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*Research article*

# $(F, h)$-upper class type functions for cyclic admissible contractions in metric spaces

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**Abstract:** In this paper, we introduce the notions of $T$-cyclic $(\alpha, \beta, H, F)$-contractive mappings using a pair $(F, h)$-upper class functions type in order to obtain new common fixed point results in the settings of metric spaces. The presented results generalize and extend existing results in the literature.

**Keywords:** common fixed point; point of coincidence; $T$-cyclic $(\alpha, \beta)$-admissible mapping; $T$-cyclic $(\alpha, \beta, H, F)$-contractive mappings; pair $(F, h)$-upper class
**Mathematics Subject Classification:** 47H10, 54H25

## 1. Introduction

Fixed point results have a crucial role to construct methods for solving problems in applied mathematics and other sciences. A large number of mathematicians have focused on this interesting topic. The Banach contraction mapping principle is the most important result in fixed point theory. It is considered the source of metric fixed point theory. Metric spaces form a natural environment for exploring fixed points of single and multivalued mappings which can be noted to Banach contraction principle [7], that is, a very interesting useful and pivotal result in fixed point theory. The important feature of the Banach contraction principle is that it gives the existence, uniqueness and the covergence of the sequence of the successive approximation to a solution of the problem. Banach contraction principle is generalized in many different ways. Reader can see two short survey of the development of fixed point theory in [15, 18].

Recently, Samet *et al.* [24] introduced the notions of $\alpha$-$\psi$-contractive mappings and $\alpha$-admissible mappings. Also, Alizadeh *et al.* [2] offered the concept of cyclic $(\alpha,\beta)$-admissible mappings and obtain some new fixed point results. For more information on fixed point results, see [1, 4, 5, 8, 9, 11, 12, 17, 19, 20, 22, 23, 25, 26].

**Definition 1.1.** [16] A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:
$(i)$ $\varphi$ is non-decreasing and continuous,
$(ii)$ $\varphi(t) = 0$ if and only if $t = 0$.

**Definition 1.2.** [2] Let $f : X \to X$ and $\alpha,\beta : X \to [0, +\infty)$. We say that $f$ is a cyclic $(\alpha,\beta)$-admissible mapping if
$(i)$ $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(fx) \geq 1$;
$(ii)$ $\beta(x) \geq 1$ for some $x \in X$ implies $\alpha(fx) \geq 1$.

**Definition 1.3.** [14] Let $X$ be a nonempty set and $f, T : X \to X$. The pair $(f, T)$ is said to be weakly compatible if $f$ and $T$ commute at their coincidence points (i.e., $fTx = Tfx$ whenever $fx = Tx$). A point $y \in X$ is called a point of coincidence of $f$ and $T$ if there exists a point $x \in X$ such that $y = fx = Tx$.

Following the direction in [10], we denote by $\Psi$ the family of all functions $\psi : \mathbb{R}_+^4 \to \mathbb{R}_+$ such that
$(\psi_1)$ $\psi$ is nondecreasing in each coordinate and continuous;
$(\psi_2)$ $\psi(t, t, t, t) \leq t$, $\psi(t, 0, 0, t) \leq t$ and $\psi\left(0, 0, t, \frac{t}{2}\right) \leq t$ for all $t > 0$;
$(\psi_3)$ $\psi(t_1, t_2, t_3, t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$.

**Definition 1.4.** [3] A mapping $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type $I$ if $x \geq 1$, then $h(1, y) \leq h(x, y)$.

**Example 1.5.** [3] The following are some examples of function of subclass of type $I$, for all $x, y \in \mathbb{R}^+$ and positive integers $m, n$,
(1) $h(x, y) = (y + l)^x, l > 1$;
(2) $h(x, y) = (x + l)^y, l > 1$;
(3) $h(x, y) = x^m y$;
(4) $h(x, y) = \frac{x^n+x^{n-1}+\cdots+x^1+1}{n+1}y$;
(5) $h(x, y) = (\frac{x^n+x^{n-1}+\cdots+x^1+1}{n+1} + l)^y, l > 1$.

**Definition 1.6.** [3] Suppose that $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. A pair $(F, h)$ is called an upper class of type $I$ if $h$ is a subclass of type $I$ and
(1) $0 \leq s \leq 1 \Longrightarrow F(s, t) \leq F(1, t)$;
(2) $h(1, y) \leq F(s, t) \Longrightarrow y \leq st$.

**Example 1.7.** [3] The following are some examples of upper class of type $I$, for all $s, t \in \mathbb{R}^+$ and positive integers $m, n$,

(1) $h(x,y) = (y+l)^x, l > 1, F(s,t) = st + l$;
(2) $h(x,y) = (x+l)^y, l > 1, F(s,t) = (1+l)^{st}$;
(3) $h(x,y) = x^m y, F(s,t) = st$;
(4) $h(x,y) = \frac{x^n + x^{n-1} + \cdots + x^1 + 1}{n+1} y, F(s,t) = st$;
(5) $h(x,y) = (\frac{x^n + x^{n-1} + \cdots + x^1 + 1}{n+1} + l)^y, l > 1, F(s,t) = (1+l)^{st}$.

**Definition 1.8.** [3] A mapping $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type $II$ if for $x, y \geq 1$, $h(1,1,z) \leq h(x,y,z)$.

**Example 1.9.** [3] The following are some examples of subclass of type $II$, for all $x, y, z \in \mathbb{R}^+$,
(1) $h(x,y,z) = (z+l)^{xy}, l > 1$;
(2) $h(x,y,z) = (xy+l)^z, l > 1$;
(3) $h(x,y,z) = z$;
(4) $h(x,y,z) = x^m y^n z^p, m, n, p \in \mathbb{N}$;
(5) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}$.

**Definition 1.10.** [3] Let $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. Then we say that the pair $(F, h)$ is called an upper class of type $II$ if $h$ is a subclass of type $II$ and
(1) $0 \leq s \leq 1 \Longrightarrow F(s,t) \leq F(1,t)$;
(2) $h(1,1,z) \leq F(s,t) \Longrightarrow z \leq st$.

**Example 1.11.** [3] The following are some examples of upper class of type $II$, for all $s, t \in \mathbb{R}^+$,
(1) $h(x,y,z) = (z+l)^{xy}, l > 1, F(s,t) = st + l$;
(2) $h(x,y,z) = (xy+l)^z, l > 1, F(s,t) = (1+l)^{st}$;
(3) $h(x,y,z) = z, F(s,t) = st$;
(4) $h(x,y,z) = x^m y^n z^p, m, n, p \in \mathbb{N}, F(s,t) = s^p t^p$;
(5) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}, F(s,t) = (st)^k$.

**Definition 1.12.** [13] Let $f$, $T : X \to X$ and $\alpha, \beta : X \to [0, +\infty)$. We say that $f$ is a $T$-cyclic $(\alpha, \beta)$-admissible mapping if
($i$) $\alpha(Tx) \geq 1$ for some $x \in X$ implies $\beta(fx) \geq 1$;
($ii$) $\beta(Tx) \geq 1$ for some $x \in X$ implies $\alpha(fx) \geq 1$.

**Example 1.13.** [13] Let $f, T : \mathbb{R} \to \mathbb{R}$ be defined by $fx = x$ and $Tx = -x$. Suppose that $\alpha, \beta : \mathbb{R} \to \mathbb{R}^+$ are given by $\alpha(x) = e^{-x}$ for all $x \in \mathbb{R}$ and $\beta(y) = e^y$ for all $y \in \mathbb{R}$. Then $f$ is a $T$-cyclic $(\alpha, \beta)$ admissible mapping. Indeed, if $\alpha(Tx) = e^x \geq 1$, then $x \geq 0$ implies $fx \geq 0$ and so $\beta(fx) = e^{fx} \geq 1$. Also, if $\beta(Ty) = e^{-y} \geq 1$, then $y \leq 0$ which implies $fy \leq 0$ and so $\alpha(fy) = e^{-fy} \geq 1$.

The following result will be used in the sequel.

**Lemma 1.14.** *[6, 21] Let $(X, d)$ be a metric space and $\{x_n\}$ be a sequence in $X$ such that*

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0.$$

*If $\{x_n\}$ is not a Cauchy sequence in $X$, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to $\varepsilon^+$ when $k \to +\infty$:*

$$d(x_{m(k)}, x_{n(k)}),\ d(x_{m(k)}, x_{n(k)+1}),\ d(x_{m(k)-1}, x_{n(k)}),$$

$$d(x_{m(k)-1}, x_{n(k)+1}),\ d(x_{m(k)+1}, x_{n(k)+1}).$$

In this paper, we introduce new notions of $T$-cyclic $(\alpha,\beta,H,F)$-contractive and $T$-cyclic $(\alpha,\beta,H,F)$-rational contractive using a pair $(F,h)$ upper class functions to obtain new fixed point and common fixed point theorems.

## 2. Main results

The following definitions will be used efficiently in the proof of main results.

**Definition 2.1.** Let $f$, $T : X \to X$ and $\lambda,\gamma : X \to [0,+\infty)$. We say that $f$ is a $T$-cyclic $(\lambda,\gamma)$-subadmissible mapping if

($i$) $\lambda(Tx) \leq 1$ for some $x \in X$ implies $\gamma(fx) \leq 1$;

($ii$) $\gamma(Tx) \leq 1$ for some $x \in X$ implies $\lambda(fx) \leq 1$.

**Definition 2.2.** Let $(X,d)$ be a metric space and $f$ be a $T$-cyclic $(\alpha,\beta)$-admissible and $T$-cyclic $(\lambda,\gamma)$-subadmissible mapping. We say that $f$ is a $T$-cyclic $(\alpha,\beta,H,F)$-contractive mapping if

$$H(\alpha(Tx),\beta(Ty),\varphi(d(fx,fy))) \leq F(\gamma(Tx)\lambda(Ty),\eta(M(x,y))), \tag{2.1}$$

for all $x,y \in X$, where

$$M(x,y) = \psi\left(d(Tx,Ty),d(Tx,fx),d(Ty,fy),\frac{1}{2}[d(Tx,fy)+d(Ty,fx)]\right)$$

for $\psi \in \Psi$, the pair $(F,h)$ is an upper class of type $II$, $\varphi$ is an altering distance function and $\eta : [0,+\infty) \to [0,+\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$.

**Theorem 2.3.** *Let $(X,d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha,\beta,H,F)$-contractive mapping. Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

*($i$) there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

*($ii$) if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

*($iii$) $\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$. Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

*Proof.* Let $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ by

$$y_n = fx_n = Tx_{n+1}, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

If $y_n = y_{n+1}$, then $y_{n+1}$ is a point of coincidence of $f$ and $T$. Suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Since $f$ is a $T$-cyclic $(\alpha,\beta)$-admissible mapping and $\alpha(Tx_0) \geq 1, \beta(fx_0) = \beta(Tx_1) \geq 1$ which implies $\alpha(Tx_2) = \alpha(fx_1) \geq 1$. By continuing this process, we get $\alpha(Tx_{2n}) \geq 1$ and $\beta(Tx_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Similarly, since $f$ is a $T$-cyclic $(\alpha,\beta)$-admissible mapping and $\beta(Tx_0) \geq 1$, we have $\beta(Tx_{2n}) \geq 1$ and $\alpha(Tx_{2n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $\alpha(Tx_n) \geq 1$ and $\beta(Tx_n) \geq 1$ for all

$n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(Tx_n)\beta(Tx_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since $f$ is a $T$-cyclic $(\lambda, \gamma)$-subadmissible mapping and $\lambda(Tx_0) \leq 1$, $\gamma(fx_0) = \gamma(Tx_1) \leq 1$ which implies $\lambda(Tx_2) = \lambda(fx_1) \leq 1$. By continuing this process, we get $\lambda(Tx_{2n}) \leq 1$ and $\gamma(Tx_{2n+1}) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Similarly, since $f$ is a $T$-cyclic $(\lambda, \gamma)$-admissible mapping and $\gamma(Tx_0) \leq 1$, we have $\gamma(Tx_{2n}) \leq 1$ and $\lambda(Tx_{2n+1}) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $\lambda(Tx_n) \geq 1$ and $\beta(Tx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\lambda(Tx_n)\gamma(Tx_{n+1}) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, by (2.1) and using (2.2), we get

$$\begin{aligned} H(1,1,\varphi(d(y_n, y_{n+1}))) &= H(1,1,\varphi(d(fx_n, fx_{n+1}))) \\ &\leq H(\alpha(Tx_n), \beta(Tx_{n+1}), \varphi(d(fx_n, fx_{n+1}))) \\ &\leq F(\lambda(Tx_n)\gamma(Tx_{n+1}), \eta(M(x_n, x_{n+1}))) \\ &\leq F(1, \eta(M(x_n, x_{n+1}))). \end{aligned}$$

This implies that

$$\varphi(d(y_n, y_{n+1})) \leq \eta(M(x_n, x_{n+1})) < \varphi(M(x_n, x_{n+1})). \tag{2.3}$$

Since $\varphi$ is nondecreasing, we have

$$d(y_n, y_{n+1}) < M(x_n, x_{n+1}), \tag{2.4}$$

where

$$\begin{aligned} &M(x_n, x_{n+1}) \\ &= \psi\left(d(Tx_n, Tx_{n+1}), d(Tx_n, fx_n), d(Tx_{n+1}, fx_{n+1}), \frac{1}{2}[d(Tx_n, fx_{n+1}) + d(Tx_{n+1}, fx_n)]\right) \\ &= \psi\left(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_{n+1}) + d(y_n, y_n)]\right) \\ &\leq \psi\left(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\right). \end{aligned} \tag{2.5}$$

Thus, from (2.4), we obtain

$$\begin{aligned} d(y_n, y_{n+1}) &< M(x_n, x_{n+1}) \\ &\leq \psi\left(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\right). \end{aligned}$$

If $d(y_{n-1}, y_n) \leq d(y_n, y_{n+1})$ for some $n \in \mathbb{N}$, then

$$\begin{aligned} d(y_n, y_{n+1}) &< \psi\left(d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\right) \\ &\leq \psi(d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_n, y_{n+1})) \\ &\leq d(y_n, y_{n+1}), \end{aligned}$$

which is a contradiction and hence $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$ for all $n \in \mathbb{N}$. Therefore, the sequence $\{d(y_n, y_{n+1})\}$ is decreasing and bounded below. Thus, there exists $r \geq 0$ such that $\lim_{n\to+\infty} d(y_n, y_{n+1}) = r$. Assume $r > 0$. Also, from (2.3), (2.5) and using the properties of $\psi$, we deduce

$$\varphi(d(y_n, y_{n+1})) \leq \eta(M(x_n, x_{n+1}))$$

$$
\begin{aligned}
&\leq \eta\left(\psi\left(d\left(y_{n-1}, y_{n}\right), d\left(y_{n-1}, y_{n}\right), d\left(y_{n}, y_{n+1}\right), \frac{1}{2}\left[d\left(y_{n-1}, y_{n}\right)+d\left(y_{n}, y_{n+1}\right)\right]\right)\right) \\
&\leq \eta\left(\psi\left(d\left(y_{n-1}, y_{n}\right), d\left(y_{n-1}, y_{n}\right), d\left(y_{n-1}, y_{n}\right), d\left(y_{n-1}, y_{n}\right)\right)\right) \\
&\leq \eta\left(d\left(y_{n-1}, y_{n}\right)\right). \qquad (2.6)
\end{aligned}
$$

Consider the properties of $\varphi$ and $\eta$. Letting $n \to +\infty$ in (2.6), we get

$$
\begin{aligned}
\varphi(r) &= \lim_{n\to+\infty} \varphi\left(d\left(y_{n}, y_{n+1}\right)\right) \\
&\leq \lim_{n\to+\infty} \eta\left(d\left(y_{n-1}, y_{n}\right)\right) = \eta(r) < \varphi(r),
\end{aligned}
$$

which implies $r = 0$ and so

$$
\lim_{n\to+\infty} d\left(y_{n}, y_{n+1}\right) = 0. \qquad (2.7)
$$

Now, we prove that $\{y_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{y_n\}$ is not a Cauchy sequence. Then, by Lemma 1.14, there exist an $\varepsilon > 0$ and two subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ with $m_k > n_k > k$ such that $d(y_{m(k)}, y_{n(k)}) \geq \varepsilon$, $d(y_{m(k)-1}, y_{n(k)}) < \varepsilon$ and

$$
\lim_{k\to+\infty} d\left(y_{n_k}, y_{m_k}\right) = \lim_{k\to+\infty} d\left(y_{n_k-1}, y_{m_k}\right) = \lim_{k\to+\infty} d\left(y_{m_k-1}, y_{n_k}\right) = \lim_{k\to+\infty} d\left(y_{m_k-1}, y_{n_k-1}\right) = \varepsilon. \qquad (2.8)
$$

From (2.1), we get

$$
\begin{aligned}
H(1,1,\varphi(d(y_{n_k}, y_{m_k}))) &= H(1,1,\varphi(d(fx_{n_k}, fx_{m_k}))) \\
&\leq H(\alpha(Tx_{n_k}), \beta(Tx_{m_k}), \varphi(d(fx, fy))) \\
&\leq F(\lambda(Tx_{n_k})\gamma(Tx_{m_k}), \eta(M(x_{n_k}, x_{m_k}))) \\
&\leq F(1, \eta(M(x, y))).
\end{aligned}
$$

This implies that

$$
\varphi\left(d\left(y_{n_k}, y_{m_k}\right)\right) \leq \eta\left(M\left(x_{n_k}, x_{m_k}\right)\right) < \varphi\left(M\left(x_{n_k}, x_{m_k}\right)\right) \qquad (2.9)
$$

where

$$
\begin{aligned}
M\left(x_{n_k}, x_{m_k}\right) &= \psi\Big(d\left(Tx_{n_k}, Tx_{m_k}\right), d\left(Tx_{n_k}, fx_{n_k}\right), d\left(Tx_{m_k}, fx_{m_k}\right), \\
&\qquad \frac{1}{2}\left[d\left(Tx_{n_k}, fx_{m_k}\right) + d\left(Tx_{m_k}, fx_{n_k}\right)\right]\Big) \\
&\leq \psi\Big(\max\{\varepsilon, d\left(y_{n_k-1}, y_{m_k-1}\right)\}, d\left(y_{n_k-1}, y_{n_k}\right), d\left(y_{m_k-1}, y_{m_k}\right), \\
&\qquad \max\left\{\varepsilon, \frac{1}{2}\left[d\left(y_{n_k-1}, y_{m_k}\right) + d\left(y_{m_k-1}, y_{n_k}\right)\right]\right\}\Big).
\end{aligned}
$$

Now, from the properties of $\varphi, \psi$ and $\eta$ and using (2.8) and the above inequality, as $k \to +\infty$ in (2.9), we have

$$
\varphi(\varepsilon) \leq \eta(\psi(\varepsilon, 0, 0, \varepsilon)) \leq \eta(\varepsilon) < \varphi(\varepsilon),
$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus, $\{y_n\}$ is a Cauchy sequence in $X$. From the completeness of $(X, d)$, there exists $z \in X$ such that

$$\lim_{n\to+\infty} y_n = z. \tag{2.10}$$

From (2.2) and (2.10), we obtain

$$fx_n \to z \quad \text{and} \quad Tx_{n+1} \to z. \tag{2.11}$$

Since $TX$ is closed, by (2.11), $z \in TX$. Therefore, there exists $u \in X$ such that $Tu = z$. Since $y_n \to z$ and $\beta(y_n) = \beta(Tx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, by $(ii)$, $\beta(z) = \beta(Tu) \geq 1$. Similarly, $\gamma(z) = \gamma(Tu) \leq 1$. Thus, $\lambda(Tx_n)\gamma(Tu) \leq 1$ for all $n \in \mathbb{N}$.

Now, applying (2.1), we get

$$\begin{aligned} H(1, 1, \varphi(d(fx_n, fu))) &\leq H(\alpha(Tx_n), \beta(Tu), \varphi(d(fx_n, fu)) \\ &\leq F(\lambda(Tx_n)\gamma(Tu), \eta(M(x_n, u))) \\ &\leq F(1, \eta(M(x_n, u))). \end{aligned}$$

This implies that

$$\varphi(d(fx_n, fu)) \leq \eta(M(x_n, u)), \tag{2.12}$$

where

$$\begin{aligned} &M(x_n, u) \\ = \ &\psi\left(d(Tx_n, Tu), d(Tx_n, fx_n), d(Tu, fu), \frac{1}{2}[d(Tx_n, fu) + d(Tu, fx_n)]\right) \\ \leq \ &\psi\left(d(Tx_n, Tu), d(Tx_n, fx_n), d(Tu, fu), \frac{1}{2}\max\{d(Tu, fu), [d(Tx_n, fu) + d(Tu, fx_n)]\}\right). \end{aligned}$$

Taking $k \to \infty$ in the inequality (2.12) and using the properties of $\varphi, \psi, \eta$ and the above inequality, we have

$$\begin{aligned} \varphi(d(z, fu)) &\leq \eta\left(\psi\left(0, 0, d(z, fu), \frac{1}{2}d(z, fu)\right)\right) \\ &\leq \eta(d(z, fu)) < \varphi(d(z, fu)), \end{aligned}$$

which implies $d(z, fu) = 0$, that is, $z = fu$. Thus we deduce

$$z = fu = Tu \tag{2.13}$$

and so $z$ is a point of coincidence for $f$ and $T$. The uniqueness of the point of coincidence is a consequence of the conditions (2.1) and $(iii)$, and so we omit the details.

By (2.13) and using weakly compatibility of $f$ and $T$, we obtain

$$fz = fTu = Tfu = Tz$$

and so $fz = Tz$. Uniqueness of the coincidence point implies $z = fz = Tz$. Consequently, $z$ is a unique common fixed point of $f$ and $T$. □

**Corollary 2.4.** *Let $(X,d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha,\beta,H,F)$-admissible mapping and $T$-cyclic $(\lambda,\gamma)$-subadmissible mapping such that*

$$H(\alpha(Tx),\beta(Ty),\varphi(d(fx,fy))) \leq F(\gamma(Tx)\lambda(Ty),\eta(M(x,y)))$$

*for all $x,y \in X$, where $\varphi$ is an altering distance function and $\eta : [0,+\infty) \to [0,+\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$M(x,y) = \psi(d(Tx,Ty), d(Tx,fx), d(Ty,fy), \frac{1}{2}[d(Tx,fy) + d(Ty,fx)])$$

*for $\psi \in \Psi$. Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

*$(i)$ there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

*$(ii)$ if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

*$(iii)$ $\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$.*

*Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.5.** *Let $(X,d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha,\beta,H,F)$-admissible mapping and $T$-cyclic $(\lambda,\gamma)$-subadmissible mapping such that*

$$(\alpha(Tx)\beta(Ty) + l)^{\varphi(d(fx,fy))} \leq (1+l)^{\gamma(Tx)\lambda(Ty)\eta(M(x,y))}$$

*for all $x,y \in X$, where $\varphi$ is an altering distance function and $\eta : [0,+\infty) \to [0,+\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$M(x,y) = \psi(d(Tx,Ty), d(Tx,fx), d(Ty,fy), \frac{1}{2}[d(Tx,fy) + d(Ty,fx)])$$

*for $\psi \in \Psi$. Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

*$(i)$ there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

*$(ii)$ if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

*$(iii)$ $\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$.*

*Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.6.** *Let $(X,d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha,\beta,H,F)$-admissible mapping and $T$-cyclic $(\lambda,\gamma)$-subadmissible mapping such that*

$$(\varphi(d(fx,fy)) + l)^{\alpha(Tx)\beta(Ty)} \leq (\gamma(Tx)\lambda(Ty)\eta(M(x,y)) + l$$

*for all $x,y \in X$, where $\varphi$ is an altering distance function and $\eta : [0,+\infty) \to [0,+\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$M(x,y) = \psi(d(Tx,Ty), d(Tx,fx), d(Ty,fy), \frac{1}{2}[d(Tx,fy) + d(Ty,fx)])$$

*for $\psi \in \Psi$. Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

($iii$) *$\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$. Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.7.** *Let $(X, d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha, \beta, H, F)$-admissible mapping and $T$-cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$H(\alpha(Tx), \beta(Ty), \varphi(d(fx, fy))) \leq F(\gamma(Tx)\lambda(Ty), \eta(M(x, y))) \tag{2.14}$$

*for all $x, y \in X$, where $\varphi$ is an altering distance function and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$M(x, y) = \max\{d(Tx, Ty), d(Tx, fx), d(Ty, fy), \frac{1}{2}[d(Tx, fy) + d(Ty, fx)]\}.$$

*Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

($iii$) *$\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$. Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.8.** *Let $(X, d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha, \beta, H, F)$-admissible mapping and $T$-cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$(\alpha(Tx)\beta(Ty) + l)^{\varphi(d(fx, fy))} \leq (1 + l)^{\gamma(Tx)\lambda(Ty)\eta(M(x,y))}$$

*for all $x, y \in X$, where $\varphi$ is an altering distance function and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$M(x, y) = \max\{d(Tx, Ty), d(Tx, fx), d(Ty, fy), \frac{1}{2}[d(Tx, fy) + d(Ty, fx)]\}.$$

*Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

($iii$) *$\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$. Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.9.** *Let $(X,d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha,\beta,H,F)$-admissible mapping and $T$-cyclic $(\lambda,\gamma)$-subadmissible mapping such that*

$$(\varphi(d(fx,fy))+l)^{\alpha(Tx)\beta(Ty)} \leq \gamma(Tx)\lambda(Ty)\eta(M(x,y))+l$$

*for all $x,y \in X$, where $\varphi$ is an altering distance function and $\eta:[0,+\infty)\to[0,+\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t)>\eta(t)$ for all $t>0$ and*

$$M(x,y)=\max\{d(Tx,Ty),d(Tx,fx),d(Ty,fy),\frac{1}{2}[d(Tx,fy)+d(Ty,fx)]\}.$$

*Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(Tx_0)\geq 1$ and $\beta(Tx_0)\geq 1$, and $\lambda(Tx_0)\leq 1$ and $\gamma(Tx_0)\leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $x_n\to x$ and $\beta(x_n)\geq 1$ for all $n$, then $\beta(x)\geq 1$, and $\gamma(x_n)\leq 1$ for all $n$, then $\gamma(x)\leq 1$;*

($iii$) *$\alpha(Tu)\geq 1$ and $\beta(Tv)\geq 1$, and $\lambda(Tu)\leq 1$ and $\gamma(Tv)\leq 1$ whenever $fu=Tu$ and $fv=Tv$.*

*Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.10.** *Let $(X,d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha,\beta,H,F)$-admissible mapping such that*

$$H(\alpha(Tx),\beta(Ty),\varphi(d(fx,fy))) \leq F(1,\eta(M(x,y))) \tag{2.15}$$

*for all $x,y \in X$, where $\varphi$ is an altering distance function and $\eta:[0,+\infty)\to[0,+\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t)>\eta(t)$ for all $t>0$ and*

$$M(x,y)=\max\left\{d(Tx,Ty),d(Tx,fx),d(Ty,fy),\frac{1}{2}[d(Tx,fy)+d(Ty,fx)]\right\}.$$

*Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(Tx_0)\geq 1$ and $\beta(Tx_0)\geq 1$, and $\lambda(Tx_0)\leq 1$ and $\gamma(Tx_0)\leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $x_n\to x$ and $\beta(x_n)\geq 1$ for all $n$, then $\beta(x)\geq 1$;*

($iii$) *$\alpha(Tu)\geq 1$ and $\beta(Tv)\geq 1$ whenever $fu=Tu$ and $fv=Tv$.*

*Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

*Proof.* Take $\gamma(Tx)\lambda(Ty)=1$, for $x,y\in X$. If we take $\psi(t_1,t_2,t_3,t_4)=\max\{t_1,t_2,t_3,t_4\}$ in Corollary 2.7, then from (2.15), we have

$$\alpha(Tx)\beta(Ty)\varphi(d(fx,fy)) \leq \gamma(Tx)\lambda(Ty)\eta(M(x,y)).$$

This implies that the inequality (2.14) holds. Therefore, the proof follows from Corollary 2.7. □

If we choose $T=I_X$ in Theorem 2.3, then we have the following corollary.

**Corollary 2.11.** *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a cyclic $(\alpha,\beta,H,F)$-admissible mapping and a cyclic $(\lambda,\gamma)$-subadmissible mapping such that*

$$H(\alpha(x),\beta(y),\varphi(d(fx,fy))) \leq F(\gamma(x)\lambda(y),\eta(M(x,y)))$$

*for all $x,y \in X$, where the pair $(F,h)$ is an upper class of type II, $\varphi$ is an altering distance function and $\eta : [0,+\infty) \to [0,+\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$M_f(x,y) = \psi\left(d(x,y), d(x,fx), d(y,fy), \frac{1}{2}[d(x,fy) + d(y,fx)]\right).$$

*Assume that the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$;*

($iii$) *$\alpha(u) \geq 1$ and $\beta(v) \geq 1$ whenever $fu = u$ and $fv = v$.*

*Then $f$ has a unique fixed point.*

If we take $\eta(t) = \varphi(t) - \eta^1(t)$ in Corollary 2.5, then we have the following corollary.

**Corollary 2.12.** *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a cyclic $(\alpha,\beta,H,F)$-admissible mapping and a cyclic $(\lambda,\gamma)$-subadmissible mapping such that*

$$H(\alpha(x),\beta(y),\varphi(d(fx,fy))) \leq F(\gamma(x)\lambda(y),\varphi\left(M_f(x,y)\right) - \eta^1\left(M_f(x,y)\right))$$

*for all $x,y \in X$, where the pair $(F,h)$ is an upper class of type II, $\varphi$ is an altering distance function and $\eta^1 : [0,+\infty) \to [0,+\infty)$ is such that $\varphi(t) - \eta^1(t)$ is nondecreasing and $\eta^1(t)$ is continuous from the right with the condition $\varphi(t) > \eta^1(t)$ for all $t > 0$.*

*Assume that the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$;*

($iii$) *$\alpha(u) \geq 1$ and $\beta(v) \geq 1$ whenever $fu = u$ and $fv = v$.*

*Then $f$ has a unique fixed point.*

If we take $\varphi(t) = t$ in Corollary 2.12, then we have the following corollary.

**Corollary 2.13.** *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a cyclic $(\alpha,\beta,H,F)$-admissible mapping and a cyclic $(\lambda,\gamma)$-subadmissible mapping such that*

$$H(\alpha(x),\beta(y),d(fx,fy)) \leq F(\gamma(x)\lambda(y),M_f(x,y) - \eta^1\left(M_f(x,y)\right))$$

*for all $x,y \in X$, where the pair $(F,h)$ is an upper class of type II and $\eta^1 : [0,+\infty) \to [0,+\infty)$ is such that $t - \eta^1(t)$ is nondecreasing and $\eta^1(t)$ is continuous from the right with the condition $\eta^1(t) > 0$ for all $t > 0$.*

*Assume that the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$;*

($iii$) *$\alpha(u) \geq 1$ and $\beta(v) \geq 1$ whenever $fu = u$ and $fv = v$.*

*Then $f$ has a unique fixed point.*

**Example 2.14.** Let $X = \mathbb{R}$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$. Let $H : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $H(x,y,z) = z$ and $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $F(s,t) = st$ for all $x,y,s,t \in \mathbb{R}^+$ and $\varphi(t) = t$, $\eta(t) = \frac{1}{5}t$ for all $t \geq 0$, and $\psi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$ for all $t_1,t_2,t_3,t_4 \geq 0$.
Now, we define the self-mappings $f$ and $T$ on $X$ by

$$fx = \begin{cases} -\frac{x}{5} & \text{if } x \in [0,1], \\ \frac{x}{25} & \text{if } x \in \mathbb{R}\setminus[0,1] \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{x}{5} & \text{if } x \in [-1,0], \\ \frac{x}{6} & \text{if } \mathbb{R}\setminus[-1,0]. \end{cases}$$

and the mappings $\alpha,\beta,\gamma,\lambda : X \to [0,\infty)$ by

$$\alpha(x) = \beta(x) = \begin{cases} 1 & \text{if } x \in \left[-\frac{1}{4},0\right], \\ 0 & \text{otherwise.} \end{cases} \gamma(x) = 1 \text{and } \lambda(x) = \frac{1}{2}.$$

Then it is clear that $fX \subset TX$.

Let $x \in X$ such that $\alpha(Tx) \geq 1$ so that $Tx \in \left[-\frac{1}{5},0\right]$ and hence $x \in [-1,0]$. By the definitions of $f$ and $\beta$, we have $fx \in \left[-\frac{1}{5},0\right]$ and so $\beta(fx) \geq 1$.

Similarly, one can show that if $\beta(Tx) \geq 1$ then $\alpha(fx) \geq 1$. Thus, $f$ is a $T$-cyclic $(\alpha,\beta)$-admissible mapping. Moreover, the conditions $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$ are satisfied with $x_0 = -1$.
Now, let $\{x_n\}$ be a sequence in $X$ such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x$ as $n \to +\infty$. Then, by the definition of $\beta$, we have $x_n \in \left[-\frac{1}{5},0\right]$ for all $n \in \mathbb{N}$ and so $x \in \left[-\frac{1}{5},0\right]$, that is, $\beta(x) \geq 1$.

Next, we prove that $f$ is a $T$-cyclic $(\alpha,\beta)$-contractive mapping. By the definitions of the mappings we get

$$H(\alpha(Tx), \beta(Ty), \varphi(d(fx,fy))) \leq F(\gamma(Tx)\lambda(Ty), \eta(M(x,y)))$$

and so

$$\varphi(d(fx,fy))) \leq \gamma(Tx) \cdot \lambda(Ty) \cdot \eta(M(x,y)).$$

Let $\alpha(Tx)\beta(Ty) \geq 1$. Then $Tx, Ty \in \left[-\frac{1}{5},0\right]$ and so $x,y \in [-1,0]$. Thus, we get

$$\varphi(d(fx,fy)) = (d(fx,fy)) = |fx - fy| = \frac{1}{25}|x-y| \leq \frac{2}{25}|x-y| = \frac{2}{5}|Tx - Ty| =$$

$$= 1 \cdot 2 \cdot \frac{1}{5}M(x,y) = \gamma(Tx) \cdot \lambda(Ty) \cdot \eta(M(x,y)).$$

Obviously, the assumption (*iii*) of Corollary 2.10 is satisfied. Consequently, all the conditions of Corollary 2.10 hold and hence $f$ and $T$ have a unique common fixed point. Here, 0 is the common fixed point of $f$ and $T$.

**Definition 2.15.** Let $(X,d)$ be a metric space and let $f$ be a $T$-cyclic $(\alpha,\beta)$-admissible mapping and a cyclic $(\lambda,\gamma)$-subadmissible mapping. We say that $f$ is a $T$-cyclic $(\alpha,\beta,H,F)$-rational contractive mapping if

$$H(\alpha(Tx), \beta(Ty), \varphi(d(fx,fy))) \leq F(\gamma(Tx)\lambda(Ty), \eta(N(x,y))) \tag{2.16}$$

for all $x,y \in X$, where

$$N(x,y) = \phi\left(d(Tx,Ty), \frac{1}{2}d(Tx,fy), d(Ty,fx), \frac{[1+d(Tx,fx)]\,d(Ty,fy)}{1+d(Tx,Ty)}\right)$$

for $\phi \in \Phi$, $\varphi$ is an altering distance function, pair $(F, h)$ is an upper class of type $II$ and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$.

**Theorem 2.16.** *Let $(X, d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha, \beta, H, F)$-rational contractive mapping. Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

*$(i)$ there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

*$(ii)$ if $\{x_n\}$ is a sequence in $X$ such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

*$(iii)$ $\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$whenever $fu = Tu$ and $fv = Tv$. Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

*Proof.* Similar to the proof of Theorem 2.3, we define sequences $\{x_n\}$ and $\{y_n\}$ in $X$ by $y_n = fx_n = Tx_{n+1}$ and note that $\alpha(Tx_n), \beta(Tx_{n+1}) \geq 1$ and also $\lambda(Tx_n)\gamma(Tx_{n+1}) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Also, we assume that $y_n \neq y_{n-1}$ for all $n \in \mathbb{N}$. Then by (2.16), we have

$$\begin{aligned} H(1, 1, \varphi(d(y_n, y_{n+1}))) &= H(1, 1, \varphi(d(fx_n, fx_{n+1}))) \\ &\leq H(\alpha(Tx_n), \beta(Tx_{n+1}), \varphi(d(fx_n, fx_{n+1}))) \\ &\leq F(\lambda(Tx_n)\gamma(Tx_{n+1}), \eta(N(x_n, x_{n+1}))) \leq F(1, \eta(N(x_n, x_{n+1}))). \end{aligned}$$

This implies that

$$\varphi(d(y_n, y_{n+1})) \leq \eta(N(x_n, x_{n+1})) < \varphi(N(x_n, x_{n+1})). \tag{2.17}$$

Since $\varphi$ is nondecreasing, we get

$$d(y_n, y_{n+1}) < N(x_n, x_{n+1}), \tag{2.18}$$

where

$$\begin{aligned} N(x_n, x_{n+1}) &= \phi\Big(d(Tx_n, Tx_{n+1}), \frac{1}{2}d(Tx_n, fx_{n+1}), d(Tx_{n+1}, fx_n), \\ &\qquad \frac{[1 + d(Tx_n, fx_n)]\, d(Tx_{n+1}, fx_{n+1})}{1 + d(Tx_n, Tx_{n+1})}\Big) \\ &= \phi\left(d(y_{n-1}, y_n), \frac{1}{2}d(y_{n-1}, y_{n+1}), d(y_n, y_n), \frac{[1 + d(y_{n-1}, y_n)]\, d(y_n, y_{n+1})}{1 + d(y_{n-1}, y_n)}\right) \\ &\leq \phi\left(d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], 0, d(y_n, y_{n+1})\right). \end{aligned} \tag{2.19}$$

Thus, from (2.18), we deduce

$$\begin{aligned} d(y_n, y_{n+1}) &< N(x_n, x_{n+1}) \\ &\leq \phi\left(d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], 0, d(y_n, y_{n+1})\right). \end{aligned}$$

If $d(y_{n-1}, y_n) \leq d(y_n, y_{n+1})$ for some $n \in \mathbb{N}$, then

$$d(y_n, y_{n+1}) < \phi\left(d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], 0, d(y_n, y_{n+1})\right)$$

$$\begin{aligned}
&\leq \phi\left(d\left(y_n, y_{n+1}\right), d\left(y_n, y_{n+1}\right), d\left(y_n, y_{n+1}\right), d\left(y_n, y_{n+1}\right)\right) \\
&\leq d\left(y_n, y_{n+1}\right),
\end{aligned}$$

which is a contradiction and hence $d\left(y_n, y_{n+1}\right) < d\left(y_{n-1}, y_n\right)$ for all $n \in \mathbb{N}$. Therefore, the sequence $\{d\left(y_n, y_{n+1}\right)\}$ is decreasing and bounded from below. Thus, there exists $\delta \geq 0$ such that $\lim_{n\to+\infty} d\left(y_n, y_{n+1}\right) = \delta$. Also, from (2.17), (2.19) and using the properties of $\varphi$ and $\eta$, we obtain

$$\begin{aligned}
&\varphi\left(d\left(y_n, y_{n+1}\right)\right) \\
&\leq \eta\left(N\left(x_n, x_{n+1}\right)\right) \\
&\leq \eta\left(\phi\left(d\left(y_{n-1}, y_n\right), \frac{1}{2}\left[d\left(y_{n-1}, y_n\right) + d\left(y_n, y_{n+1}\right)\right], 0, d\left(y_n, y_{n+1}\right)\right)\right) \\
&\leq \eta\left(\phi\left(d\left(y_{n-1}, y_n\right), d\left(y_{n-1}, y_n\right), d\left(y_{n-1}, y_n\right), d\left(y_{n-1}, y_n\right)\right)\right) \\
&\leq \eta\left(d\left(y_{n-1}, y_n\right)\right) < \varphi\left(d\left(y_{n-1}, y_n\right)\right).
\end{aligned} \tag{2.20}$$

Consider the properties of $\varphi$ and $\eta$. Letting $n \to +\infty$ in (2.20), we get

$$\begin{aligned}
\varphi(\delta) &= \lim_{n\to+\infty} \varphi\left(d\left(y_n, y_{n+1}\right)\right) \\
&\leq \lim_{n\to+\infty} \eta\left(d\left(y_{n-1}, y_n\right)\right) = \eta(\delta) < \varphi(\delta),
\end{aligned}$$

which implies $\delta = 0$ and so

$$\lim_{n\to+\infty} d\left(y_n, y_{n+1}\right) = 0. \tag{2.21}$$

Now, we want to show that $\{y_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{y_n\}$ is not a Cauchy sequence. Then, by Lemma 1.14, there exist an $\varepsilon > 0$ and two subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ with $m_k > n_k > k$ such that $d(y_{m(k)}, y_{n(k)}) \geq \varepsilon$, $d(y_{m(k)-1}, y_{n(k)}) < \varepsilon$ and

$$\lim_{k\to+\infty} d\left(y_{n_k}, y_{m_k}\right) = \lim_{k\to+\infty} d\left(y_{n_k-1}, y_{m_k}\right) = \lim_{k\to+\infty} d\left(y_{m_k-1}, y_{n_k}\right) = \lim_{k\to+\infty} d\left(y_{m_k-1}, y_{n_k-1}\right) = \varepsilon. \tag{2.22}$$

From (2.16), we get

$$\begin{aligned}
H(1, 1, \varphi\left(d\left(y_{n_k}, y_{m_k}\right)\right)) &= H(1, 1, \varphi\left(d\left(f x_{n_k}, f x_{m_k}\right)\right)) \\
&\leq H(\alpha\left(T x_{n_k}\right), \beta\left(T x_{m_k}\right), \varphi\left(d\left(f x_{n_k}, f x_{m_k}\right)\right)) \\
&\leq F(\lambda\left(T x_{n_k}\right)\gamma\left(T x_{m_k}\right), \eta\left(N\left(x_{n_k}, x_{m_k}\right)\right)) \leq F(1, \eta\left(N\left(x_{n_k}, x_{m_k}\right)\right)).
\end{aligned}$$

This implies

$$\varphi\left(d\left(y_{n_k}, y_{m_k}\right)\right) \leq \eta\left(N\left(x_{n_k}, x_{m_k}\right)\right) < \varphi\left(N\left(x_{n_k}, x_{m_k}\right)\right), \tag{2.23}$$

where

$$\begin{aligned}
N\left(x_{n_k}, x_{m_k}\right) &= \phi\Bigg(d\left(T x_{n_k}, T x_{m_k}\right), \frac{1}{2} d\left(T x_{n_k}, f x_{m_k}\right), d\left(T x_{m_k}, f x_{n_k}\right), \\
&\qquad \frac{\left[1 + d\left(T x_{n_k}, f x_{n_k}\right)\right] d\left(T x_{m_k}, f x_{m_k}\right)}{1 + d\left(T x_{n_k}, T x_{m_k}\right)}\Bigg) \\
&= \phi\Bigg(d\left(y_{n_k-1}, y_{m_k-1}\right), \frac{1}{2} d\left(y_{n_k-1}, y_{m_k}\right), d\left(y_{m_k-1}, y_{n_k}\right),
\end{aligned}$$

$$\frac{\left[1+d\left(y_{n_k-1},y_{n_k}\right)\right]d\left(y_{m_k-1},y_{m_k}\right)}{1+d\left(y_{n_k-1},y_{m_k-1}\right)}\Bigg)$$
$$\begin{aligned}
&\leq \max\{\varepsilon, N\left(x_{n_k},x_{m_k}\right)\}\\
&= \phi\Bigg(\max\{\varepsilon, d\left(y_{n_k-1},y_{m_k-1}\right)\}, \frac{1}{2}\max\{\varepsilon, d\left(y_{n_k-1},y_{m_k}\right)\},\\
&\quad \max\{\varepsilon, d\left(y_{m_k-1},y_{n_k}\right)\}, \frac{\left[1+d\left(y_{n_k-1},y_{n_k}\right)\right]d\left(y_{m_k-1},y_{m_k}\right)}{1+d\left(y_{n_k-1},y_{m_k-1}\right)}\Bigg).
\end{aligned}$$

Therefore, $\lim_{k\to+\infty}\max\{\varepsilon, N\left(x_{n_k},x_{m_k}\right)\} = \phi\left(\varepsilon,\frac{\varepsilon}{2},\varepsilon,0\right)\leq\varepsilon$.

Now, from the properties of $\varphi$ and $\eta$ and using (2.22) and the previous inequality, as $k\to+\infty$ in (2.23), we have

$$\varphi\left(\varepsilon\right) = \lim_{k\to+\infty}\varphi\left(d\left(y_{m_k},y_{n_k}\right)\right) \leq \lim_{k\to+\infty}\eta\left(\max\{\varepsilon, N\left(x_{n_k},x_{m_k}\right)\}\right) \leq \eta\left(\varepsilon\right) < \varphi\left(\varepsilon\right),$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus, $\{y_n\}$ is a Cauchy sequence in $X$. From the completeness of $(X,d)$, there exists $w\in X$ such that

$$\lim_{n\to+\infty} y_n = w \tag{2.24}$$

and so by (2.24), we obtain

$$fx_n \to w \quad \text{and} \quad Tx_{n+1}\to w. \tag{2.25}$$

Since $TX$ is closed, by (2.25), $w\in TX$. Therefore, there exists $v\in X$ such that $Tv = w$. Since $y_n\to w$ and $\beta\left(y_n\right) = \beta\left(Tx_{n+1}\right)\geq 1$ for all $n\in\mathbb{N}$, by $(ii)$, $\beta\left(w\right) = \beta\left(Tv\right)\geq 1$. Similarly, $\gamma\left(z\right) = \gamma\left(Tu\right)\leq 1$. Thus, $\lambda\left(Tx_n\right)\gamma\left(Tv\right)\leq 1$ for all $n\in\mathbb{N}$.

Now, applying (2.16), we get

$$\begin{aligned}
H(1,1,\varphi\left(d\left(fx_n,fv\right)\right)) &\leq H(\alpha\left(Tx_n\right),\beta\left(Tv\right),\varphi\left(d\left(fx_n,fv\right)\right))\\
\leq F(\lambda\left(Tx_n\right)\gamma\left(Tv\right),\eta\left(N\left(x_n,v\right)\right)) &\leq F(1,\eta\left(N\left(x_n,v\right)\right)),
\end{aligned}$$

which implies that

$$\varphi\left(d\left(fx_n,fv\right)\right) \leq \eta\left(N\left(x_n,v\right)\right), \tag{2.26}$$

where

$$\begin{aligned}
N\left(x_n,v\right) \leq \phi\Bigg(&d\left(Tx_n,Tv\right), \frac{1}{2}\max\{d(v,fv), d\left(Tx_n,fv\right)\}, d\left(Tv,fx_n\right),\\
&d\left(Tv,fv\right)\max\left\{\frac{\left[1+d\left(Tx_n,fx_n\right)\right]}{1+d\left(Tx_n,Tv\right)},1\right\}\Bigg).
\end{aligned}$$

Taking $k\to+\infty$ in the inequality (2.26), using the properties of $\varphi$, $\eta$ and the previous inequality, we have

$$\begin{aligned}
\varphi\left(d\left(w,fv\right)\right) &\leq \eta\left(\phi\left(0,\frac{1}{2}d\left(w,fv\right),0,d\left(w,fv\right)\right)\right)\\
&\leq \eta\left(d\left(w,fv\right)\right) < \varphi\left(d\left(w,fv\right)\right),
\end{aligned}$$

which implies $d(w, fv) = 0$, that is, $w = fv$. Thus, we deduce

$$w = fv = Tv, \tag{2.27}$$

and so $w$ is a point of coincidence for $f$ and $T$. The uniqueness of the point of coincidence is a consequence of the conditions (2.16) and (*iii*), and so we omit the details.

By (2.27) and using weakly compatibility of $f$ and $T$, we obtain

$$fw = fTv = Tfv = Tw.$$

The uniqueness of the point of coincidence implies $w = fw = Tw$. Consequently, $w$ is the unique common fixed point of $f$ and $T$. □

**Corollary 2.17.** *Let $(X, d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha, \beta, H, F)$-admissible mapping and $T$-cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$H(\alpha(Tx), \beta(Ty), \varphi(d(fx, fy))) \leq F(\gamma(Tx)\lambda(Ty), \eta(N(x, y)))$$

*for all $x, y \in X$, where $\varphi$ is an altering distance function and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$N(x, y) = \phi\left(d(Tx, Ty), \frac{1}{2}d(Tx, fy), d(Ty, fx), \frac{[1 + d(Tx, fx)]\, d(Ty, fy)}{1 + d(Tx, Ty)}\right)$$

*for $\phi \in \Phi$. Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

(*i*) *there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

(*ii*) *if $\{x_n\}$ is a sequence in $X$ such that $\{x_n\} \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

(*iii*) *$\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$.*

*Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.18.** *Let $(X, d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha, \beta, H, F)$-admissible mapping and $T$-cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$(\alpha(Tx)\beta(Ty) + l)^{\varphi(d(fx, fy))} \leq (1 + l)^{\gamma(Tx)\lambda(Ty)\eta(N(x, y))}$$

*for all $x, y \in X$, where $\varphi$ is an altering distance function and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$N(x, y) = \phi\left(d(Tx, Ty), \frac{1}{2}d(Tx, fy), d(Ty, fx), \frac{[1 + d(Tx, fx)]\, d(Ty, fy)}{1 + d(Tx, Ty)}\right)$$

*for $\phi \in \Phi$. Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

(*i*) *there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

(*ii*) *if $\{x_n\}$ is a sequence in $X$ such that $\{x_n\} \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

(*iii*) *$\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$.*

*Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.19.** *Let $(X, d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha, \beta, H, F)$-admissible mapping and $T$-cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$(\varphi(d(fx, fy)) + l)^{\alpha(Tx)\beta(Ty)} \leq (\gamma(Tx)\lambda(Ty)\eta(N(x, y)) + l$$

*for all $x, y \in X$, where $\varphi$ is an altering distance function and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$N(x, y) = \phi\left(d(Tx, Ty), \frac{1}{2}d(Tx, fy), d(Ty, fx), \frac{[1 + d(Tx, fx)]\, d(Ty, fy)}{1 + d(Tx, Ty)}\right)$$

*for $\phi \in \Phi$. Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $\{x_n\} \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

($iii$) *$\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$.*

*Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.20.** *Let $(X, d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha, \beta, H, F)$-admissible mapping and $T$-cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$H(\alpha(Tx), \beta(Ty), \varphi(d(fx, fy))) \leq F(\gamma(Tx), \lambda(Ty), \eta(N(x, y)))$$

*for all $x, y \in X$, where $\varphi$ is an altering distance function and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$N(x, y) = \max\{d(Tx, Ty), \frac{1}{2}d(Tx, fy), d(Ty, fx), \frac{[1 + d(Tx, fx)]\, d(Ty, fy)}{1 + d(Tx, Ty)}\}.$$

*Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(Tx_0) \geq 1$ and $\beta(Tx_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $\{x_n\} \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$, and $\gamma(x_n) \leq 1$ for all $n$, then $\gamma(x) \leq 1$;*

($iii$) *$\alpha(Tu) \geq 1$ and $\beta(Tv) \geq 1$, and $\lambda(Tu) \leq 1$ and $\gamma(Tv) \leq 1$ whenever $fu = Tu$ and $fv = Tv$.*

*Then $f$ and $T$ have a unique point of coincidence in $X$. Moreover, if $f$ and $T$ are weakly compatible, then $f$ and $T$ have a unique common fixed point.*

**Corollary 2.21.** *Let $(X, d)$ be a complete metric space and let $f$ and $T$ be self-mappings on $X$ such that $fX \subset TX$. Let $f$ be a $T$-cyclic $(\alpha, \beta, H, F)$-admissible mapping and $T$-cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$(\alpha(Tx)\beta(Ty) + l)^{\varphi(d(fx, fy))} \leq (1 + l)^{\gamma(Tx)\lambda(Ty)\eta(N(x, y))}$$

*for all $x, y \in X$, where $\varphi$ is an altering distance function and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$N(x, y) = \max\{d(Tx, Ty), \frac{1}{2}d(Tx, fy), d(Ty, fx), \frac{[1 + d(Tx, fx)]\, d(Ty, fy)}{1 + d(Tx, Ty)}\}.$$

*Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists* $x_0 \in X$ *such that* $\alpha(Tx_0) \geq 1$ *and* $\beta(Tx_0) \geq 1$, *and* $\lambda(Tx_0) \leq 1$ *and* $\gamma(Tx_0) \leq 1$;

($ii$) *if* $\{x_n\}$ *is a sequence in* $X$ *such that* $\{x_n\} \to x$ *and* $\beta(x_n) \geq 1$ *for all* $n$, *then* $\beta(x) \geq 1$, *and* $\gamma(x_n) \leq 1$ *for all* $n$, *then* $\gamma(x) \leq 1$;

($iii$) $\alpha(Tu) \geq 1$ *and* $\beta(Tv) \geq 1$, *and* $\lambda(Tu) \leq 1$ *and* $\gamma(Tv) \leq 1$ *whenever* $fu = Tu$ *and* $fv = Tv$.

*Then* $f$ *and* $T$ *have a unique point of coincidence in* $X$. *Moreover, if* $f$ *and* $T$ *are weakly compatible, then* $f$ *and* $T$ *have a unique common fixed point.*

**Corollary 2.22.** *Let* $(X, d)$ *be a complete metric space and let* $f$ *and* $T$ *be self-mappings on* $X$ *such that* $fX \subset TX$. *Let* $f$ *be a* $T$*-cyclic* $(\alpha, \beta, H, F)$*-admissible mapping and* $T$*-cyclic* $(\lambda, \gamma)$*-subadmissible mapping such that*

$$(\varphi(d(fx, fy)) + l)^{\alpha(Tx)\beta(Ty)} \leq \gamma(Tx)\lambda(Ty)\eta(N(x, y)) + l$$

*for all* $x, y \in X$, *where* $\varphi$ *is an altering distance function and* $\eta : [0, +\infty) \to [0, +\infty)$ *is a nondecreasing and right-continuous function with the condition* $\varphi(t) > \eta(t)$ *for all* $t > 0$ *and*

$$N(x, y) = \max\{d(Tx, Ty), \frac{1}{2}d(Tx, fy), d(Ty, fx), \frac{[1 + d(Tx, fx)]\, d(Ty, fy)}{1 + d(Tx, Ty)}\}.$$

*Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists* $x_0 \in X$ *such that* $\alpha(Tx_0) \geq 1$ *and* $\beta(Tx_0) \geq 1$, *and* $\lambda(Tx_0) \leq 1$ *and* $\gamma(Tx_0) \leq 1$;

($ii$) *if* $\{x_n\}$ *is a sequence in* $X$ *such that* $\{x_n\} \to x$ *and* $\beta(x_n) \geq 1$ *for all* $n$, *then* $\beta(x) \geq 1$, *and* $\gamma(x_n) \leq 1$ *for all* $n$, *then* $\gamma(x) \leq 1$;

($iii$) $\alpha(Tu) \geq 1$ *and* $\beta(Tv) \geq 1$, *and* $\lambda(Tu) \leq 1$ *and* $\gamma(Tv) \leq 1$ *whenever* $fu = Tu$ *and* $fv = Tv$.

*Then* $f$ *and* $T$ *have a unique point of coincidence in* $X$. *Moreover, if* $f$ *and* $T$ *are weakly compatible, then* $f$ *and* $T$ *have a unique common fixed point.*

If we take $\psi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$ and $\gamma(Tx)\lambda(Ty) = 1$ for all $x, y \in X$, then we have the following result.

**Corollary 2.23.** *Let* $(X, d)$ *be a complete metric space and let* $f$ *and* $T$ *be self-mappings on* $X$ *such that* $fX \subset TX$. *Let* $f$ *be a* $T$*-cyclic* $(\alpha, \beta, H, F)$*-admissible mapping such that*

$$H(\alpha(Tx), \beta(Ty), \varphi(d(fx, fy))) \leq F(1, \eta(N(x, y)))$$

*for all* $x, y \in X$, *where* $\varphi$ *is an altering distance function and* $\eta : [0, +\infty) \to [0, +\infty)$ *is a nondecreasing and right-continuous function with the condition* $\varphi(t) > \eta(t)$ *for all* $t > 0$ *and*

$$N(x, y) = \max\left\{d(Tx, Ty), \frac{1}{2}d(Tx, fy), d(Ty, fx), \frac{[1 + d(Tx, fx)]\, d(Ty, fy)}{1 + d(Tx, Ty)}\right\}.$$

*Assume that $TX$ is a closed subset of $X$ and the following conditions are satisfied:*

($i$) *there exists* $x_0 \in X$ *such that* $\alpha(Tx_0) \geq 1$ *and* $\beta(Tx_0) \geq 1$, *and* $\lambda(Tx_0) \leq 1$ *and* $\gamma(Tx_0) \leq 1$;

($ii$) *if* $\{x_n\}$ *is a sequence in* $X$ *such that* $\{x_n\} \to x$ *and* $\beta(x_n) \geq 1$ *for all* $n$, *then* $\beta(x) \geq 1$;

($iii$) $\alpha(Tu) \geq 1$ *and* $\beta(Tv) \geq 1$ *whenever* $fu = Tu$ *and* $fv = Tv$.

*Then* $f$ *and* $T$ *have a unique point of coincidence in* $X$. *Moreover, if* $f$ *and* $T$ *are weakly compatible, then* $f$ *and* $T$ *have a unique common fixed point.*

If we choose $T = I_X$ in Theorem 2.3, then we have the following corollary.

**Corollary 2.24.** *Let $(X, d)$ be a complete metric space and $f : X \to X$ be a cyclic $(\alpha,\beta, H, F)$-admissible mapping and a cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$H(\alpha(x), \beta(y), \varphi(d(fx, fy))) \leq F(\gamma(x)\lambda(y), \eta\left(N_f(x, y)\right))$$

*for all $x, y \in X$, where pair $(F, h)$ is an upper class of type II, $\varphi$ is an altering distance function and $\eta : [0, +\infty) \to [0, +\infty)$ is a nondecreasing and right-continuous function with the condition $\varphi(t) > \eta(t)$ for all $t > 0$ and*

$$N_f(x, y) = \phi\left(d(x, y), \frac{1}{2}d(x, fy), d(y, fx), \frac{[1 + d(x, fx)]\, d(y, fy)}{1 + d(x, y)}\right).$$

*Assume that the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $\{x_n\} \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$;*

($iii$) *$\alpha(u) \geq 1$ and $\beta(v) \geq 1$ whenever $fu = u$ and $fv = v$.*

*Then $f$ has a unique fixed point.*

If we take $\eta(t) = \varphi(t) - \eta^1(t)$ in Corollary 2.5, then we have the following corollary.

**Corollary 2.25.** *Let $(X, d)$ be a complete metric space and $f : X \to X$ be a cyclic $(\alpha,\beta)$-admissible mapping and a cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$H(\alpha(x), \beta(y), \varphi(d(fx, fy))) \leq F(\gamma(x)\lambda(y), \varphi\left(N_f(x, y)\right) - \eta^1\left(N_f(x, y)\right))$$

*for all $x, y \in X$, where pair $(F, h)$ is an upper class of type II, $\varphi$ is an altering distance function and $\eta^1 : [0, +\infty) \to [0, +\infty)$ is such that $\varphi(t) - \eta^1(t)$ is nondecreasing and $\eta^1(t)$ is continuous from the right with the condition $\varphi(t) > \eta^1(t)$ for all $t > 0$.*

*Assume that the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, and $\lambda(Tx_0) \leq 1$ and $\gamma(Tx_0) \leq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $\{x_n\} \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$;*

($iii$) *$\alpha(u) \geq 1$ and $\beta(v) \geq 1$ whenever $fu = u$ and $fv = v$.*

*Then $f$ has a unique fixed point.*

If we take $\varphi(t) = t$ in Corollary 2.6, then we have the following corollary.

**Corollary 2.26.** *Let $(X, d)$ be a complete metric space and $f : X \to X$ be a cyclic $(\alpha,\beta, H, F)$-admissible mapping and a cyclic $(\lambda, \gamma)$-subadmissible mapping such that*

$$H(\alpha(x), \beta(y), d(fx, fy)) \leq F(\gamma(x)\lambda(y), N_f(x, y) - \eta^1\left(N_f(x, y)\right))$$

*for all $x, y \in X$, where pair $(F, h)$ is an upper class of type II and $\eta^1 : [0, +\infty) \to [0, +\infty)$ is such that $t - \eta^1(t)$ is nondecreasing and $\eta^1(t)$ is continuous from the right with the condition $\eta^1(t) > 0$ for all $t > 0$.*

*Assume that the following conditions are satisfied:*

($i$) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;*

($ii$) *if $\{x_n\}$ is a sequence in $X$ such that $\{x_n\} \to x$ and $\beta(x_n) \geq 1$ for all $n$, then $\beta(x) \geq 1$;*

($iii$) *$\alpha(u) \geq 1$ and $\beta(v) \geq 1$ whenever $fu = u$ and $fv = v$.*

*Then $f$ has a unique fixed point.*

## 3. Conclusion

In this paper, we have introduced the notions of $T$-cyclic $(\alpha, \beta, H, F)$-contractive mappings using a pair $(F, h)$-upper class functions type in order to obtain new common fixed point results in the settings of metric spaces. The presented results have generalized and extended existing results in the literature.

## Conflict of interest

All the authors of this paper contributed equally. They have read and approved the final version of the paper.

The authors of this paper declare that they have no conflict of interest.

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