.

Let $\vartheta, h : [a, b] \to \mathbb{R}$ be the functions such that h is strictly monotone and continuous, and $\vartheta h(f) \in \mathcal{L}$ for $f \in \mathcal{L}$ with $f(x) \in [a, b]$ and $\mathcal{G}(\vartheta) > 0$. Then the generalized quasi arithmetic mean is defined by

$$M_h(\vartheta, f; \mathcal{G}) = h^{-1}\left(\frac{\mathcal{G}(\vartheta h(f))}{\mathcal{G}(\vartheta)}\right). \tag{1.5}$$

The following Theorem 1.3 on the monotonicity of the generalized quasi arithmetic mean is given in [40].

Theorem 1.3. Let the above hypotheses hold and $g : [a, b] \to \mathbb{R}$ be a strictly monotone and continuous function such that $\vartheta g(f) \in \mathcal{L}$ for $f \in \mathcal{L}$ with $f(x) \in [a, b]$ and $\mathcal{G}(\vartheta) > 0$. Then one has

$$M_g(\vartheta, f; \mathcal{G}) \geq M_h(\vartheta, f; \mathcal{G}). \tag{1.6}$$

if $g \circ h^{-1}$ is a convex function.

In 1937, McShane [41] extended the above functional version of Jensen's inequality from convex functions with one variable to the convex functions with several variables. The following Theorem 1.4 is a weighted version of McShane's result.

Theorem 1.4. Let the hypothesis **H** be true, $\mathcal{G} : \mathcal{L} \to \mathbb{R}$ be an isotonic linear functional, C be a convex closed subset of $\mathbb{R}^n$, ψ be a convex and continuous function defined on C, $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \vartheta(x)$ be the functions from $\mathcal{L}$ such that $\boldsymbol{\phi}(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_n(x)) \in C$ for all $x \in \mathcal{M}$, $\vartheta\psi(\boldsymbol{\phi}(x)), \vartheta\phi_i \in \mathcal{L}$ ($i = 1, 2, \ldots, n$) and $\mathcal{G}(\vartheta) > 0$. Then one has

$$\psi\left(\frac{\mathcal{G}(\vartheta\phi_1)}{\mathcal{G}(\vartheta)}, \frac{\mathcal{G}(\vartheta\phi_2)}{\mathcal{G}(\vartheta)}, \ldots, \frac{\mathcal{G}(\vartheta\phi_n)}{\mathcal{G}(\vartheta)}\right) \leq \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(\vartheta\psi(\phi_1, \phi_2, \ldots, \phi_n)). \tag{1.7}$$

The following generalization of Beck's inequality can be found in the literature [40] by using McShane's inequality.

Theorem 1.5. Let the hypothesis **H** be valid, $\mathcal{G} : \mathcal{L} \to \mathbb{R}$ be an isotonic linear functional, $\psi_i : I_i \to \mathbb{R}$ ($i = 1, 2, \ldots, n$) be continuous and strictly monotonic, $\tau : I \to \mathbb{R}$ be continuous and increasing, and $g_1, g_2, \ldots, g_n : \mathcal{M} \to \mathbb{R}$ and $\psi : I_1 \times I_2 \times \cdots \times I_n \to \mathbb{R}$ be the real-valued functions such that $g_1(\mathcal{M}) \subset I_1$, $g_2(\mathcal{M}) \subset I_2, \ldots, g_n(\mathcal{M}) \subset I_n$, $\psi_1(g_1), \psi_2(g_2), \ldots, \psi_n(g_n), \tau(\psi(g_1, g_2, \ldots, g_n)), \vartheta \in \mathcal{L}$ and $\mathcal{G}(\vartheta) > 0$. Then the inequality

$$\psi\left(M_{\psi_1}(\vartheta, g_1; \mathcal{G}), M_{\psi_2}(\vartheta, g_2; \mathcal{G}), \ldots, M_{\psi_n}(\vartheta, g_n; \mathcal{G})\right) \geq M_\tau(\vartheta, \psi(g_1, g_2, \ldots, g_n); \mathcal{G}) \tag{1.8}$$

holds if the function H defined by $H(s_1, s_2, \ldots, s_n) = -\tau(\psi(\psi_1^{-1}(s_1), \psi_2^{-1}(s_2), \ldots, \psi_n^{-1}(s_n))$ is convex.

Remark 1.1. It is important to note that Beck [42] gave the special case of Theorem 1.5 for discrete functionals with $n = 2$.

The main purpose of the article is to refine the Jessen's and McShane's inequalities associated to certain functions from the linear functions space, improve the generalized means, Hölder and McShane's inequalities, and generalize the Jessen's and McShane's inequalities containing n certain functions.

2. Refinement of Jessen's inequality with applications

We first present a refinement of the Jessen's inequality.

Theorem 2.1. Under the assumptions of Theorem 1.1, if $u, v \in \mathcal{L}$ such that $u(t) + v(t) = 1$ for $t \in \mathcal{M}$ and $u\vartheta f, v\vartheta f, u\vartheta, v\vartheta \in \mathcal{L}$ with $\mathcal{G}(u\vartheta), \mathcal{G}(v\vartheta) > 0$, then

$$\psi\left(\frac{\mathcal{G}(\vartheta f)}{\mathcal{G}(\vartheta)}\right) \leq \frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(u\vartheta f)}{\mathcal{G}(u\vartheta)}\right) + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(v\vartheta f)}{\mathcal{G}(v\vartheta)}\right) \leq \frac{\mathcal{G}(\vartheta\psi(f))}{\mathcal{G}(\vartheta)}. \tag{2.1}$$

Proof. It follows from $u(t) + v(t) = 1$ for $t \in \mathcal{M}$ and the linearity of $\mathcal{G}$ that

$$\begin{aligned}
\psi\left(\frac{\mathcal{G}(\vartheta f)}{\mathcal{G}(\vartheta)}\right) &= \psi\left(\frac{\mathcal{G}((u+v)\vartheta f)}{\mathcal{G}(\vartheta)}\right) \\
&= \psi\left(\frac{\mathcal{G}(u\vartheta f)}{\mathcal{G}(\vartheta)} + \frac{\mathcal{G}(v\vartheta f)}{\mathcal{G}(\vartheta)}\right) \\
&= \psi\left(\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}(u\vartheta f)}{\mathcal{G}(u\vartheta)} + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}(v\vartheta f)}{\mathcal{G}(v\vartheta)}\right)
\end{aligned} \tag{2.2}$$

and

$$\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)} + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)} = \frac{\mathcal{G}(u\vartheta + v\vartheta)}{\mathcal{G}(\vartheta)} = \frac{\mathcal{G}(\vartheta)}{\mathcal{G}(\vartheta)} = 1.$$

Making use of the convexity of ψ on the right hand side of inequality (2.2) we obtain

$$\psi\left(\frac{\mathcal{G}(\vartheta f)}{\mathcal{G}(\vartheta)}\right) \leq \frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(u\vartheta f)}{\mathcal{G}(u\vartheta)}\right) + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(v\vartheta f)}{\mathcal{G}(v\vartheta)}\right) \tag{2.3}$$

Applying Jessen's inequality (1.2) on both sides of inequality (2.3), we have

$$\begin{aligned}
&\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(u\vartheta f)}{\mathcal{G}(u\vartheta)}\right) + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(v\vartheta f)}{\mathcal{G}(v\vartheta)}\right) \\
&\leq \frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}(u\vartheta\psi(f))}{\mathcal{G}(u\vartheta)} + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}(v\vartheta\psi(f))}{\mathcal{G}(v\vartheta)} \\
&= \frac{\mathcal{G}(u\vartheta\psi(f))}{\mathcal{G}(\vartheta)} + \frac{\mathcal{G}(v\vartheta\psi(f))}{\mathcal{G}(\vartheta)} = \frac{\mathcal{G}(\vartheta\psi(f))}{\mathcal{G}(\vartheta)}.
\end{aligned} \tag{2.4}$$

Therefore, inequality (2.1) follows from inequalities (2.3) and (2.4). □

From Theorem 2.1 we get Corollary 2.1, which is the refinement of inequality (1.4).

Corollary 2.1. Let the hypothesis **H** be valid, f, ϑ, u, v be the functions defined on $\mathcal{M}$ such that $f, \vartheta, u, v, u\vartheta, v\vartheta, u\vartheta f^r, v\vartheta f^r \vartheta f^r \in \mathcal{L}$ $(r \in \mathbb{R})$ and $f(x) > 0$ for $x \in \mathcal{M}$, $\mathcal{G}$ be an isotonic linear functional on $\mathcal{L}$ such that $\mathcal{G}(\vartheta), \mathcal{G}(u\vartheta), \mathcal{G}(v\vartheta) > 0$, $u(x) + v(x) = 1$ for $x \in \mathcal{M}$, $p, l \in \mathbb{R}$ with $p \leq l$. Then one has

$$M_l(\vartheta, f; \mathcal{G}) \geq \Big[M_1(\vartheta, u; \mathcal{G}) M_p^l(u\vartheta, f; \mathcal{G}) + M_1(\vartheta, v; \mathcal{G}) M_p^l(v\vartheta, f; \mathcal{G})\Big]^{\frac{1}{l}} \geq M_p(\vartheta, f; \mathcal{G}) \tag{2.5}$$

for $l \neq 0$,

$$M_0(\vartheta, f; \mathcal{G}) \geq \exp\Big(M_1(\vartheta, u; \mathcal{G}) \log M_p(u\vartheta, f; \mathcal{G}) + M_1(\vartheta, v; \mathcal{G}) \log M_p(v\vartheta, f; \mathcal{G})\Big) \geq M_p(\vartheta, f; \mathcal{G}) \tag{2.6}$$

for $p \leq 0$,

$$M_p(\vartheta, f; \mathcal{G}) \leq \Big[M_1(\vartheta, u; \mathcal{G}) M_l^p(u\vartheta, f; \mathcal{G}) + M_1(\vartheta, v; \mathcal{G}) M_l^p(v\vartheta, f; \mathcal{G})\Big]^{\frac{1}{p}} \leq M_l(\vartheta, f; \mathcal{G}) \tag{2.7}$$

for $p \neq 0$, and

$$M_0(\vartheta, f; \mathcal{G}) \leq \exp\Big(M_1(\vartheta, u; \mathcal{G}) \log M_l(u\vartheta, f; \mathcal{G}) + M_1(\vartheta, v; \mathcal{G}) \log M_l(v\vartheta, f; \mathcal{G})\Big) \leq M_l(\vartheta, f; \mathcal{G}) \tag{2.8}$$

for $l \geq 0$.

Proof. Let $p, l \neq 0$. Then using (2.1) for $\psi(z) = z^{\frac{l}{p}}$ $(z > 0)$, $f \to f^p$ and taking the power $\frac{1}{l}$ we get inequality (2.5).

Next, we prove inequality (2.6) by taking limit $l \to 0$ in (2.5). Let

$$B(l, p, \vartheta, u, v, f; \mathcal{G}) = M_1(\vartheta, u; \mathcal{G}) M_p^l(u\vartheta, f; \mathcal{G}) + M_1(\vartheta, v; \mathcal{G}) M_p^l(v\vartheta, f; \mathcal{G}).$$

Then it follows from linearity of $\mathcal{G}$ and $u + v = 1$ that

$$B(0, p, \vartheta, u, v, f; \mathcal{G}) = M_1(\vartheta, u; \mathcal{G}) + M_1(\vartheta, v; \mathcal{G}) = \frac{\mathcal{G}(\vartheta u)}{\mathcal{G}(\vartheta)} + \frac{\mathcal{G}(\vartheta v)}{\mathcal{G}(\vartheta)}$$

$$= \frac{\mathcal{G}(\vartheta u) + \mathcal{G}(\vartheta v)}{\mathcal{G}(\vartheta)} = \frac{\mathcal{G}(\vartheta(u + v))}{\mathcal{G}(\vartheta)} = 1.$$

Let

$$K(l, B) = (B(l, p, \vartheta, u, v, f; \mathcal{G}))^{\frac{1}{l}}.$$

Then

$$\log(K(l, B)) = \frac{\log B(l, p, \vartheta, u, v, f; \mathcal{G})}{l}.$$

Taking $l \to 0$ and using l'Hôpital rule we obtain

$$\lim_{l \to 0} \log(K(l, B)) = M_1(\vartheta, u; \mathcal{G}) \log M_p(u\vartheta, f; \mathcal{G}) + M_1(\vartheta, v; \mathcal{G}) \log M_p(v\vartheta, f; \mathcal{G}),$$

that is

$$\lim_{l\to 0} K(l,B) = \exp\Big(M_1(\vartheta,u;\mathcal{G})\log M_p(u\vartheta,f;\mathcal{G}) + M_1(\vartheta,v;\mathcal{G})\log M_p(v\vartheta,f;\mathcal{G})\Big). \tag{2.9}$$

Taking $l \to 0$ in (2.5) and using (2.9) we obtain the desired inequality (2.6).

Similarly utilizing inequality (2.1) for $\psi(z) = z^{\frac{p}{l}}$, $(z > 0, p, l \neq 0)$, $f \to f^l$ and taking power $\frac{1}{p}$ we get (2.7). Taking $p \to 0$ in (2.7) leads to (2.8) □

As an application of Theorem 2.1, we derive an refinement of inequality inequality (1.5).

Corollary 2.2. Let all the above hypotheses hold, $g : [a,b] \to \mathbb{R}$ be a strictly monotone and continuous function such that $\vartheta g(f) \in \mathcal{L}$ for $f \in \mathcal{L}$ with $f(x) \in [a,b]$, $u, v \in \mathcal{L}$ such that $u(x) + v(x) = 1$ for $x \in \mathcal{M}$ and $\mathcal{G}(\vartheta), \mathcal{G}(u\vartheta), \mathcal{G}(v\vartheta) > 0$. Then the inequality

$$\begin{aligned} M_g(\vartheta,f;\mathcal{G}) &\geq g^{-1}\Big(M_1(\vartheta,u;\mathcal{G})g(M_h(\vartheta u,f;\mathcal{G})) \\ &+ M_1(\vartheta,v;\mathcal{G})g(M_h(\vartheta v,f;\mathcal{G}))\Big) \geq M_h(\vartheta,f;\mathcal{G}) \end{aligned} \tag{2.10}$$

holds if $g \circ h^{-1}$ is a convex function.

Proof. Inequality (2.10) can be derived by use of inequality (2.1) for $f \to h \circ f$ and $\psi \to g \circ h^{-1}$. □

As applications of Theorem 2.1, we present the refinements of the Hölder inequality in the following Corollaries 2.3 and 2.4.

Corollary 2.3. Let the hypothesis **H** be valid, $\mathcal{G} : \mathcal{L} \to \mathbb{R}$ be an isotonic linear functional, $r_1 > 1$, $r_2 = \frac{r_1}{r_1-1}$, u, v, w, g_1 and g_2 be non-negative functions defined on $\mathcal{M}$ such that $wg_1^{r_1}, wg_2^{r_2}, uwg_2^{r_2}, vwg_2^{r_2}, uwg_1g_2, vwg_1g_2, wg_1g_2 \in \mathcal{L}$ and $u(x) + v(x) = 1$ for $x \in \mathcal{M}$. Then

$$\begin{aligned} \mathcal{G}(wg_1g_2) &\leq \mathcal{G}^{\frac{1}{r_2}}(wg_2^{r_2})\Big\{\Big(\mathcal{G}(uwg_2^{r_2})\Big)^{1-r_1}(\mathcal{G}(uwg_1g_2))^{r_1} \\ &+ \Big(\mathcal{G}(vwg_2^{r_2})\Big)^{1-r_1}(\mathcal{G}(vwg_1g_2))^{r_1}\Big\}^{\frac{1}{r_1}} \\ &\leq \mathcal{G}^{\frac{1}{r_1}}(wg_1^{r_1})\mathcal{G}^{\frac{1}{r_2}}(wg_2^{r_2}). \end{aligned} \tag{2.11}$$

In the case of $0 < r_1 < 1$ and $r_2 = \frac{r_1}{r_1-1}$ with $\mathcal{G}(wg_2^{r_2}) > 0$ or $r_1 < 0$ and $\mathcal{G}(wg_1^{r_1}) > 0$, we have

$$\begin{aligned} \mathcal{G}(wg_1g_2) &\geq \mathcal{G}^{\frac{1}{r_2}}(uwg_2^{r_2})\mathcal{G}^{\frac{1}{r_1}}(uwg_1^{r_1}) + \mathcal{G}^{\frac{1}{r_2}}(vwg_2^{r_2})\mathcal{G}^{\frac{1}{r_1}}(vwg_1^{r_1}) \\ &\geq \mathcal{G}^{\frac{1}{r_1}}(wg_1^{r_1})\mathcal{G}^{\frac{1}{r_2}}(wg_2^{r_2}). \end{aligned} \tag{2.12}$$

Proof. Assume that $\mathcal{G}(wg_2^{r_2}) > 0$. Since $wg_2^{r_2}g_1g_2^{\frac{-r_2}{r_1}} = wg_1g_2 \in \mathcal{L}$ and $wg_2^{r_2}g_1^{r_1}g_2^{-r_2} = wg_1^{r_1} \in \mathcal{L}$, therefore by using Theorem 2.1 for $\psi(z) = z^{r_1}$ $(z > 0, r_1 > 1)$, $\vartheta = wg_2^{r_2}$, $f = g_1g_2^{\frac{-r_2}{r_1}}$, we obtain inequality (2.11). If $\mathcal{G}(wg_1^{r_1}) > 0$, then applying the same procedure but taking r_1, r_2, g_1, g_2 instead of r_2, r_1, g_2, g_1, we also obtain inequality (2.11).

If $\mathcal{G}(wg_2^{r_2}) = \mathcal{G}(wg_1^{r_1}) = 0$, then as we know that

$$0 \leq wg_1g_2 \leq \frac{1}{r_1}wg_1^{r_1} + \frac{1}{r_2}wg_2^{r_2}. \tag{2.13}$$

It gives that $\mathcal{G}(wg_1g_2) = 0$. The proof for the case $r_1 > 1$ is completed.

If $0 < r_1 < 1$, then $M = \frac{1}{r_1} > 1$, and the desired result can be obtained by using (2.11) for M, $N = (1 - r_1)^{-1}$, $\overline{g}_1 = (g_1g_2)^{r_1}$, and $\overline{g}_2 = g_2^{-r_1}$ instead of r_1, r_2, g_1 and g_2,

Finally, if $r_1 < 0$, then $0 < r_2 < 1$, we may use the similar arguments with r_1, r_2, g_1, g_2 replaced by r_2, r_1, g_2, g_1 provided that $\mathcal{G}(wg_1^{r_1}) > 0$ to get the desired result. □

Corollary 2.4. Let the hypothesis **H** be true, $\mathcal{G} : \mathcal{L} \to \mathbb{R}$ be an isotonic linear functional, $r_1 > 1$, $r_2 = \frac{r_1}{r_1-1}$, u, v, w, g_1 and g_2 be non-negative functions defined on $\mathcal{M}$ such that $wg_1^{r_1}, wg_2^{r_2}, uwg_2^{r_2}, vwg_2^{r_2}, uwg_1g_2, vwg_1g_2, wg_1g_2 \in \mathcal{L}$ and $u(x) + v(x) = 1$ for $x \in \mathcal{M}$. Then

$$\begin{aligned}\mathcal{G}(wg_1g_2) &\leq \mathcal{G}^{\frac{1}{r_1}}(uwg_1^{r_1})\mathcal{G}^{\frac{1}{r_2}}(uwg_2^{r_2}) + \mathcal{G}^{\frac{1}{r_1}}(vwg_1^{r_1})\mathcal{G}^{\frac{1}{r_2}}(vwg_2^{r_2}) \\ &\leq \mathcal{G}^{\frac{1}{r_1}}(wg_1^{r_1})\mathcal{G}^{\frac{1}{r_2}}(wg_2^{r_2}). \end{aligned} \tag{2.14}$$

In the case of $0 < r_1 < 1$ and $r_2 = \frac{r_1}{r_1-1}$ with $\mathcal{G}(wg_2^{r_2}) > 0$ or $r_1 < 0$ and $\mathcal{G}(wg_1^{r_1}) > 0$, we get

$$\begin{aligned}&\mathcal{G}^{\frac{1}{r_1}}(wg_1^{r_1})\mathcal{G}^{\frac{1}{r_2}}(wg_2^{r_2}) \\ &\leq \mathcal{G}^{\frac{1}{r_2}}(wg_2^{r_2})\left\{\left(\mathcal{G}(uwg_2^{r_2})\right)^{1-r_1}(\mathcal{G}(uwg_1g_2))^{r_1} \right. \\ &\left. +\left(\mathcal{G}(vwg_2^{r_2})\right)^{1-r_1}(vwg_1g_2))^{r_1}\right\}^{\frac{1}{r_1}} \leq \mathcal{G}(wg_1g_2). \end{aligned} \tag{2.15}$$

Proof. If $\mathcal{G}(wg_2^{r_2}) > 0$, then let $\psi(z) = -z^{\frac{1}{r_1}}$ $(z > 0, r_1 > 1)$, we clearly see that the function ψ is convex. Therefore, using Theorem 2.1 for $\psi(z) = -z^{\frac{1}{r_1}}$, $\vartheta = wg_2^{r_2}$ and $f = g_1^{r_1}g_2^{-r_2}$ we obtain (2.14). If $\mathcal{G}(wg_1^{r_1}) > 0$, then applying the same procedure but taking r_1, r_2, g_1 and g_2 instead of r_2, r_1, g_2 and g_1 we also get (2.14).

For the case of $\mathcal{G}(wg_2^{r_2}) = \mathcal{G}(wg_1^{r_1}) = 0$, we can complete the proof by use of the similar argument as in the proof of Corollary 2.3.

If $0 < r_1 < 1$, then $M = \frac{1}{r_1} > 1$ and applying (2.14) for M and $N = (1 - r_1)^{-1}$, $\overline{g}_1 = (g_1g_2)^{r_1}$ and $\overline{g}_2 = g_2^{-r_1}$ instead of r_1, r_2, g_1 and g_2 we get (2.15).

Finally, if $r_1 < 0$, then $0 < r_2 < 1$, and we can get the desired result if we use the same method as above but instead r_1, r_2, g_1 and g_2 by r_2, r_1, g_2 and g_1 provided that $\int_a^b w(\varrho)g_1^{r_1}(\varrho)d\varrho > 0$. □

Remark 2.1. In the above results if we consider $\mathcal{G}(\vartheta g) = \int_a^b p(x)f(x)dx$ and $\mathcal{G}(\vartheta) = \int_a^b p(x)dx$, then we obtain the results presented in [43], where $p, g : [a, b] \to \mathbb{R}$ are integrable functions with $p(x) > 0$ for $x \in [a, b]$.

3. Refinement of McShane's inequality with applications

We begin this section by giving a refinement of the McShane's inequality.

Theorem 3.1. Under the assumptions of Theorem 1.4, if $u, v \in \mathcal{L}$ such that $u(x) + v(x) = 1$ for $x \in \mathcal{M}$ and $\vartheta\psi(\boldsymbol{\phi}(x)), u\vartheta\phi_i, v\vartheta\phi_i, u\vartheta, v\vartheta \in \mathcal{L}$ $(i = 1, 2, \ldots, n)$ with $\mathcal{G}(u\vartheta), \mathcal{G}(v\vartheta) > 0$, then

$$\begin{aligned}
&\psi\left(\frac{\mathcal{G}(\vartheta\phi_1)}{\mathcal{G}(\vartheta)}, \frac{\mathcal{G}(\vartheta\phi_2)}{\mathcal{G}(\vartheta)}, \cdots, \frac{\mathcal{G}(\vartheta\phi_n)}{\mathcal{G}(\vartheta)}\right) \\
&\leq \frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(u\vartheta\phi_1)}{\mathcal{G}(u\vartheta)}, \cdots, \frac{\mathcal{G}(u\vartheta\phi_n)}{\mathcal{G}(u\vartheta)}\right) + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(v\vartheta\phi_1)}{\mathcal{G}(v\vartheta)}, \cdots, \frac{\mathcal{G}(v\vartheta\phi_n)}{\mathcal{G}(v\vartheta)}\right) \\
&\leq \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(\vartheta\psi(\phi_1, \phi_2, \cdots, \phi_n). \qquad (3.1)
\end{aligned}$$

Proof. It follows from the convexity, (1.7), $u(x) + v(x) = 1$ for $x \in \mathcal{M}$ and the linearity of $\mathcal{G}$ that

$$\begin{aligned}
&\psi\left(\frac{\mathcal{G}(\vartheta\phi_1)}{\mathcal{G}(\vartheta)}, \frac{\mathcal{G}(\vartheta\phi_2)}{\mathcal{G}(\vartheta)}, \cdots, \frac{\mathcal{G}(\vartheta\phi_n)}{\mathcal{G}(\vartheta)}\right) \\
&= \psi\left(\frac{\mathcal{G}(u\vartheta\phi_1)}{\mathcal{G}(\vartheta)} + \frac{\mathcal{G}(v\vartheta\phi_1)}{\mathcal{G}(\vartheta)}, \cdots, \frac{\mathcal{G}(u\vartheta\phi_n)}{\mathcal{G}(\vartheta)} + \frac{\mathcal{G}(v\vartheta\phi_n)}{\mathcal{G}(\vartheta)}\right) \\
&= \psi\left(\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}(u\vartheta\phi_1)}{\mathcal{G}(u\vartheta)} + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}(v\vartheta\phi_1)}{\mathcal{G}(v\vartheta)}, \cdots, \frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}(u\vartheta\phi_n)}{\mathcal{G}(u\vartheta)} + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}(v\vartheta\phi_n)}{\mathcal{G}(v\vartheta)}\right) \\
&= \psi\left(\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\left(\frac{\mathcal{G}(u\vartheta\phi_1)}{\mathcal{G}(u\vartheta)}, \cdots, \frac{\mathcal{G}(u\vartheta\phi_n)}{\mathcal{G}(u\vartheta)}\right) + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\left(\frac{\mathcal{G}(v\vartheta\phi_1)}{\mathcal{G}(v\vartheta)}, \cdots, \frac{\mathcal{G}(v\vartheta\phi_n)}{\mathcal{G}(v\vartheta)}\right)\right) \\
&\leq \frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(u\vartheta\phi_1)}{\mathcal{G}(u\vartheta)}, \cdots, \frac{\mathcal{G}(u\vartheta\phi_n)}{\mathcal{G}(u\vartheta)}\right) + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}(v\vartheta\phi_1)}{\mathcal{G}(v\vartheta)}, \cdots, \frac{\mathcal{G}(v\vartheta\phi_n)}{\mathcal{G}(v\vartheta)}\right) \\
&\leq \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(u\vartheta\psi(\phi_1, \phi_1, \cdots, \phi_n) + \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(v\vartheta\psi(\phi_1, \phi_1, \cdots, \phi_n) \\
&= \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(\vartheta\psi(\phi_1, \phi_2, \cdots, \phi_n).
\end{aligned}$$

□

The following Theorem 3.2 provides a refinement of the generalized Beck's inequality (1.8).

Theorem 3.2. Let the hypothesis **H** be valid, $\mathcal{G} : \mathcal{L} \to \mathbb{R}$ be an isotonic linear functional, $\psi_i : I_i \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ be continuous and strictly monotonic, $\tau : I \to \mathbb{R}$ be a continuous and increasing function, $g_1, g_2, \cdots, g_n : \mathcal{M} \to \mathbb{R}$, $\psi : I_1 \times I_2 \times \cdots \times I_n \to \mathbb{R}$ be the real-valued functions such that $g_1(\mathcal{M}) \subset I_1$, $g_2(\mathcal{M}) \subset I_2$, $\cdots$, $g_n(\mathcal{M}) \subset I_n$, $\psi_1(g_1)$, $\psi_2(g_2)$, $\cdots$, $\psi_n(g_n)$, $\tau(\psi(g_1, g_2, \ldots, g_n))$, $u, v, \vartheta, u\vartheta, v\vartheta \in \mathcal{L}$, $u(x) + v(x) = 1$ for $x \in \mathcal{M}$ and $\mathcal{G}(\vartheta), \mathcal{G}(u\vartheta), \mathcal{G}(v\vartheta) > 0$. Then

$$\psi\left(M_{\psi_1}(\vartheta, g_1; \mathcal{G}), M_{\psi_2}(\vartheta, g_2; \mathcal{G}), \ldots, M_{\psi_n}(\vartheta, g_n; \mathcal{G})\right)$$

$$\geq \tau^{-1}\Bigg[\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(u\vartheta, g_1;\mathcal{G}),\cdots,M_{\psi_n}(u\vartheta, g_n;\mathcal{G})\right)\right)$$
$$+\frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(v\vartheta, g_1;\mathcal{G}),\cdots,M_{\psi_n}(v\vartheta, g_n;\mathcal{G})\right)\right)\Bigg]$$
$$\geq M_\tau(\vartheta,\psi(g_1,g_2,\ldots,g_n);\mathcal{G}) \tag{3.2}$$

if the function H defined by $H(s_1,s_2,\cdots,s_n)=-\tau(\psi(\psi_1^{-1}(s_1),\psi_2^{-1}(s_2),\cdots,\psi_n^{-1}(s_n))$ is convex.

Proof. Applying Theorem 3.2 for the function H instead of ψ, we obtain

$$\tau\left(\psi\left(\psi_1^{-1}\left(\frac{\mathcal{G}(\vartheta\phi_1)}{\mathcal{G}(\vartheta)}\right),\psi_2^{-1}\left(\frac{\mathcal{G}(\vartheta\phi_2)}{\mathcal{G}(\vartheta)}\right),\ldots,\psi_n^{-1}\left(\frac{\mathcal{G}(\vartheta\phi_n)}{\mathcal{G}(\vartheta)}\right)\right)\right)$$
$$\geq\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(\psi\left(\psi_1^{-1}\left(\frac{\mathcal{G}(u\vartheta\phi_1)}{\mathcal{G}(u\vartheta)}\right),\ldots,\psi_n^{-1}\left(\frac{\mathcal{G}(u\vartheta\phi_n)}{\mathcal{G}(u\vartheta)}\right)\right)\right)$$
$$+\frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(\psi\left(\psi_1^{-1}\left(\frac{\mathcal{G}(v\vartheta\phi_1)}{\mathcal{G}(v\vartheta)}\right),\ldots,\psi_n^{-1}\left(\frac{\mathcal{G}(v\vartheta\phi_n)}{\mathcal{G}(v\vartheta)}\right)\right)\right)$$
$$\geq\frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(\vartheta\tau(\psi(\psi_1^{-1}(\phi_1),\psi_2^{-1}(\phi_2),\cdots,\psi_n^{-1}(\phi_n))). \tag{3.3}$$

Let $\phi_i=\psi_i(g_i)$ $(i=1,2,\ldots,n)$. Then (3.3) becomes

$$\tau\left(\psi\left(M_{\psi_1}(\vartheta,g_1;\mathcal{G}),M_{\psi_2}(\vartheta,g_2;\mathcal{G}),\cdots,M_{\psi_n}(\vartheta,g_n;\mathcal{G})\right)\right)$$
$$\geq\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(u\vartheta,g_1;\mathcal{G}),\cdots,M_{\psi_n}(u\vartheta,g_n;\mathcal{G})\right)\right)$$
$$+\frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(v\vartheta,g_1;\mathcal{G}),\cdots,M_{\psi_n}(v\vartheta,g_n;\mathcal{G})\right)\right)$$
$$\geq\frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(\vartheta\tau(\psi(g_1,g_2,\ldots,g_n))), \tag{3.4}$$

which is equivalent to (3.2). □

A refinement of the Beck's inequality is given in the following Corollary 3.1.

Corollary 3.1. Under the assumptions of Theorem 3.2 for $n=2$, we have the following inequalities

$$\psi\left(M_{\psi_1}(\vartheta,g_1;\mathcal{G}),M_{\psi_2}(\vartheta,g_2;\mathcal{G})\right)$$
$$\geq\tau^{-1}\Bigg[\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(u\vartheta,g_1;\mathcal{G}),M_{\psi_2}(u\vartheta,g_2;\mathcal{G})\right)\right)$$
$$+\frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(\psi\left(M_{\psi_1}(v\vartheta,g_1;\mathcal{G}),M_{\psi_2}(v\vartheta,g_2;\mathcal{G})\right)\right)\Bigg]$$
$$\geq M_\tau(\vartheta,\psi(g_1,g_2);\mathcal{G}) \tag{3.5}$$

if the function H defined by $H(s_1,s_2)=-\tau(\psi(\psi_1^{-1}(s_1),\psi_2^{-1}(s_2))$ is convex.

Next, we discuss some particular cases of Corollary 3.1.

Corollary 3.2. Let all the assumptions of Theorem 3.2 hold for $n = 2$ with $\psi(z_1, z_2) = z_1 + z_2$, and ψ_1, ψ_2, τ be the twice continuous differentiable functions such that $\psi_1', \psi_2', \tau', \psi_1'', \psi_2''$ and τ'' are positive. Then the inequality

$$\begin{aligned} &M_{\psi_1}(\vartheta, g_1; \mathcal{G}) + M_{\psi_2}(\vartheta, g_2; \mathcal{G}) \\ &\geq \tau^{-1}\Bigg[\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(M_{\psi_1}(u\vartheta, g_1; \mathcal{G}) + M_{\psi_2}(u\vartheta, g_2; \mathcal{G})\right) \\ &\quad + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(M_{\psi_1}(v\vartheta, g_1; \mathcal{G}) + M_{\psi_2}(v\vartheta, g_2; \mathcal{G})\right)\Bigg] \\ &\geq M_\tau(\vartheta, g_1 + g_2; \mathcal{G}) \end{aligned} \tag{3.6}$$

holds if $G(z_1) + H(z_2) \leq K(z_1 + z_2)$, where $G = \frac{\psi_1'}{\psi_1''}$, $H = \frac{\psi_2'}{\psi_2''}$ and $K = \frac{\tau'}{\tau''}$.

Proof. The idea of the proof is similar to the proof of Corollary 3.2 given in [44]. □

Similar to the idea of Corollary 3.2, we state the following Corollary 3.3.

Corollary 3.3. Let all the assumptions of Theorem 3.2 hold for $n = 2$ with $\psi(z_1, z_2) = z_1 z_2$, ψ_1, ψ_2, τ be the twice continuous differentiable functions and $L_1(z) = \frac{\psi_1'(z)}{\psi_1'(z)+z\psi_1''(z)}$, $L_2(z) = \frac{\psi_2'(z)}{\psi_2'(z)+z\psi_2''(z)}$, $L_3(z) = \frac{\tau'(z)}{\tau'(z)+z\tau''(z)}$ such that $\psi_1', \psi_2', \tau', \tau'', \psi_1'', \psi_2'', L_1, L_2, L_3$ are positive. Then the inequality

$$\begin{aligned} &M_{\psi_1}(\vartheta, g_1; \mathcal{G}) M_{\psi_2}(\vartheta, g_2; \mathcal{G}) \\ &\geq \tau^{-1}\Bigg[\frac{\mathcal{G}(u\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(M_{\psi_1}(u\vartheta, g_1; \mathcal{G}) M_{\psi_2}(u\vartheta, g_2; \mathcal{G})\right) \\ &\quad + \frac{\mathcal{G}(v\vartheta)}{\mathcal{G}(\vartheta)}\tau\left(M_{\psi_1}(v\vartheta, g_1; \mathcal{G}) M_{\psi_2}(v\vartheta, g_2; \mathcal{G})\right)\Bigg] \\ &\geq M_\tau(\vartheta, g_1 g_2; \mathcal{G}) \end{aligned} \tag{3.7}$$

holds if $L_1(z_1) + L_2(z_2) \leq L_3(z_1 z_2)$.

4. Further generalizations

The following Theorem 4.1 provides further generalization for the refinement of the Jessen's inequality associated to n certain functions.

Theorem 4.1. Let all the assumptions of Theorem 1.1 hold, $u_l \in \mathcal{L}$ such that $\sum_{l=1}^{n} u_l = 1$ and $u_l \vartheta f, u_l \vartheta \in \mathcal{L}$ with $\mathcal{G}(u_l \vartheta) > 0$ for all $l \in \{1, 2, \cdots, n\}$, and $\mathcal{S}_1, \mathcal{S}_2 \subset \{1, 2, \ldots, n\}$ such that $\mathcal{S}_1$ and $\mathcal{S}_2$ are non empty, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and $\mathcal{S}_1 \cup \mathcal{S}_2 = \{1, 2, \ldots, n\}$. Then one has

$$\psi\left(\frac{\mathcal{G}(\vartheta f)}{\mathcal{G}(\vartheta)}\right) \leq \frac{\mathcal{G}\left(\sum_{l \in \mathcal{S}_1} u_l \vartheta\right)}{\mathcal{G}(\vartheta)} \psi\left(\frac{\mathcal{G}\left(\sum_{l \in \mathcal{S}_1} u_l \vartheta f\right)}{\mathcal{G}\left(\sum_{l \in \mathcal{S}_1} u_l \vartheta\right)}\right)$$

$$+\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\right)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta f\right)}{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\right)}\right)\leq\frac{\mathcal{G}(\vartheta\psi(f))}{\mathcal{G}(\vartheta)}.$$

Proof. It follows from the linearity of $\mathcal{G}$, $\sum\limits_{l=1}^{n} u_l = 1$, Jessen's inequality and the definition of the convexity that

$$\psi\left(\frac{\mathcal{G}(\vartheta f)}{\mathcal{G}(\vartheta)}\right)=\psi\left(\frac{\mathcal{G}\left(\sum\limits_{l=1}^{n} u_l\vartheta f\right)}{\mathcal{G}(\vartheta)}\right)=\psi\left(\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta f+\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta f\right)}{\mathcal{G}(\vartheta)}\right)$$

$$=\psi\left(\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta f\right)+\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta f\right)}{\mathcal{G}(\vartheta)}\right)$$

$$=\psi\left(\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\right)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta f\right)}{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\right)}+\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\right)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta f\right)}{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\right)}\right)$$

and

$$\psi\left(\frac{\mathcal{G}(\vartheta f)}{\mathcal{G}(\vartheta)}\right)\leq\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\right)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta f\right)}{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\right)}\right)+\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\right)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta f\right)}{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\right)}\right)$$

$$\leq\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\right)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\psi(f)\right)}{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\right)}+\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\right)}{\mathcal{G}(\vartheta)}\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\psi(f)\right)}{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\right)}$$

$$=\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\psi(f)\right)}{\mathcal{G}(\vartheta)}+\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\psi(f)\right)}{\mathcal{G}(\vartheta)}$$

$$=\frac{\mathcal{G}\left(\sum\limits_{l\in\mathcal{S}_1} u_l\vartheta\psi(f)+\sum\limits_{l\in\mathcal{S}_2} u_l\vartheta\psi(f)\right)}{\mathcal{G}(\vartheta)}=\frac{\mathcal{G}(\vartheta\psi(f))}{\mathcal{G}(\vartheta)}.$$

□

Similar to the above Theorem 4.1, in the following Theorem 4.2 we give further generalization of the McShane's inequality.

Theorem 4.2. Let all the assumptions of Theorem 1.4 hold, $u_l \in \mathcal{L}$ such that $\sum_{l=1}^{n} u_l = 1$ and $u_l\vartheta\psi(\boldsymbol{\phi}(x))$, $v_l\vartheta\psi(\boldsymbol{\phi}(x))$, $u_l\vartheta\phi_l$, $u_l\vartheta \in \mathcal{L}$ with $\mathcal{G}(u_l\vartheta) > 0$ for all $l \in \{1, 2, \cdots, n\}$, and $\mathcal{S}_1$ and $\mathcal{S}_2$ are non empty and disjoint subsets of $\{1, 2, \cdots, n\}$ such that $\mathcal{S}_1 \cup \mathcal{S}_2 = \{1, 2, \cdots, n\}$. Then

$$\begin{aligned}
&\psi\left(\frac{\mathcal{G}(\vartheta\phi_1)}{\mathcal{G}(\vartheta)}, \frac{\mathcal{G}(\vartheta\phi_2)}{\mathcal{G}(\vartheta)}, \cdots, \frac{\mathcal{G}(\vartheta\phi_n)}{\mathcal{G}(\vartheta)}\right)\\
&\leq \frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_1)}{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}, \cdots, \frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_n)}{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}\right)\\
&+\frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_1)}{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}, \cdots, \frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_n)}{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}\right)\\
&\leq \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(\vartheta\psi(\phi_1, \phi_2, \ldots, \phi_n)).
\end{aligned}$$

Proof. It follows from (1.7), $\sum_{l=1}^{n} u_l(x) = 1$ for $x \in \mathcal{M}$, the linearity of $\mathcal{G}$ and the definition of the convexity that

$$\begin{aligned}
&\psi\left(\frac{\mathcal{G}(\vartheta\phi_1)}{\mathcal{G}(\vartheta)}, \frac{\mathcal{G}(\vartheta\phi_2)}{\mathcal{G}(\vartheta)} \cdots, \frac{\mathcal{G}(\vartheta\phi_n)}{\mathcal{G}(\vartheta)}\right)\\
&= \psi\left(\frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_1)}{\mathcal{G}(\vartheta)} + \frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_1)}{\mathcal{G}(\vartheta)}, \cdots, \frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_n)}{\mathcal{G}(\vartheta)} + \frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_n)}{\mathcal{G}(\vartheta)}\right)\\
&= \psi\Bigg(\frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}{\mathcal{G}(\vartheta)}\frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_1)}{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)} + \frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}{\mathcal{G}(\vartheta)}\frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_1)}{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}, \cdots,\\
&\qquad \frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}{\mathcal{G}(\vartheta)}\frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_n)}{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)} + \frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}{\mathcal{G}(\vartheta)}\frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_n)}{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}\Bigg)\\
&= \psi\Bigg(\frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}{\mathcal{G}(\vartheta)}\left(\frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_1)}{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}, \cdots, \frac{\mathcal{G}(\sum_{l\in\mathcal{S}_1}u_l\vartheta\phi_n)}{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}\right)\\
&\qquad +\frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}{\mathcal{G}(\vartheta)}\left(\frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_1)}{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}, \cdots, \frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_n)}{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}\right)\Bigg)\\
&\leq \frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_1)}{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}, \ldots, \frac{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta\phi_n)}{\sum_{l\in\mathcal{S}_1}\mathcal{G}(u_l\vartheta)}\right)\\
&\qquad +\frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}{\mathcal{G}(\vartheta)}\psi\left(\frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_1)}{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}, \cdots, \frac{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta\phi_n)}{\sum_{l\in\mathcal{S}_2}\mathcal{G}(v_l\vartheta)}\right)\\
&\leq \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}\left(\sum_{l\in\mathcal{S}_1}u_l\vartheta\psi(\phi_1, \phi_1, \ldots, \phi_n)\right) + \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}\left(\sum_{l\in\mathcal{S}_2}v_l\vartheta\psi(\phi_1, \phi_1, \ldots, \phi_n)\right)\\
&= \frac{1}{\mathcal{G}(\vartheta)}\mathcal{G}(\vartheta\psi(\phi_1, \phi_2, \cdots, \phi_n)).
\end{aligned}$$

□

Remark 4.1. Analogously as in the section 2, we may give some applications of Theorems 4.1 and 4.2.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article.

The research is supported by the Natural Science Foundation of China (Grant No. 61772025).

Conflict of interest

The authors declare that they have no competing interests.

References

1. M. Adil Khan, M. Hanif, Z. A. Khan, et al. *Association of Jensen's inequality for s-convex function with Csiszár divergence*, J. Inequal. Appl., **2019** (2019), 1–14.
2. S. Khan, M. Adil Khan, Y. M. Chu, *Converses of the Jensen inequality derived from the Green functions with applications in information theory*, Math. Method. Appl. Sci., **43** (2020), 2577–2587.
3. S. Khan, M. Adil Khan, Y. M. Chu, *New converses of Jensen inequality via Green functions with applications*, RACSAM, **114** (2020), 114.
4. X. S. Chen, *New convex functions in linear spaces and Jensen's discrete inequality*, J. Inequal. Appl., **2013** (2013), 1–8.
5. S. S. Dragomir, *On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese J. Math., **5** (2001), 775–788.
6. G. M. Eshaghi, M. Rostanian Delavar, M. De La Sen, *On ϕ-convex functions*, J. Math. Inequal., **10** (2016), 173–183.
7. K. Krulić, J. Pečarić, K. Smoljak, *$\alpha(x)$-convex functions and their inequalities*, Bull. Malays. Math. Sci. Soc., **35** (2012), 695–716.
8. N. Merentes, K. Nikodem, *Remarks on strongly convex functions*, Aequationes Math., **80** (2010), 193–199.
9. X. M. Hu, J. F. Tian, Y. M. Chu, et al. *On Cauchy-Schwarz inequality for N-tuple diamond-alpha integral*, J. Inequal. Appl., **2020** (2020), 1–15.
10. S. Rashid, M. A. Noor, K. I. Noor, et al. *Ostrowski type inequalities in the sense of generalized $\mathcal{K}$-fractional integral operator for exponentially convex functions*, AIMS Mathematics, **5** (2020), 2629–2645.
11. S. Rashid, F. Jarad, Y. M. Chu, *A note on reverse Minkowski inequality via generalized proportional fractional integral operator with respect to another function*, Math. Probl. Eng., **2020** (2020), 1–12.
12. S. Rashid, M. A. Noor, K. I. Noor, et al. *Hermite-Hadamrad type inequalities for the class of convex functions on time scale*, Mathematics, **7** (2019), 1–20.
13. M. A. Latif, S. Rashid, S. S. Dragomir, et al. *Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications*, J. Inequal. Appl., **2019** (2019), 1–33.

14. M. Adil Khan, N. Mohammad, E. R. Nwaeze, et al. *Quantum Hermite-Hadamard inequality by means of a Green function*, Adv. Differ. Equ., **2020** (2020), 1–20.

15. A. Iqbal, M. Adil Khan, S. Ullah, et al. *Some new Hermite-Hadamard-type inequalities associated with conformable fractional integrals and their applications*, J. Funct. Space., **2020** (2020), 1–18.

16. M. U. Awan, S. Talib, Y. M. Chu, et al. *Some new refinements of Hermite-Hadamard-type inequalities involving* Ψ_k*-Riemann-Liouville fractional integrals and applications*, Math. Probl. Eng., **2020** (2020), 1–20.

17. M. U. Awan, N. Akhtar, S. Iftikhar, et al. *Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions*, J. Inequal. Appl., **2020** (2020), 1–12.

18. T. H. Zhao, L. Shi, Y. M. Chu, *Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means*, RACSAM, **114** (2020), 1–14.

19. I. Abbas Baloch, Y. M. Chu, *Petrović-type inequalities for harmonic h-convex functions*, J. Funct. Space., **2020** (2020), 1–7.

20. S. Rashid, F. Jarad, H. Kalsoom, et al. *On Pólya-Szegö and Ćebyšev type inequalities via generalized k-fractional integrals*, Adv. Differ. Equ., **2020** (2020), 1–18.

21. S. Rashid, R. Ashraf, M. A. Noor, et al. *New weighted generalizations for differentiable exponentially convex mappings with application*, AIMS Mathematics, **5** (2020), 3525–3546.

22. S. Zaheer Ullah, M. Adil Khan, Z. A. Khan, et al. *Integral majorization type inequalities for the functions in the sense of strong convexity*, J. Funct. Space., **2019** (2019), 1–11.

23. S. Rashid, F. Jarad, M. A. Noor, et al. *Inequalities by means of generalized proportional fractional integral operators with respect another function*, Mathematics, **7** (2019), 1–18.

24. W. M. Qian, Z. Y. He, Y. M. Chu, *Approximation for the complete elliptic integral of the first kind*, RACSAM, **114** (2020), 1–12.

25. Z. H. Yang, W. M. Qian, W. Zhang, et al. *Notes on the complete elliptic integral of the first kind*, Math. Inequal. Appl., **23** (2020), 77–93.

26. M. K. Wang, H. H. Chu, Y. M. Li, et al. *Answers to three conjectures on convexity of three functions involving complete elliptic integrals of the first kind*, Appl. Anal. Discrete Math., **14** (2020), 255–271.

27. S. Rafeeq, H. Kalsoom, S. Hussain, et al. *Delay dynamic double integral inequalities on time scales with applications*, Adv. Differ. Equ., **2020** (2020), 1–32.

28. H. Z. Xu, Y. M. Chu, W. M. Qian, *Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means*, J. Inequal. Appl., **2018** (2018), 1–13.

29. W. M. Qian, Y. Y. Yang, H. W. Zhang, et al. *Optimal two-parameter geometric and arithmetic mean bounds for the Sándor-Yang mean*, J. Inequal. Appl., **2019** (2019), 1–12.

30. W. M. Qian, W. Zhang, Y. M. Chu, *Bounding the convex combination of arithmetic and integral means in terms of one-parameter harmonic and geometric means*, Miskolc Math. Notes, **20** (2019), 1157–1166.

31. M. K. Wang, Z. Y. He, Y. M. Chu, *Sharp power mean inequalities for the generalized elliptic integral of the first kind*, Comput. Meth. Funct. Th., **20** (2020), 111–124.

32. B. Wang, C. L. Luo, S. H. Li, et al. *Sharp one-parameter geometric and quadratic means bounds for the Sándor-Yang means*, RACSAM, **114** (2020), 1–10.

33. T. H. Zhao, Y. M. Chu, H. Wang, *Logarithmically complete monotonicity properties relating to the gamma function*, Abstr. Appl. Anal., **2011** (2011), 1–13.

34. S. Zaheer Ullah, M. Adil Khan, Y. M. Chu, *A note on generalized convex functions*, J. Inequal. Appl., **2019** (2019), 1–10.

35. M. K. Wang, M. Y. Hong, Y. F. Xu, et al. *Inequalities for generalized trigonometric and hyperbolic functions with one parameter*, J. Math. Inequal., **14** (2020), 1–21.

36. M. Adil Khan, D. Pečarić, J. Pečarić, *Bounds for Csiszár divergence and hybrid Zipf-Mandelbrot entropy*, Math. Method. Appl. Sci., **42** (2019), 7411–7424.

37. J. G. Liao, A. Berg, *Sharpening Jensen's inequality*, Am. Stat., **79** (2019), 278–281.

38. Q. Lin, *Jensen inequality for superlinear expectations*, Stat. Probab. Lett., **151** (2019), 79–83.

39. B. Jessen, *Bemrkninger om konvekse Funktioner og Uligheder imellem Middelvrdier I*, Mat. Tidsskr. B, **2** (1931), 17–26.

40. J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Boston, 1992.

41. E. J. McShane, *Jensen's inequality*, B. Am. Math. Soc., **43** (1937), 521–527.

42. E. Beck, *Über Ungleichungen von der Form* $f(M_\phi(x;\alpha), M_\psi(y;\alpha)) \geq M_\chi(f(x,y);\alpha)$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., **320–328** (1970), 1–14.

43. M. Adil Khan, D. Pečarić, J. Pečarić, *New refinement of the Jensen inequality associated to certain functions with applications*, J. Inequal. Appl., **2020** (2020), 1–11.

44. V. Čuljak, B. Ivanković, J. Pečarić, *On Jensen-Mcshane's inequality*, Period. Math. Hungar., **58** (2019), 139–154.