Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4830–4848.
DOI: 10.3934/math.2020308
Received: 27 February 2020
Accepted: 20 May 2020
Published: 02 June 2020

Research article

New subclass of q-starlike functions associated with generalized conic domain

Xiaoli Zhang[1], **Shahid Khan**[2], **Saqib Hussain**[3], **Huo Tang**[1,*] **and Zahid Shareef**[4]

[1] School of Mathematics and Computer Sciences, Chifeng University, Chifeng 024000, Inner Mongolia, People's Republic of China
[2] Department of Mathematics, Riphah International University, 44000 Islamabad, Pakistan
[3] Department of Mathematics, COMSATS University Islamabad, Abbottabad campus 22060, Pakistan
[4] Mathematics and Natural Science division, Higher Colleges of Technology, Fujairah Men's, Fujairah, United Arab Emirates

* **Correspondence:** Email: thth2009@163.com.

Abstract: In this paper, the concepts of quantum (or q-) calculus and conic regions are combined to define a new domain $\Omega_{k,q,\gamma}$ which represents the generalized conic regions. Then by using a certain generalized conic domain $\Omega_{k,q,\gamma}$ we define and investigate a new subclass of normalized analytic functions in open unit disk E. We also investigate a number of useful properties and characteristics of this subclass such as, structural formula, necessary and sufficient condition, coefficient estimates, Feketo-Szego problem, distortion inequalities, closure theorem, and subordination result. We also highlight some known consequences of our main results as corollaries.

Keywords: analytic functions; quantum (or q-) calculus; q-derivative operator; q-starlike functions; conic and generalized conic domains; subordination
Mathematics Subject Classification: 30C45, 30C50

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of all functions f which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ and has the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$ which are univalent in E (see [1]). Goodman [2] introduced $\mathcal{UCV}$ of the uniformly convex functions and $\mathcal{ST}$ of starlike functions. A function $f \in \mathcal{A}$ is called uniformly convex if every (positively oriented) circular arc of the form $\{z \in E : |z - \xi| = r\}$ and $\xi \in E$, the arc $f(\xi)$ is convex. For more details of the class $\mathcal{UCV}$ and $\mathcal{ST}$ see [3].

Later in [4] Kanas and Wisniowska introduced the class $k\text{–}\mathcal{UCV}$ and the class $k - \mathcal{ST}$, defined as:

$$f(z) \in k - \mathcal{ST} \Longleftrightarrow f(z) \in \mathcal{A} \text{ and } 1 > k\left|\frac{zf'(z)}{f'(z)} - 1\right| - \Re\left\{\frac{zf'(z)}{f'(z)}\right\}, \quad z \in E$$

and

$$f(z) \in k - \mathcal{UCV} \Longleftrightarrow f(z) \in \mathcal{A} \text{ and } 1 > k\left|\frac{zf''(z)}{f'(z)}\right| - \Re\left\{\frac{zf''(z)}{f'(z)}\right\}, \quad z \in E.$$

Note that $f(z) \in k - \mathcal{UCV} \Longleftrightarrow zf'(z) \in k - \mathcal{ST}$.

In [4], if $k \geq 0$, the class $k - \mathcal{UCV}$ is defined geometrically as a subclass of univalent functions which map the intersection of E with any disk center et ζ, $|\zeta| \leq k$, onto a convex domain. Therefore, the notion of k- uniform convexity is a generalization of the notion of convexity. For $k = 0$, the center ζ is the origin and the class $k - \mathcal{UCV}$ reduces to the class C of convex univalent functions, (see [1]). Moreover for $k = 1$ it coincides with the class of uniformly convex functions ($\mathcal{UCV}$) introduced by Goodman [2] and studied extensively by Ronning [5] and Ma and Minda [3]. We note that the class $k - \mathcal{UCV}$ started much earlier in [6] with some additional conditions but without the geometric interpretation.

We say that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_{k,\gamma}$, $k \geq 0$, $\gamma \in \mathbb{C}\backslash\{0\}$, if and only if

$$1 > k\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| - \Re\left\{\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\}, \quad z \in E.$$

For more detail about the class $\mathcal{S}^*_{k,\gamma}$, (see [7]).

If $f(z)$ and $g(z)$ are analytic in E, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in E, then we have the following equivalence, (see [1]).

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

For two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad (z \in E).$$

The convolution (Hadamard product) of $f(z)$ and $g(z)$ is defined as:

$$f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let $\mathcal{P}$ denote the well-known Carathéodory class of functions p, analytic in the open unit disk E, which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

such that

$$\Re\left(p(z)\right) > 0.$$

We have discussed above that Kanas and Wisniowska [4] introduced and studied the class $k-\mathcal{UCV}$ of k-uniformly convex functions and the corresponding class $k-\mathcal{ST}$ of k-starlike functions. Then Kanas and Wisniowska [4] defined these classes subject to the conic domain Ω_k, $(k \geq 0)$ as follows:

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\},$$

or

$$\Omega_k = \{w : \Re w > k\,|w - 1|\}.$$

This domain represents the right half plane for $k = 0$, a hyperbola for $0 < k < 1$, a parabola for $k = 1$ and an ellipse for $k > 1$. Deniz et al. [8] defined new subclasses of analytic functions subject to the conic domain $\Omega_{k,}$, (also see [9]). Theses classes were then generalized to $\mathcal{KD}(k,\gamma)$ and $\mathcal{SD}(k,\gamma)$ respectively by Shams et al. [10] subject to the conic domain $\Omega_{k,\gamma}$ $(k \geq 0)$, $0 \leq \gamma < 1$, which is

$$\Omega_{k,\gamma} = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2} + \gamma\right\},$$

or

$$\Omega_{k,\gamma} = \{w : \Re w > k\,|w - 1| + \gamma\}.$$

For this conic domains, the following function play the role of extremal function.

$$p_{k,\gamma}(z) = \begin{cases} \frac{1+z}{1-z} & \text{for } k = 0 \\ 1 + \frac{2\gamma}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & \text{for } k = 1 \\ 1 + \frac{2\gamma}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\} & \text{for } 0 < k < 1 \\ 1 + \frac{\gamma}{k^2-1}\sin\left(\frac{\pi}{2K(i)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(ix)^2}}dx\right) + \frac{\gamma}{1-k^2} & \text{for } k > 1, \end{cases} \tag{1.2}$$

where $i \in (0,1)$, $k = \cosh\left(\frac{\pi K'(i)}{4K(i)}\right)$, $K(i)$ is the first kind of Legendre's complete elliptic integral. For details (see [4]). Indeed, from (1.2), we have

$$p_{k,\gamma}(z) = 1 + Q_1 z + Q_2 z^2 + ..., \tag{1.3a}$$

where

$$Q_1 = \begin{cases} \frac{2\gamma\left(\frac{2}{\pi}\arccos k\right)^2}{1-k^2} & \text{for } 0 \leq k < 1, \\ \frac{8\gamma}{\pi^2} & \text{for } k = 1, \\ \frac{\pi^2\gamma}{4(1+t)\sqrt{t}K^2(t)\left(k^2-1\right)} & \text{for } k > 1, \end{cases} \tag{1.4}$$

$$Q_2 = \begin{cases} \frac{\left(\frac{2}{\pi}\arccos k\right)^2+2}{3} Q_1 & \text{for } 0 \le k < 1, \\ \frac{2}{3} Q_1 & \text{for } k = 1, \\ \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24K^2(t)(1+t)\sqrt{t}} Q_1 & \text{for } k > 1. \end{cases} \tag{1.5}$$

The quantum (or q-) calculus is an important tools used to study various families of analytic functions and has inspired the researchers due to its applications in mathematics and some related areas. Srivastava [11] studied univalent functions using q-calculus. The quantum (or q-)calculus is also widely applied in the approximation theory, especially for various operators, which include convergence of operators to functions in a real and complex domains. Jackson [12] was among the first few researchers who defined the q-analogue of derivative and integral operator as well as provided some of their applications. Later on, Aral and Gupta [13] introduced the q-Baskakov-Durrmeyer operator by using q-beta function while [14] studied the q-generalization of complex operators known as q-Picard and q-Gauss-Weierstrass singular integral operators. Kanas and Raducanu [15] introduced the q-analogue of Ruscheweyh differential operator and Arif et al. [16] discussed some of its applications for multivalent functions while [17] studied q-calculus by using the concept of convolution. Authors in [18] and [19] studied q-differential and q-integral operators for the class of analytic functions. Here we will present the basic definitions of quantum (or q-) calculus which will help us in onwards study.

Definition 1. *([20]). The q-number* $[t]_q$ *for* $q \in (0, 1)$ *is defined as:*

$$[t]_q = \begin{cases} \frac{1-q^t}{1-q}, & (t \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + ... + q^{n-1} & (t = n \in \mathbb{N}). \end{cases} \tag{1.6}$$

Definition 2. *The q-factorial* $[n]_q!$ *for* $q \in (0, 1)$ *is defined as:*

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases} \tag{1.7}$$

Definition 3. *The q-generalized Pochhammer symbol* $[t]_{n,q}$, $t \in \mathbb{C}$, *is defined as:*

$$[t]_{n,q} = \frac{(q^t, q)_n}{(1-q)n} = \begin{cases} 1 & (n = 0) \\ [t]_q[t+1]_q[t+2]_q...[t+n-1]_q & (n \in \mathbb{N}). \end{cases}$$

Furthermore, the q-Gamma function be defined as:

$$\Gamma_q(t+1) = [t]_q\Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Definition 4. *([12]). For* $f \in \mathcal{A}$, *the q-derivative operator or q-difference operator be defined as:*

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \in E. \tag{1.8}$$

From (1.1) and (1.8), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

For $n \in \mathbb{N}$ and $z \in E$, we have

$$D_q z^n = [n]_q z^{n-1}, \quad D_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

We can observe that

$$\lim_{q \to 1^-} D_q f(z) = f^{'}(z).$$

Definition 5. *([21]). A function $f \in \mathcal{A}$ is said to belong to the class S_q^* if*

$$f(0) = f^{'}(0) = 1, \tag{1.9}$$

and

$$\left| \frac{zD_q f(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}. \tag{1.10}$$

Equivalently, we can rewrite the conditions in (1.9) and (1.10) as follows, (see [22]).

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1+z}{1-qz}.$$

Now, making use of quantum (or q-) calculus and principle of subordination we present the following definition as:

Definition 6. *Let $k \in [0, \infty)$, $q \in (0, 1)$ and $\gamma \in \mathbb{C} \backslash \{0\}$. A function $p(z)$ is said to be in the class $k - \mathcal{P}_{q,\gamma}$ if and only if*

$$p(z) \prec p_{k,\gamma,q}(z), \tag{1.11}$$

where

$$p_{k,\gamma,q}(z) = \frac{2p_{k,\gamma}(z)}{(1+q) + (1-q)\, p_{k,\gamma}(z)}, \tag{1.12}$$

and $p_{k,\gamma}(z)$ is given by (1.2).

Geometrically, the function $p(z) \in k - \mathcal{P}_{q,\gamma}$ takes all values from the domain $\Omega_{k,q,\gamma}$ which is defined as follows:

$$\Omega_{k,q,\gamma} = \gamma \Omega_{k,q} + (1 - \gamma), \tag{1.13}$$

where

$$\Omega_{k,q} = \left\{ w : \Re \left(\frac{(1+q)\,w}{(q-1)\,w + 2} \right) > k \left| \frac{(1+q)\,w}{(q-1)\,w + 2} - 1 \right| \right\}.$$

The domain $\Omega_{k,q,\gamma}$ represents a generalized conic region.

Remark 1. *When $q \to 1^-$, then $\Omega_{k,q,\gamma} = \Omega_{k,\gamma}$, where $\Omega_{k,\gamma}$ is the conic domain considered by Shams et al [10].*

Remark 2. *When $\gamma = 1$, $q \to 1^-$, then $\Omega_{k,q,\gamma} = \Omega_k$, where Ω_k is the conic domain considered by by Kanas and Wisniowska [7].*

Remark 3. *For $\gamma = 1$, $q \to 1^-$, then $k - \mathcal{P}_{q,\gamma} = \mathcal{P}(p_k)$, where $\mathcal{P}(p_k)$ is the well-known class introduced by Kanas and Wisniowska [7].*

Remark 4. *For $\gamma = 1$, $k = 0$, $q \to 1^-$, then $k - \mathcal{P}_{q,\gamma} = \mathcal{P}$, where $\mathcal{P}$ is the well-known class of analytic functions with positive real part.*

Definition 7. *A function $f \in \mathcal{A}$ is said to be in class $k - \mathcal{UST}(q,\gamma)$ if it satisfies the condition*

$$\Re\left\{1 + \frac{1}{\gamma}\left(\mathcal{J}(q, f(z)) - 1\right)\right\} > k\left|\frac{1}{\gamma}\left(\mathcal{J}(q, f(z)) - 1\right)\right|, \tag{1.14}$$

or equivalently

$$\mathcal{J}(q, f(z)) \in k - \mathcal{P}_{q,\gamma}, \tag{1.15}$$

where

$$\mathcal{J}(q, f(z)) = \frac{(1+q)\frac{zD_q f(z)}{f(z)}}{(q-1)\frac{zD_q f(z)}{f(z)} + 2}. \tag{1.16}$$

Special cases:
i. For $q \to 1^-$, then the class $k - \mathcal{UST}(q,\gamma)$ reduces to the $\mathcal{S}^*_{k,\gamma}$ (see [7]).
ii. For $\gamma = 1$ and $q \to 1^-$, then the class $k - \mathcal{UST}(q,\gamma)$ reduces to the $k - \mathcal{UCV}$ (see [4]).

Geometrically a function $f(z) \in \mathcal{A}$ is said to be in the class $k - \mathcal{UST}(q,\gamma)$, if and only if the function $\mathcal{J}(q, f(z))$ takes all values in the conic domain $\Omega_{k,q,\gamma}$ given by (1.13). Taking this geometrical interpretation into consideration, one can rephrase the above definition as:

Definition 8. *A function $f \in \mathcal{A}$ is said to be in the class $k - \mathcal{UST}(q,\gamma)$ if and only if*

$$\mathcal{J}(q, f(z)) \prec p_{k,\gamma,q}(z), \tag{1.17}$$

where $p_{k,\gamma,q}(z)$ is defined by (1.12).

We also set $k - \mathcal{UST}^-(q,\gamma) = k - \mathcal{UST}(q,\gamma) \cap T$, T is the subclass of $k - \mathcal{UST}(q,\gamma)$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \text{ for all } n \geq 2. \tag{1.18}$$

2. Set of lemmas

In order to prove our main results in this paper, we need each of the following lemmas.

Lemma 1. *(see [23]). Let $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n \prec F(z) = 1 + \sum\limits_{n=1}^{\infty} C_n z^n$. If $F(z)$ is convex univalent in E, then*

$$|p_n| \leq |C_1|, \quad n \geq 1.$$

Lemma 2. *Let $k \in [0,\infty)$ be fixed and*

$$p_{k,\gamma,q}(z) = \frac{2p_{k,\gamma}(z)}{(1+q)+(1-q)\,p_{k,\gamma}(z)}.$$

Then

$$p_{k,\gamma,q}(z) = 1 + \frac{2}{1+q}Q_1 z + \left\{\frac{2}{1+q}Q_2 - \frac{2(1-q)}{1+q}Q_1^2\right\}z^2 + \cdots,$$

where Q_1, and Q_2 is given by (1.4) *and* (1.5) .

Proof. From (1.12), we have

$$\begin{aligned} p_{k,\gamma,q}(z) &= \frac{2p_{k,\gamma}(z)}{(1+q)+(1-q)\,p_{k,\gamma}(z)} \\ &= \frac{2}{(1+q)}\left\{p_{k,\gamma}(z)\right\} - \frac{2(1-q)}{(1+q)^2}\left\{p_{k,\gamma}^2(z)\right\} + \frac{2(1-q)^2}{(1+q)^3}\left\{p_{k,\gamma}^3(z)\right\} \\ &\quad - \frac{2(1-q)^3}{(1+q)^4}\left\{p_{k,\gamma}^4(z)\right\} + \cdots. \end{aligned} \tag{2.1}$$

By using (1.3a) in (2.1), we have

$$\begin{aligned} p_{k,\gamma,q}(z) &= \sum_{n=1}^{\infty}\frac{2(-1)^{n-1}(1-q)^{n-1}}{(1+q)^n} + \sum_{n=1}^{\infty}\frac{2n(-1)^{n-1}(1-q)^{n-1}}{(1+q)^{n+1}}Q_1 z \\ &\quad + \left\{\sum_{n=1}^{\infty}\frac{2n(-1)^{n-1}(1-q)^{n-1}}{(1+q)^{n+1}}Q_2 \right. \\ &\quad \left. - \sum_{n=1}^{\infty}\frac{2(2n-1)(-1)^{n-1}(1-q)^{n}}{(1+q)^{n+1}}Q_1^2\right\}z^2 + \cdots. \end{aligned} \tag{2.2}$$

The series $\sum_{n=1}^{\infty}\frac{2(-1)^{n-1}(1-q)^{n-1}}{(1+q)^n}$, $\sum_{n=1}^{\infty}\frac{2n(-1)^{n-1}(1-q)^{n-1}}{(1+q)^{n+1}}$, and $\sum_{n=1}^{\infty}\frac{2(2n-1)(-1)^{n-1}(1-q)^{n}}{(1+q)^{n+1}}$ are convergent and convergent to 1, $\frac{2}{1+q}$, and $\frac{2(1-q)}{(1+q)}$.

Therefore (2.2) becomes

$$p_{k,\gamma,q}(z) = 1 + \frac{2}{1+q}Q_1 z + \left\{\frac{2}{1+q}Q_2 - \frac{2(1-q)}{1+q}Q_1^2\right\}z^2 + \cdots. \tag{2.3}$$

This complete the proof of Lemma 2. □

Remark 5. *When $q \to 1^-$, the Lemma 2, reduces to the lemma which was introduced by Sim et. al [24].*

Lemma 3. *Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in k - \mathcal{P}_{q,\gamma}$, then*

$$|p_n| \le \frac{2}{1+q}|Q_1|, \quad n \ge 1.$$

Proof. By definition (6), a function $p(z) \in k - \mathcal{P}_{q,\gamma}$ if and only if

$$p(z) \prec p_{k,\gamma,q}(z), \tag{2.4}$$

where $k \in [0, \infty)$, and $p_{k,\gamma}(z)$ is given by (1.2).

By using (2.3) in (2.4), we have

$$p(z) \prec 1 + \frac{2}{1+q} Q_1 z + \left\{ \frac{2}{1+q} Q_2 - \frac{2(1-q)}{1+q} Q_1^2 \right\} z^2 + \cdots. \tag{2.5}$$

Now by using Lemma 1 on (2.5), we have

$$|p_n| \leq \frac{2}{1+q} |Q_1|.$$

Hence the proof of Lemma 3 is complete. □

Remark 6. *When $q \to 1^-$, then Lemma 3 reduces to the lemma which was introduced by Noor et. al [25].*

Lemma 4. *[26]. Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $h(z)$ be analytic in E and satisfy $Re\{h(z)\} > 0$ for z in E, then the following sharp estimate holds;*

$$\left| c_2 - v c_1^2 \right| \leq 2 \max\{1, |2v - 1|\}, \quad \forall v \in \mathbb{C}.$$

3. Main results

Theorem 1. *If a function $f \in \mathcal{A}$ of the form (1.1) and it satisfies*

$$\sum_{n=2}^{\infty} \left\{ 2(k+1) q [n-1]_q + |\gamma| \left\{ \left| (q-1) [n]_q \right| + 2 \right\} \right\} |a_n| \leq (q+1) |\gamma|, \tag{3.1}$$

then $f(z) \in k - \mathcal{UST}(q, \gamma)$.

Proof. Assume that (3.1) is holds, then it is suffice to show that

$$\left| \frac{k}{\gamma} (\mathcal{J}(q, f(z) - 1) \right| - \Re \left\{ \frac{1}{\gamma} (\mathcal{J}(q, f(z)) - 1) \right\} \leq 1.$$

Using (1.16), we have

$$\begin{aligned}
&\left|\frac{k}{\gamma}\left(\frac{(1+q)\frac{zD_qf(z)}{f(z)}}{(q-1)\frac{zD_qf(z)}{f(z)}+2}-1\right)\right|-\Re\left\{\frac{1}{\gamma}\left(\frac{(1+q)\frac{zD_qf(z)}{f(z)}}{(q-1)\frac{zD_qf(z)}{f(z)}+2}-1\right)\right\}\\
\leq\ &\frac{k}{|\gamma|}\left|\frac{(1+q)\frac{zD_qf(z)}{f(z)}}{(q-1)\frac{zD_qf(z)}{f(z)}+2}-1\right|+\frac{1}{|\gamma|}\left|\frac{(1+q)\frac{zD_qf(z)}{f(z)}}{(q-1)\frac{zD_qf(z)}{f(z)}+2}-1\right|,\\
\leq\ &\frac{(k+1)}{|\gamma|}\left|\frac{(1+q)\frac{zD_qf(z)}{f(z)}}{(q-1)\frac{zD_qf(z)}{f(z)}+2}-1\right|,\\
=\ &\frac{2(k+1)}{|\gamma|}\left|\frac{\sum_{n=2}^{\infty}q[n-1]_q a_n z^n}{(q+1)+\sum_{n=2}^{\infty}\left\{(q-1)[n]_q+2\right\}a_n z^n}\right|,\\
\leq\ &\frac{2(k+1)}{|\gamma|}\left\{\frac{\sum_{n=2}^{\infty}\left|q[n-1]_q\right||a_n|}{(q+1)-\sum_{n=2}^{\infty}\left|(q-1)[n]_q+2\right||a_n|}\right\}.
\end{aligned}$$

The last expression is bounded above by 1.

$$\frac{2(k+1)}{|\gamma|}\left\{\frac{\sum_{n=2}^{\infty}\left|q[n-1]_q\right||a_n|}{(q+1)-\sum_{n=2}^{\infty}\left|(q-1)[n]_q+2\right||a_n|}\right\}<1.$$

After some simple calculation we have

$$\sum_{n=2}^{\infty}\left\{2(k+1)q[n-1]_q+|\gamma|\left\{\left|(q-1)[n]_q\right|+2\right\}\right\}|a_n|\leq(q+1)|\gamma|.$$

Hence we complete the proof of Theorem 1. □

When $q\to 1^-$ and $\gamma=1-\alpha$ with $0\leq\alpha<1$, we have the following known result proved by Shams et. al in [10].

Corollary 1. *A function $f\in\mathcal{A}$ and of the form (1.1) is in the class $k-\mathcal{UST}(1-\alpha)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\alpha)\}|a_n|\leq 1-\alpha,$$

where $0\leq\alpha<1$ and $k\geq 0$.

Inequality (3.1) gives us a tool to obtain some special member of $k-\mathcal{UST}(q,\gamma)$. Thus we have the following corollary:

Corollary 2. *Let $0\leq k<\infty$, $q\in(0,1)$ and $\gamma\in\mathbb{C}\backslash\{0\}$. If the inequality*

$$|a_n|\leq\frac{(q+1)|\gamma|}{\left\{2(k+1)q[n-1]_q+|\gamma|\left\{\left|(q-1)[n]_q\right|+2\right\}\right\}},\quad n\geq 2,$$

holds for $f(z)=z+a_nz^n$, then $k-\mathcal{UST}(q,\gamma)$. In particular,

$$f(z)=z+\frac{(q+1)|\gamma|}{\left\{2q(k+1)+|\gamma|\left\{\left|1-q^2\right|+2\right\}\right\}}z^2\in k-\mathcal{UST}(q,\gamma),$$

and

$$|a_2|=\frac{(q+1)|\gamma|}{\left\{2q(k+1)+|\gamma|\left\{\left|1-q^2\right|+2\right\}\right\}}.$$

Theorem 2. *If* $f(z)\in k-\mathcal{UST}(q,\gamma)$ *and is of the form* (1.1). *Then*

$$|a_2|\leq\frac{|Q_1|\varphi_0}{q(1+q)}\tag{3.2}$$

and

$$|a_n|\leq\prod_{j=0}^{n-2}\left(\frac{\left|Q_1-q[j]_q\right|}{q(q+1)[j+1]_q}\right)\varphi_j\quad for\quad n\geq3,\tag{3.3}$$

where Q_1 *and* φ_j *are defined by* (1.4) *and* (3.6).

Proof. Let

$$\frac{(1+q)\frac{zD_qf(z)}{f(z)}}{(q-1)\frac{zD_qf(z)}{f(z)}+2}=p(z).\tag{3.4}$$

Now from (3.4), we have

$$(1+q)\,zD_qf(z)=\left\{(q-1)\,zD_qf(z)+2f(z)\right\}p(z),$$

which implies that

$$\begin{aligned}&z+\sum\nolimits_{n=2}^{\infty}\left(\frac{2q[n-1]_q}{q+1}\right)a_nz^n\\=\;&\left(1+\sum\nolimits_{n=1}^{\infty}c_nz^n\right)\left(z+\sum\nolimits_{n=2}^{\infty}\left(\frac{[n]_q(q-1)+2}{q+1}\right)a_nz^n\right).\end{aligned}$$

Equating coefficients of z^n on both sides, we have

$$\left(\frac{2q[n-1]_q}{q+1}\right)a_n=\sum\nolimits_{j=1}^{n-1}\left(\frac{[j-1]_q(q-1)+2}{q+1}\right)a_{n-j}c_j,\quad a_1=1.$$

This implies that

$$|a_n|\leq\frac{1}{2q[n-1]_q}\sum\nolimits_{j=1}^{n-1}\left\{[j-1]_q(q-1)+2\right\}\left|a_{n-j}\right|\left|c_j\right|.$$

By using Lemma 3, we have

$$|a_n| \leq \frac{|Q_1|}{q(1+q)[n-1]_q} \sum_{j=1}^{n-1} \left\{[j-1]_q (q-1)+2\right\} \left|a_j\right|,$$

$$|a_n| \leq \frac{|Q_1|}{q(1+q)[n-1]_q} \sum_{j=1}^{n-1} \varphi_{j-1} \left|a_j\right|, \tag{3.5}$$

where

$$\varphi_{j-1} = [j-1]_q (q-1)+2. \tag{3.6}$$

Now we prove that

$$\frac{|Q_1|}{q(1+q)[n-1]_q} \sum_{j=1}^{n-1} \varphi_{j-1} \left|a_j\right| \leq \prod_{j=0}^{n-2} \left(\frac{\left|Q_1 - q[j]_q\right|}{q(1+q)[j+1]_q}\right) \varphi_j. \tag{3.7}$$

For this we use the induction method. For $n=2$ from (3.5) we have

$$|a_2| \leq \frac{|Q_1|\varphi_0}{q(1+q)}.$$

From (3.3) we have

$$|a_2| \leq \frac{|Q_1|\varphi_0}{q(1+q)}.$$

For $n=3$, from (3.5), we have

$$\begin{aligned} |a_3| &\leq \frac{|Q_1|}{q(1+q)[2]_q} \left(\varphi_0 + \varphi_1 |a_2|\right), \\ &\leq \frac{|Q_1|\varphi_0}{q(1+q)[2]_q} \left(1 + \frac{|Q_1|\varphi_1}{q(1+q)}\right). \end{aligned}$$

From (3.3), we have

$$\begin{aligned} |a_3| &\leq \frac{|Q_1|\varphi_0}{q(1+q)} \left\{\left(\frac{\left|Q_1 - q[1]_q\right|}{q(1+q)[2]_q}\right)\varphi_1\right\}, \\ &\leq \frac{|Q_1|\varphi_0}{q(1+q)} \left\{\left(\frac{|Q_1| + q[1]_q}{q(1+q)[2]_q}\right)\varphi_1\right\}, \\ &= \frac{|Q_1|\varphi_1}{q(1+q)[2]_q} \left(\frac{|Q_1|\varphi_0}{q(1+q)} + \frac{\varphi_0}{(1+q)}\right), \\ &= \frac{|Q_1|\varphi_1}{q(1+q)[2]_q} \left(\frac{|Q_1|\varphi_0}{q(1+q)} + \frac{2}{(1+q)}\right). \end{aligned}$$

Let the hypothesis be true for $n=m$. From (3.5), we have

$$|a_m| \leq \frac{|Q_1|}{q(1+q)[m-1]_q} \sum_{j=1}^{m-1} \varphi_{j-1} \left|a_j\right|.$$

From (3.3) we have

$$
\begin{aligned}
|a_m| &\leq \prod_{j=0}^{m-2}\left(\frac{\left|Q_1 - q[j]_q\right|}{q\,(1+q)\,[j+1]_q}\right)\varphi_j, \quad n \geq 2,\\
&\leq \prod_{j=0}^{m-2}\left(\frac{|Q_1| + q[j]_q}{q\,(1+q)\,[j+1]_q}\right)\varphi_j, \quad n \geq 2.
\end{aligned}
$$

By the induction hypothesis, we have

$$
\frac{|Q_1|}{q\,(1+q)\,[m-1]_q}\sum_{j=1}^{m-1}\varphi_{j-1}\left|a_j\right| \leq \prod_{j=0}^{m-2}\left(\frac{|Q_1| + q[j]_q}{q\,(1+q)\,[j+1]_q}\right)\varphi_j. \tag{3.8}
$$

Multiplying $\frac{|Q_1|+q(q+1)[m-1,q]}{q(1+q)[m-1]_q}$ on both sides of (3.8), we have

$$
\begin{aligned}
&\prod_{j=0}^{m-2}\left(\frac{|Q_1| + q[j]_q}{q\,(1+q)\,[j+1]_q}\right)\varphi_j,\\
\geq\ & \frac{|Q_1| + q(q+1)[m-1,q]}{q\,(1+q)\,[m-1]_q}\left\{\frac{|Q_1|}{q\,(1+q)\,[m-1]_q}\sum_{j=1}^{m-1}\varphi_{j-1}\left|a_j\right|\right\},\\
=\ & \frac{|Q_1|}{q\,(1+q)\,[m-1]_q}\left\{\frac{|Q_1|}{q\,(1+q)\,[m-1]_q}+1\right\}\sum_{j=1}^{m-1}\varphi_{j-1}\left|a_j\right|,\\
\geq\ & \frac{|Q_1|}{q\,(1+q)\,[m-1]_q}\left\{|a_m| + \sum_{j=1}^{m-1}\varphi_{j-1}\left|a_j\right|\right\},\\
=\ & \frac{|Q_1|}{q\,(1+q)\,[m-1]_q}\sum_{j=1}^{m}\varphi_{j-1}\left|a_j\right|.
\end{aligned}
$$

That is,

$$
\frac{|Q_1|}{q\,(1+q)\,[m-1]_q}\sum_{j=1}^{m}\varphi_{j-1}\left|a_j\right| \leq \prod_{j=0}^{m-2}\left(\frac{|Q_1| + q[j]_q}{q\,(1+q)\,[j+1]_q}\right)\varphi_j,
$$

which shows that inequality (3.8) is true for $n = m + 1$. Hence the proof of Theorem 2 is complete □

When $q \to 1^-$, then we have the following known result, proved by Kanas and Wisniowska in [4].

Corollary 3. *If* $f(z) \in k - \mathcal{UST}(q,\gamma)$ *and is of the form* (1.1). *Then*

$$
|a_n| \leq \prod_{j=0}^{n-2}\left(\frac{|Q_1 - j|}{(j+1)}\right) \ \textit{for} \ \ n \geq 3.
$$

Theorem 3. *Let* $0 \leq k < \infty$, $q \in (0,1)$, *be fixed and let* $f(z) \in k - \mathcal{UST}(q,\gamma)$ *and is of the form* (1.1). *Then for a complex number* μ,

$$
\left|a_3 - \mu a_2^2\right| \leq \frac{|Q_1|}{2q[2]_q}\max\{1, |2v-1|\}, \tag{3.9}
$$

where v *is given by* (3.13).

Proof. If $f(z) \in k-\mathcal{UST}(q,\gamma)$, then there exist a Schwarz function $w(z)$ with $w(0)=0$ and $|w(z)|<1$, such that

$$\begin{aligned}\mathcal{J}(q,f(z)) &\prec p_{k,\gamma,q}(z),\\ \frac{(1+q)\frac{zD_qf(z)}{f(z)}}{(q-1)\frac{zD_qf(z)}{f(z)}+2} &= p_{k,\gamma,q}(w(z)).\end{aligned} \tag{3.10}$$

Let $h(z) \in \mathcal{P}$ be a function defined as:

$$h(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots,$$

This gives

$$w(z)=\frac{c_1}{2}z+\frac{1}{2}(c_2-\frac{c_1^2}{2})z^2+\cdots$$

and

$$p_{k,\gamma,q}(w(z))=1+\frac{Q_1c_1}{(1+q)}z+\frac{1}{(1+q)}\left\{\frac{Q_2c_1^2}{2}+(c_2-\frac{c_1^2}{2})Q_1-\frac{(1-q)Q_1^2c_1^2}{2}\right\}z^2+\cdots. \tag{3.11}$$

By using (3.11) in (3.10) we obtain

$$a_2=\frac{Q_1c_1}{2q},$$

and

$$a_3=\frac{1}{2q\,[2]_q}\left\{\frac{Q_1^2c_1^2}{2}+\left\{\left(c_2-\frac{c_1^2}{2}\right)Q_1-\frac{(1-q)}{2}Q_1^2c_1^2\right\}+\frac{\left\{(q-1)\,[2]_q+2\right\}Q_1^2c_1^2}{2q(1+q)}\right\}.$$

For any complex number μ we have

$$\left|a_3-\mu a_2^2\right|=\frac{|Q_1|}{2q[2]_q}\left\{c_2-vc_1^2\right\}, \tag{3.12}$$

where

$$v=\frac{1}{2}\left\{1-\frac{Q_2}{Q_1}+(1-q)Q_1-\frac{\left\{(q-1)\,[2]_q+2\right\}Q_1}{q(1+q)}+\mu\frac{[2]_q\,Q_1}{q}\right\}. \tag{3.13}$$

Now by using Lemma 4 on (3.12) we have

$$\left|a_3-\mu a_2^2\right|\leq\frac{|Q_1|}{2q[2]_q}\max\{1,|2v-1|\}.$$

Hence we complete the proof of Theorem 3. □

Theorem 4. *Let $k \in [0,\infty)$, $q \in (0,1)$ and $\gamma \in \mathbb{C}\backslash\{0\}$. A necessary and sufficient condition for $f(z)$ of the form (1.18) to be in the class $k-\mathcal{UST}^-(q,\gamma)$ can be formulated as follows:*

$$\sum_{n=2}^{\infty}\left\{2(k+1)q[n-1]_q+|\gamma|\left\{\left|(q-1)\,[n]_q\right|+2\right\}\right\}a_n\leq(q+1)\,|\gamma|. \tag{3.14}$$

The result is sharp for the function

$$f(z) = z - \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}} z^n.$$

Proof. In view of Theorem 1, it remains to prove the necessity. If $f \in k - \mathcal{UST}^{-}(q,\gamma)$, then in virtue of the fact that $\left|\Re(z)\right| \le |z|$, for any z, we have

$$\left|1 + \frac{1}{\gamma\,(q+1)}\left(\frac{\sum_{n=2}^{\infty} 2q[n-1]_q a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{1}{(q+1)}\left\{(q-1)\,[n]_q + 2\right\} a_n z^{n-1}}\right)\right|$$
$$\geq \left|\frac{k}{\gamma\,(q+1)}\left\{\frac{\sum_{n=2}^{\infty} 2q[n-1]_q a_n z^{n-1}}{1 - \sum_{n=2}^{\infty}\left\{(q-1)\,[n]_q + 2\right\} a_n z^{n-1}}\right\}\right|. \quad (3.15)$$

Letting $z \to 1^{-}$, along the real axis, we obtain the desired inequality (3.14). Hence we complete the proof of Theorem 4. □

Corollary 4. *Let the function $f(z)$ of the form (1.18) be in the class $k - \mathcal{UST}^{-}(q,\gamma)$. Then*

$$a_n \le \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}}, \quad n \ge 2. \quad (3.16)$$

Corollary 5. *Let the function $f(z)$ of the form (1.18) be in the class $k - \mathcal{UST}^{-}(q,\gamma)$. Then*

$$a_2 = \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|1-q^2\right| + 2\right\}\right\}}. \quad (3.17)$$

Theorem 5. *Let $k \in [0,\infty)$, $q \in (0,1)$ and $\gamma \in \mathbb{C}\backslash\{0\}$ and let*

$$f_1(z) = z,$$

and

$$f_n(z) = z - \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}} z^n, \quad n \ge 3. \quad (3.18)$$

Then $f \in k - \mathcal{UST}^{-}(q,\gamma)$, if and only if f can be expressed in the form of

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \quad \lambda_n > 0, \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = 1. \quad (3.19)$$

Proof. Suppose that

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty}\lambda_n f_n(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty}\lambda_n f_n(z),\\
&= \lambda_1 f_1(z) + \sum_{n=2}^{\infty}\lambda_n\left\{z - \frac{(q+1)\,|\gamma|}{\left\{2q(k+1)[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}}z^n\right\},\\
&= \lambda_1 z + \sum_{n=2}^{\infty}\lambda_n z - \sum_{n=2}^{\infty}\lambda_n\frac{(q+1)\,|\gamma|}{\left\{2q(k+1)[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}}z^n,\\
&= \left(\sum_{n=1}^{\infty}\lambda_n\right)z - \sum_{n=2}^{\infty}\lambda_n\frac{(q+1)\,|\gamma|}{\left\{2q(k+1)[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}}z^n,\\
&= z - \sum_{n=2}^{\infty}\lambda_n\frac{(q+1)\,|\gamma|}{\left\{2q(k+1)[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}}z^n.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{n=2}^{\infty}\lambda_n\frac{(q+1)\,|\gamma|\left\{2q(k+1)[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}}{\left\{2q(k+1)[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}(q+1)\,|\gamma|}\\
= &\sum_{n=2}^{\infty}\lambda_n = \sum_{n=1}^{\infty}\lambda_n - \lambda_1 = 1 - \lambda_1 \leq 1,
\end{aligned}$$

and we find $k - \mathcal{UST}^{-}(q,\gamma)$.

Conversely, assume that $k - \mathcal{UST}^{-}(q,\gamma)$. Since

$$|a_n| \leq \frac{(q+1)\,|\gamma|}{\left\{2q(k+1)[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}},$$

we can set

$$\lambda_n = \frac{\left\{2q(k+1)[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}}{(q+1)\,|\gamma|}\,|a_n|\,,$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n.$$

Then

$$\begin{aligned}
f(z) &= z + \sum_{n=2}^{\infty}a_n z^n = z + \sum_{n=2}^{\infty}\lambda_n\frac{(q+1)\,|\gamma|}{\left\{2(k+1)q[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right| + 2\right\}\right\}}z^n,\\
&= z + \sum_{n=2}^{\infty}\lambda_n(z + f_n(z)) = z + \sum_{n=2}^{\infty}\lambda_n z + \sum_{n=2}^{\infty}\lambda_n f_n(z),\\
&= \left(1 - \sum_{n=2}^{\infty}\lambda_n\right)z + \sum_{n=2}^{\infty}\lambda_n f_n(z) = \lambda_1 z + \sum_{n=2}^{\infty}\lambda_n f_n(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z).
\end{aligned}$$

The proof of Theorem 5 is complete. □

Theorem 6. *Let* $k \in [0,\infty)$, $q \in (0,1)$, *and* $\gamma \in \mathbb{C}\backslash\{0\}$. *Let* f *defined by (1.18) belongs to the class* $k-\mathcal{UST}^{-}(q,\gamma)$. *Thus for* $|z| = r < 1$, *the following inequality is true:*

$$r - \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|1-q^2\right|+2\right\}\right\}} r^2 \le |f(z)| \le r + \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|1-q^2\right|+2\right\}\right\}} r^2. \tag{3.20}$$

Equality in (3.20) is attained for the function f given by the formula

$$f(z) = z + \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|1-q^2\right|+2\right\}\right\}} z^2. \tag{3.21}$$

Proof. Since $f \in k-\mathcal{UST}^{-}(q,\gamma)$, in view of Theorem 4 we find

$$\begin{aligned} &\left\{2(k+1)q + |\gamma|\left\{\left|(q-1)\,[2]_q\right|+2\right\}\right\} \sum_{n=2}^{\infty} a_n \\ \le\ &\sum_{n=2}^{\infty} \left\{2(k+1)q[n-1]_q + |\gamma|\left\{\left|(q-1)\,[n]_q\right|+2\right\}\right\} |a_n| \le (q+1)\,|\gamma|\,. \end{aligned}$$

This gives

$$\sum_{n=2}^{\infty} a_n \le \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|(q-1)\,[2]_q\right|+2\right\}\right\}}. \tag{3.22}$$

Therefore

$$|f(z)| \le |z| + \sum_{n=2}^{\infty} a_n\,|z|^n \le r + \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|(q-1)\,[2]_q\right|+2\right\}\right\}} r^2,$$

and

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} a_n\,|z|^n \ge r - \frac{(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|(q-1)\,[2]_q\right|+2\right\}\right\}} r^2.$$

The required results follows by letting $r \to 1^-$. Hence the proof of Theorem 6 is complete. □

Theorem 7. *Let* $k \in [0,\infty)$, $q \in (0,1)$, *and* $\gamma \in \mathbb{C}\backslash\{0\}$. *Let* f *defined by (1.18) belongs to the class* $k-\mathcal{UST}^{-}(q,\gamma)$. *Thus, for* $|z| = r < 1$, *the following inequality is true:*

$$1 - \frac{2\,(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|1-q^2\right|+2\right\}\right\}} r \le \left|f'(z)\right| \le 1 + \frac{2\,(q+1)\,|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|1-q^2\right|+2\right\}\right\}} r. \tag{3.23}$$

Proof. Differentiating f and using triangle inequality for the modulus, we obtain

$$\left|f'(z)\right| \le 1 + \sum_{n=2}^{\infty} n a_n\,|z|^{n-1} \le 1 + r\sum_{n=2}^{\infty} n a_n, \tag{3.24}$$

and

$$\left|f'(z)\right| \geq 1 - \sum_{n=2}^{\infty} na_n |z|^{n-1} \geq 1 - r\sum_{n=2}^{\infty} na_n. \tag{3.25}$$

Assertion (3.23) follows from (3.24) and (3.25) in view of rather simple consequence of (3.22) given by the inequality

$$\sum_{n=2}^{\infty} na_n \leq \frac{2(q+1)|\gamma|}{\left\{2(k+1)q + |\gamma|\left\{\left|(q-1)[2]_q\right| + 2\right\}\right\}}.$$

Hence we complete the proof of Theorem 7. □

Theorem 8. *The class $k - \mathcal{UST}^{-}(q,\gamma)$ is closed under convex linear combination.*

Proof. Let the functions $f(z)$ and $g(z)$ are in class $k - \mathcal{UST}^{-}(q,\gamma)$. Suppose $f(z)$ is given by (1.18) and

$$g(z) = z - \sum_{n=2}^{\infty} d_n z^n, \tag{3.26}$$

where a_n, $d_n \geq 0$.

It is sufficient to prove that for $0 \leq \lambda \leq 1$, the function

$$H(z) = \lambda f(z) + (1-\lambda)g(z), \tag{3.27}$$

is also in the class $k - \mathcal{UST}^{-}(q,\gamma)$.

From (1.18), (3.26) and (3.27), we have

$$H(z) = z - \sum_{n=2}^{\infty} \{\lambda a_n + (1-\lambda)d_n\} z^n. \tag{3.28}$$

As $f(z)$ and $g(z)$ are in class $k - \mathcal{UST}^{-}(q,\gamma)$ and $0 \leq \lambda \leq 1$, so by using Theorem 4, we obtain

$$\sum_{n=2}^{\infty} \left\{2(k+1)q[n-1]_q + |\gamma|\left\{\left|(q-1)[n]_q\right| + 2\right\}\right\}\{\lambda a_n + (1-\lambda)d_n\} \leq (1+q)|\gamma|. \tag{3.29}$$

Again by Theorem 4 and inequality (3.29), we have $H(z) \in k - \mathcal{UST}^{-}(q,\gamma)$. Hence the proof of Theorem 8 is complete. □

4. Conclusions

In this paper, motivated significantly by a number of recent works, we have made use of a certain general conic domain $\Omega_{k,q,\gamma}$ and the quantum (or q-) calculus in order to define and investigate a new subclass of normalized analytic functions in the open unit disk E and we have successfully derived several properties and characteristics of newly defined subclass of analytic functions. For verification and validity of our main results we have also pointed out relevant connections of our main results with those in several earlier related works on this subject.

For further investigation, we can make obvious connections between the q-analysis and (p,q)-analysis and the results for q-analogues which we have consider in this article for $0 < q < 1$, can easily be translated into the corresponding results for the (p,q)-analogues with $(0 < q < p \leq 1)$ by applying some obvious parameter and argument variations.

Acknowledgments

This work was supported by the Natural Science Foundation of the Peoples Republic of China (Grant No. 11561001), the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region (Grant No. NJYT-18-A14), the Natural Science Foundation of Inner Mongolia of the Peoples Republic of China (Grant No. 2018MS01026), the Higher School Foundation of Inner Mongolia of the Peoples Republic of China (Grant No. NJZY18217) and the Natural Science Foundation of Chifeng of Inner Mongolia.

Conflict of interest

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

References

1. A. W. Goodman, *Univalent Functions*, vols. I, II. Polygonal Publishing House, New Jersey 1983.
2. A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., **56** (1991), 87–92.
3. W. Ma, D. Minda, *Uniformly convex functions*, Ann. Polon. Math., **57** (1992), 165–175.
4. S. Kanas, A. Wisniowska, *Conic domains and k-starlike functions*, Rev. Roum. Math. Pure Appl., **45** (2000), 647–657.
5. F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Am. Math. Soc., **118** (1993), 189–196.
6. K. G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahmanyam, et al. *Subclasses of uniformly convex and uniformly starlike functions*, Math. Jpn., **42** (1995), 517–522.
7. S. Kanas, A. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math., **105** (1999), 327–336.
8. E. Deniz, M. Caglar, H. Orhan, *The Fekete-Szego problem for a class of analytic functions defined by Dziok-Srivastava operator*, Kodai Math. J., **35** (2012), 439–462.
9. E. Deniz, H. Orhan, J. Sokol, *Classes of analytic functions defined by a differential operator related to conic domains*, Ukr. Math. J., **67** (2016), 1367–1385.
10. S. Shams, S. R. Kulkarni, J. M. Jahangiri, *Classes of uniformly starlike and convex functions*, Int. J. Math. Math. Sci., **55**, (2004), 2959–2961.
11. H. M. Srivastava, *Univalent functions, fractional calculus, and associated generalized hypergeometric functions*, In: H. M. Srivastava, S. Owa, *Univalent functions, fractional Calculus, and Their Applications*, John Wiley Sons, New York, etc. 1989.
12. F. H. Jackson, *On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh*, **46** (1908), 253–281.
13. A. Aral, V. Gupta, *On q-Baskakov type operators*, Demon-stratio Math., **42**, (2009), 109–122.

14. A. Aral, *On the generalized Picard and Gauss Weierstrass singular integrals*, J. Comput. Anal. Appl., **8** (2006), 249–261.
15. S. Kanas, D. Raducanu, *Some class of analytic functions related to conic domains*, Math. Slovaca, **64** (2014), 1183–1196.
16. M. Arif, H. M. Srivastava, S. Uma, *Some applications of a q-analogue of the Ruscheweyh type operator for multivalent functions*, RACSAM, **113** (2019), 1211–1221.
17. M. Arif, Z. G. Wang, R. Khan, et al. *Coefficient inequalities for Janowski-Sakaguchi type functions associated with conic regions*, Hacet. J. Math. Stat., **47** (2018), 261–271.
18. L. Shi, M. Raza, K. Javed, et al. *Class of analytic functions defined by q-integral operator in a symmetric region*, Symmetry, **11** (2019), 1042.
19. H. M. Srivastava, S. Khan, Q. Z. Ahmad, et al. *The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator*, Stud. Univ. Babe s-Bolyai Math., **63** (2018), 419–436.
20. G. Gasper, M. Rahman, *Basic Hpergeometric series, vol. 35 of Encyclopedia of Mathematics and its applications*, Ellis Horwood, Chichester, UK, 1990.
21. M. E. H. Ismail, E. Merkes, D. Styer, *A generalization of starlike functions*, Complex Var. Theory Appl., **14** (1990), 77–84.
22. H. E. O. Uçar, *Coefficient inequality for q-starlike Functions*, Appl. Math. Comput., **76** (2016), 122–126.
23. W. Rogosinski, *On the coefficients of subordinate functions*, Proc. Lond. Math. Soc., **48** (1943), 48–82.
24. Y. J. Sim, O. S. Kwon, N. E. Cho, et al. *Some classes of analytic functions associated with conic regions*, Taiwan J. Math., **16** (2012), 387–408.
25. K. I. Noor, M. Arif, W. Ul-Haq, *On k-uniformly close-to-convex functions of complex order*, Appl. Math. Comput., **215**, (2009), 629–635.
26. W. Ma, D. Minda, *A unified treatment of some special classes of univalent functions*. In: Z. Li, F. Ren, L. Yang, et al. (Eds.) *Proceedings of the Conferene on Complex Analysis*, Int. Press Inc. (1992), 157–169.

AIMS Press