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Research article

Derivation of bounds of integral operators via convex functions

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Abstract: Convex functions play a vital role in the derivation of inequalities. In this paper these functions are used to obtain certain bounds of a unified integral operator. A Hadamard inequality for these operators is established. Further bounds of several kinds of fractional and conformable integral operators are deduced in particular.

Keywords: integral operators; fractional integrals; convex functions; bounds
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1. Introduction

Fractional integral operators play an important part in the advancement of fractional calculus. We start with the definitions of some known generalized fractional integral operators.

Definition 1. *[1] Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ where $\Re(\mu) > 0$ are defined by:*

$$ {}_g^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (g(x) - g(t))^{\mu-1} g'(t) f(t)dt, \;\; x > a \tag{1.1}$$

and

$$ {}_g^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (g(t) - g(x))^{\mu-1} g'(t) f(t)dt, \;\; x < b, \tag{1.2}$$

where $\Gamma(.)$ *is the gamma function.*

Definition 2. *[2] Let* $f : [a,b] \to \mathbb{R}$ *be an integrable function. Also let* g *be an increasing and positive function on* $(a,b]$*, having a continuous derivative* g' *on* (a,b)*. The left-sided and right-sided fractional integrals of a function* f *with respect to another function* g *on* $[a,b]$ *of order* μ *where* $\mu \in \mathbb{C}$, $\Re(\mu), k > 0$ *are defined by:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1} g'(t) f(t)dt, \ \ x > a \tag{1.3} $$

and

$$ {}_g^{\mu}I_{b^-}^{k} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t)-g(x))^{\frac{\mu}{k}-1} g'(t) f(t)dt, \ \ x < b, \tag{1.4} $$

where $\Gamma_k(.)$ *is k-gamma function [3] defined as follows:*

$$ \Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \ \Re(x) > 0. \tag{1.5} $$

The well known Mittag-Leffler function has been generalized/extended by many mathematicians, one can see the references [4–7]. There are many fractional integral operators which have been defined by using Mittag-Leffler functions in their kernels. A generalized fractional integral operator containing an extended Mittag-Leffler function is defined as follows:

Definition 3. *[8] Let* $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0$, $\delta > 0$ *and* $0 < k \leq \delta + \Re(\mu)$*. Let* $f \in L_1[a,b]$ *and* $x \in [a,b]$*. Then the generalized fractional integral operators* $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f$ *and* $\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f$ *are defined by:*

$$ \left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x;p) = \int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-t)^{\mu};p) f(t)dt, \tag{1.6} $$

and

$$ \left(\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(x;p) = \int_x^b (t-x)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(t-x)^{\mu};p) f(t)dt, \tag{1.7} $$

where

$$ E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{t^n}{(l)_{n\delta}} \tag{1.8} $$

is the extended generalized Mittag-Leffler function and $(c)_{nk}$ is the Pochhammer symbol defined by: $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

Recently, Farid defined a unified integral operator as follows:

Definition 4. *[9] Let* $f, g : [a,b] \longrightarrow \mathbb{R}$, $0 < a < b$*, be the functions such that* f *be positive and* $f \in L_1[a,b]$*, and* g *be differentiable and strictly increasing. Also let* $\frac{\phi}{x}$ *be an increasing function on* $[a,\infty)$ *and* $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$ *and* $0 < k \leq \delta + \mu$*. Then for* $x \in [a,b]$ *the left and right integral operators are defined by:*

$$ ({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f)(x,\omega;p) = \int_a^x K_x^y(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) f(y) d(g(y)) \tag{1.9} $$

and

$$({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f)(x,\omega;p) = \int_x^b K^x_y(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi)f(y)d(g(y)), \tag{1.10}$$

where $K^y_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi) = \dfrac{\phi(g(x)-g(y))}{g(x)-g(y)} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(y))^{\mu};p)$.

For suitable settings of functions ϕ, g and certain values of parameters included in Mittag-Leffler function (1.8), interesting consequences can be obtained which are comprised in the upcoming two remarks.

Remark 1. *(i) Let* $\phi(x) = \dfrac{x^{\beta/k}\Gamma(\beta)}{k\Gamma_k(\beta)}$, $k > 0, \beta > k$ *and* $p = \omega = 0$, *in unified integral operators (1.9) and (1.10). Then generalized Riemann-Liouville fractional integral operators (1.3) and (1.4) are obtained.*
(ii) For $k = 1$, *(1.3) and (1.4) fractional integrals coincide with (1.1) and (1.2) fractional integrals, which further produce the following fractional and conformable integrals:*
(1). By taking g as identity function, (1.3) and (1.4) fractional integrals coincide with k-fractional Riemann-Liouville fractional integrals defined by Mubeen et al. in [10].
(2). For $k = 1$, *along with g as identity function, (1.3) and (1.4) fractional integrals coincide with Riemann-Liouville fractional integrals [1].*
(3). For $k = 1$ *and* $g(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$, *(1.3) and (1.4) produce fractional integrals defined by Chen et al. in [11].*
(4). For $k = 1$ *and* $g(x) = \frac{x^{\tau+s}}{\tau+s}$ *, (1.3) and (1.4) produce generalized conformable fractional integrals defined by Khan et al. in [12].*
(5). If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ *in (1.3) and* $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ *in (1.4), then conformable* (k, s)*-fractional integrals are achieved as defined by Habib et al. in [13].*
(6). If we take $g(x) = \frac{x^{1+s}}{1+s}$, *then conformable fractional integrals are achieved as defined by Sarikaya et al. in [14].*
(7). If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ *in (1.3) and* $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ *in (1.4) with* $k = 1$, *then conformable fractional integrals are achieved as defined by Jarad et al. in [15].*
(8). If we take $p = \omega = 0$ *and* $\phi(t) = \Gamma(\mu)t^{\frac{\mu}{k}}E^{\sigma,k}_{\rho,\lambda}(\omega(t)^{\rho})$ *in (1.9) and (1.10), then generalized k-fractional integral operators defined by Tunc et al. in [16].*
(9). If we take $k = 1$ *and* $\phi(t) = \Gamma(\mu)\dfrac{exp(-At)}{\mu}$, $A = \dfrac{1-\mu}{\mu}$, $\mu > 0$ *in (1.3) and (1.4), then generalized fractional integral operators with exponential kernel is obtained [17].*

Remark 2. *Let* $\phi(x) = x^{\beta}$ *and* $g(x) = x$, $\beta > 0$, *in unified integral operators (1.9) and (1.10). Then fractional integral operators (1.6) and (1.7) are obtained, which along with different settings of* k, δ, l, c, γ *in generalized Mittag-Leffler function give the following integral operators:*
(1). By setting $p = 0$, *fractional integral operators (1.6) and (1.7) reduce to the fractional integral operators defined by Salim-Faraj in [18].*
(2). By setting $l = \delta = 1$, *fractional integral operators (1.6) and (1.7) reduce to the fractional integral operators defined by Rahman et al. in [6].*
(3). By setting $p = 0$ *and* $l = \delta = 1$, *fractional integral operators (1.6) and (1.7) reduce to the fractional integral operators defined by Srivastava-Tomovski in [19].*
(4). By setting $p = 0$ *and* $l = \delta = k = 1$, *fractional integral operators (1.6) and (1.7) reduce to the fractional integral operators defined by Prabhakar in [7].*

(5). By setting $p = \omega = 0$, fractional integral operators (1.6) and (1.7) reduce to the left-sided and right-sided Riemann-Liouville fractional integrals.
(6). By setting $p = \omega = 0$ in fractional integral operators (1.9) and (1.10) we get $\frac{1}{\Gamma(\mu)}(F_{a^+}^{\phi,g} f)(x) = ({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f)(x,0;0)$ and $\frac{1}{\Gamma(\mu)}(F_{b^-}^{\phi,g} f)(x) = ({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f)(x,0;0)$ where $(F_{a^+}^{\phi,g} f)(x)$ and $(F_{b^-}^{\phi,g} f)(x)$ are defined in [20].

To derive the results of this paper we need to recall convex functions. Convex functions have significant job in numerous areas of mathematics. They are particularly useful in the study of optimization problems where they are recognized by various advantageous properties.

Definition 5. *[21] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if*

$$f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda)f(y) \tag{1.11}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Lemma 1. *[21] Let $f : I \to \mathbb{R}$ be convex and increasing function and let $g : J \to \mathbb{R}$, Rang$(g) \subseteq I$ be convex, then the composite function $f \circ g$ is convex on J.*

Lemma 2. *[22] Let $f : [a,b] \to \mathbb{R}$ be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq f(x), \quad x \in [a,b]. \tag{1.12}$$

The aim of this paper is to derive inequalities for unified integral operators by using convex functions. These inequalities investigate further results for several known integral operators. In Section 2, upper bounds of unified integral operators (1.9) and (1.10) are established by using composite convex functions. Further by using an additional condition of symmetry, two sided Hadamard type bounds are obtained. Moreover some bounds are studied by using convexity of $|f'|$. Some special cases are studied in Section 3.

2. Main results

In this section, bounds of integral operators (1.9) and (1.10) using composite convex functions are established in Theorem 1. Also using Theorem 1, boundedness of left and right integral operators is obtained. In Theorem 3 and Theorem 4, upper and lower bounds of sum of integral operators (1.9) and (1.10) in the form of Hadamard inequality and some bounds of integral operators (1.9) and (1.10) are established via composite convex functions respectively.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Also let $g : J \to \mathbb{R}$, where Range$(g) \subseteq [a,b]$ be a differentiable and strictly increasing function, $0 < a < b$. Let $\frac{\phi}{x}$ be an increasing function on $[a,b]$ and $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then for $x \in [a,b]$ we have*

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f \circ g\right)(x,\omega;p) \leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(g(x) - g(a))\frac{(f\circ g)(x) + (f \circ g)(a)}{2} \tag{2.1}$$

and

$$\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f \circ g\right)(x,\omega;p) \leq K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(g(b) - g(x))\frac{(f\circ g)(x) + (f \circ g)(b)}{2} \tag{2.2}$$

hence

$$\begin{aligned}&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\circ g\right)(x,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\circ g\right)(x,\omega;p)\\&\le K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(g(x)-g(a))\frac{(f\circ g)(x)+(f\circ g)(a)}{2}\\&+K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(g(b)-g(x))\frac{(f\circ g)(x)+(f\circ g)(b)}{2}.\end{aligned}\tag{2.3}$$

Proof. Under the assumptions of ϕ and g, we have

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)}\le\frac{\phi(g(x)-g(a))}{g(x)-g(a)}\quad t\in[a,x],x\in(a,b).\tag{2.4}$$

Multiplying on both sides with $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)g'(t)$, the following inequality is yielded:

$$K^t_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(t)\le\frac{\phi(g(x)-g(a))}{g(x)-g(a)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)g'(t).\tag{2.5}$$

Further by using $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(t))^{\mu};p)\le E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)$, the following inequality is obtained:

$$K^t_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(t)\le K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(t).\tag{2.6}$$

Using convexity of f on the identity $g(t)=\frac{g(x)-g(t)}{g(x)-g(a)}g(a)+\frac{g(t)-g(a)}{g(x)-g(a)}g(x)$ we have

$$f(g(t))\le\frac{g(x)-g(t)}{g(x)-g(a)}f(g(a))+\frac{g(t)-g(a)}{g(x)-g(a)}f(g(x)).\tag{2.7}$$

The following integral inequality can be obtained from (2.6) and (2.7):

$$\begin{aligned}&\int_a^x K^t_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)f(g(t))d(g(t))\\&\le\frac{f(g(a))}{g(x)-g(a)}K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)\int_a^x(g(x)-g(t))d(g(t))\\&+\frac{f(g(x))}{g(x)-g(a)}K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)\int_a^x(g(t)-g(a))d(g(t)).\end{aligned}\tag{2.8}$$

By using (1.9) of Definition 4 on left hand side and integrating by parts on right hand side, we get

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\circ g\right)(x,\omega;p)\le K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(g(x)-g(a))\frac{(f\circ g)(x)+(f\circ g)(a)}{2}.\tag{2.9}$$

Now on the other hand for $x\in(a,b)$ the following inequality holds true:

$$K^x_t(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(t)\le\frac{\phi(g(b)-g(x))}{g(b)-g(x)}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p)g'(t).\tag{2.10}$$

Further by using $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(t)-g(x))^{\mu};p)\le E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(x))^{\mu};p)$, the following inequality is obtained:

$$K^t_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(t)\le K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(t).\tag{2.11}$$

On the other hand using convexity of f on the identity $g(t) = \frac{g(t)-g(x)}{g(b)-g(x)}g(b) + \frac{g(b)-g(t)}{g(b)-g(x)}g(x)$ we have

$$f(g(t)) \leq \frac{g(t)-g(x)}{g(b)-g(x)}f(g(b)) + \frac{g(b)-g(t)}{g(b)-g(x)}f(g(x)). \tag{2.12}$$

The following integral inequality can be obtained by (2.11) and (2.12):

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f \circ g\right)(x,\omega;p) \leq K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)(g(b)-g(x))\frac{(f\circ g)(x)+(f\circ g)(b)}{2}. \tag{2.13}$$

By adding (2.9) and (2.13), (2.3) can be achieved.

□

Theorem 2. *Under the assumptions of Theorem 1, if $f \circ g \in L_\infty[a,b]$, then the following inequalities hold:*

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f \circ g\right)(x,\omega;p)\right| \leq K\|f\circ g\|_\infty, \tag{2.14}$$

and

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f \circ g\right)(x,\omega;p)\right| \leq K\|f\circ g\|_\infty, \tag{2.15}$$

where $K = (g(b)-g(a))K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)$.

Proof. From (2.1) we have

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f \circ g\right)(x,\omega;p)\right| \leq K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)\phi(g(b)-g(a))\|f\circ g\|_\infty \tag{2.16}$$

that is (2.14) holds. Similarly, (2.2) gives (2.15). □

The following lemma is needful to prove the upcoming theorem.

Lemma 3. *Let $f : I \to \mathbb{R}$ be convex and increasing function and let $g : J \to \mathbb{R}$, $Rang(g) \subseteq I$ be convex and symmetric about $\frac{a+b}{2}$ for $a,b \in J$. Then we have*

$$(f\circ g)\left(\frac{a+b}{2}\right) \leq (f\circ g)(x) \tag{2.17}$$

for all $x \in [a,b]$.

Proof. Since f and g are convex functions moreover f is increasing, therefore $f \circ g$ is convex. Also g is symmetric about $\frac{a+b}{2}$ so is $f \circ g$. Hence applying Lemma 2 we get (2.17) □

The following theorem provides the Hadamard type estimation of integral operators (1.9) and (1.10).

Theorem 3. *Under the assumptions of Theorem 1, and in addition if $(f \circ g)(x) = f(a + g(b) - x)$, then the following inequality holds:*

$$\begin{aligned}
&(f\circ g)\left(\frac{a+b}{2}\right)\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} 1\right)(a,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} 1\right)(b,\omega;p)\right)\\
&\leq \left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\circ g\right)(a,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\circ g\right)(b,\omega;p)\right)\\
&\leq (g(b)-g(a))K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)\left((f\circ g)(a) + (f\circ g)(b)\right).
\end{aligned} \tag{2.18}$$

Proof. For $x \in (a, b)$, under the assumptions of ϕ and g the following inequality holds:

$$K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(x) \leq K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(x). \tag{2.19}$$

Using convexity of f on the identity $g(x) = \frac{g(x)-g(a)}{g(b)-g(a)}g(b) + \frac{g(b)-g(x)}{g(b)-g(a)}g(a)$ we have

$$f(g(x)) \leq \frac{g(x) - g(a)}{g(b) - g(a)}f(g(b)) + \frac{g(b) - g(x)}{g(b) - g(a)}f(g(a)). \tag{2.20}$$

The following integral inequality can be obtained from (2.19) and (2.20):

$$\begin{aligned}
&\int_a^b K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)f(g(x))d(g(x)) \\
&\leq \frac{f(g(b))}{g(b) - g(a)}K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)\int_a^b (g(x) - g(a))d(g(x)) \\
&+ \frac{f(g(a))}{g(b) - g(a)}K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)\int_a^b (g(b) - g(x))d(g(x)).
\end{aligned}$$

By using (1.9) of Definition 4 and integrating by parts we get

$$\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f \circ g\right)(a, \omega; p) \leq K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(g(b) - g(a))\frac{(f \circ g)(a) + (f \circ g)(b)}{2}. \tag{2.21}$$

On the other hand for $x \in (a, b)$ the following inequality holds true:

$$K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(x) \leq K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(x). \tag{2.22}$$

The following integral inequality can be obtained from (2.20) and (2.22):

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f \circ g\right)(b, \omega; p) \leq K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(g(b) - g(a))\frac{(f \circ g)(a) + (f \circ g)(b)}{2}. \tag{2.23}$$

By adding (2.21) and (2.23), we have

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f \circ g\right)(b, \omega; p) + \left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f \circ g\right)(a, \omega; p) \tag{2.24}$$

$$\leq (g(b) - g(a))K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)\left((f \circ g)(a) + (f \circ g)(b)\right). \tag{2.25}$$

Multiplying both sides of (2.17) by $g'(x)K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)$, then integrating over $[a, b]$ we get

$$(f \circ g)\left(\frac{a + b}{2}\right)\int_a^b K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)d(g(x)) \leq \int_a^b K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(f \circ g)(x)d(g(x)). \tag{2.26}$$

By using (1.10) of Definition 4 we get

$$(f \circ g)\left(\frac{a + b}{2}\right)\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} 1\right)(a, \omega; p) \leq \left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f \circ g\right)(a, \omega; p). \tag{2.27}$$

Multiplying both sides of (2.17) by $K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(x)$ and integrating over $[a, b]$ we get

$$(f \circ g)\left(\frac{a+b}{2}\right)\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} 1\right)(b, \omega; p) \leq \left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f \circ g\right)(b, \omega; p), \tag{2.28}$$

by adding (2.27) and (2.28), the following inequality is obtained:

$$\begin{aligned} &(f \circ g)\left(\frac{a+b}{2}\right)\left(\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} 1\right)(a, \omega; p) + \left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} 1\right)(b, \omega; p)\right) \\ &\leq \left(\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f \circ g\right)(a, \omega; p) + \left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f \circ g\right)(b, \omega; p)\right). \end{aligned} \tag{2.29}$$

Combining (2.24) and (2.29), inequality (2.18) can be achieved. □

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $|f'|$ is convex and also let $g : J \to \mathbb{R}$, where Range$(g) \subseteq [a, b]$ be a differentiable and strictly increasing function. Let $\frac{\phi}{x}$ be an increasing function and $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then for $x \in (a, b)$ we have*

$$\begin{aligned} &\left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f' \circ g)\right)(x, \omega; p) + \left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}(f' \circ g)\right)(x, \omega; p)\right| \\ &\leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(g(x) - g(a))\frac{|(f' \circ g)(x)| + |(f' \circ g)(a)|}{2} \\ &+ K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(g(b) - g(x))\frac{|(f' \circ g)(x)| + |(f' \circ g)(b)|}{2}. \end{aligned} \tag{2.30}$$

Proof. Using convexity of $|f'|$ on the identity $g(t) = \frac{g(x)-g(t)}{g(x)-g(a)}g(a) + \frac{g(t)-g(a)}{g(x)-g(a)}g(x)$ $x \in (a, b)$
we have

$$|f'(g(t))| \leq \frac{g(x) - g(t)}{g(x) - g(a)}|f'(g(a))| + \frac{g(t) - g(a)}{g(x) - g(a)}|f'(g(x))|. \tag{2.31}$$

From which we can write

$$\begin{aligned} &-\left(\frac{g(x) - g(t)}{g(x) - g(a)}|f'(g(a))| + \frac{g(t) - g(a)}{g(x) - g(a)}|f'(g(x))|\right) \leq f'(g(t)) \\ &\leq \left(\frac{g(x) - g(t)}{g(x) - g(a)}|f'(g(a))| + \frac{g(t) - g(a)}{g(x) - g(a)}|f'(g(x))|\right), \end{aligned} \tag{2.32}$$

we consider the right hand side inequality of the above inequality i.e.,

$$f'(g(t)) \leq \left(\frac{g(x) - g(t)}{g(x) - g(a)}|f'(g(a))| + \frac{g(t) - g(a)}{g(x) - g(a)}|f'(g(x))|\right). \tag{2.33}$$

Further the following inequality holds true:

$$K_x^t(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(t) \leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(t). \tag{2.34}$$

The following integral inequality can be obtained from (2.33) and (2.34):

$$\int_a^x K_x^t(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)f'(g(t))d(g(t))$$

$$\leq \frac{|f'(g(a))|}{g(x)-g(a)} K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \int_a^x (g(x)-g(t))d(g(t))$$
$$+ \frac{|f'(g(x))|}{g(x)-g(a)} K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \int_a^x (g(t)-g(x))d(g(t))$$

which gives

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f' \circ g)\right)(x,\omega;p) \leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \times (g(x)-g(a)) \frac{|(f' \circ g)(x)| + |(f' \circ g)(a)|}{2}. \tag{2.35}$$

If we consider the left hand side inequality from the inequality (2.32) and proceed as we did for the right hand side inequality we have

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f' \circ g)\right)(x,\omega;p) \geq -K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \times (g(x)-g(a)) \frac{|(f' \circ g)(x)| + |(f' \circ g)(a)|}{2}. \tag{2.36}$$

Combining (2.35) and (2.36), the following inequality is obtained:

$$\left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f' \circ g)\right)(x,\omega;p)\right| \leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \times (g(x)-g(a)) \frac{|(f' \circ g)(x)| + |(f' \circ g)(a)|}{2}. \tag{2.37}$$

On the other hand using convexity of $|f'(t)|$ on the identity $g(t) = \frac{g(t)-g(x)}{g(b)-g(x)} g(b) + \frac{g(b)-g(t)}{g(b)-g(x)} g(x)$ we have

$$|f'(g(t))| \leq \frac{g(t)-g(x)}{g(b)-g(x)} |f'(g(b))| + \frac{g(b)-g(t)}{g(b)-g(x)} |f'(g(x))|. \tag{2.38}$$

Further the following inequality holds true:

$$K_x^t(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) g'(t) \leq K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) g'(t). \tag{2.39}$$

The following integral inequality can be obtained from (2.38) and (2.39):

$$\left|\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}(f' \circ g)\right)(x,\omega;p)\right| \leq K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \times (g(b)-g(x)) \frac{|(f' \circ g)(x)| + |(f' \circ g)(b)|}{2}. \tag{2.40}$$

Combining (2.37) and (2.40), inequality (2.30) can be achieved. □

3. Some special cases

By putting specific functions ϕ and g in Theorem 1, following results are obtained for some of the fractional and conformable integral operators:

Corollary 1. *If we put $\phi(t) = t^\tau$, $\tau \geq 1$ and $g(x) = I(x) = x$ in (2.3), then following inequality holds:*

$$\left(\epsilon_{\mu,\tau,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x,\omega;p) + \left(\epsilon_{\mu,\tau,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(x,\omega;p) \leq E_{\mu,\tau,l}^{\gamma,\delta,k,c}(\omega(x-a)^\mu;p)(x-a)^\tau \times \frac{f(x)+f(a)}{2} + E_{\mu,\tau,l}^{\gamma,\delta,k,c}(\omega(b-x)^\mu;p)(b-x)^\tau \frac{f(x)+f(b)}{2}. \tag{3.1}$$

Corollary 2. *If we put* $\phi(t) = t^\tau$, $\tau \geq 1$ *and* $g(x) = I(x) = x$ *in (2.18), then following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\left(\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,a^+}1\right)(b,\omega;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}1\right)(a,\omega;p)\right) \tag{3.2}$$
$$\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,a^+}f\right)(b,\omega;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}f\right)(a,\omega;p)$$
$$\leq (b-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(b-a)^\mu;p)(f(a)+f(b)).$$

Corollary 3. *If we put* $\phi(t) = t^\tau$, $\tau \geq 1$ *and* $g(x) = I(x) = x$ *in (2.30), then following inequality holds:*

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,a^+}f'\right)(x,\omega;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}f'\right)(x,\omega;p)\right| \leq E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(x-a)^\mu;p)(x-a)^\tau \tag{3.3}$$
$$\times \frac{|f'(x)|+|f'(a)|}{2} + E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(b-x)^\mu;p)(b-x)^\tau \frac{|f'(x)|+|f'(b)|}{2}.$$

Corollary 4. *If we put* $\phi(t) = \dfrac{\Gamma(\tau)t^{\frac{\tau}{k}}}{k\Gamma_k(\tau)}$, $\tau \geq k$, $g(x) = I(x) = x$ *and* $p = \omega = 0$ *in (2.3), then following inequality holds:*

$$({}^\tau I^k_{a^+}f)(x) + ({}^\tau I^k_{b^-}f)(x) \leq \frac{1}{k\Gamma_k(\tau)}\left((x-a)^{\frac{\tau}{k}}\frac{f(x)+f(a)}{2} + (b-x)^{\frac{\tau}{k}}\frac{f(b)+f(x)}{2}\right). \tag{3.4}$$

Remark 3. *(i) If we put* $k = 1$ *in (3.4), then [22, Corollary 1] can be obtained.*
(ii) If we put $\tau = 1$ *and* $x = a$ *or* $x = b$*, then [22, Corollary 2] can be obtained.*
(iii) If we put $\tau = 1$ *and* $x = \dfrac{a+b}{2}$*, then [22, Corollary 3] can be obtained.*

Corollary 5. *If we put* $\phi(t) = \Gamma(\tau)t^{\frac{\tau}{k}+1}$, $\tau \geq 1$, $g(x) = I(x) = x$ *and* $p = \omega = 0$ *in (2.18), then following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\left(\frac{1}{\tau+k}\right) \leq \frac{\Gamma_k(\tau+k)}{2(b-a)^{\frac{\tau}{k}+1}}(({}^{\tau+k}I^k_{a^+}f)(b) + ({}^{\tau+k}I^k_{b^-}f)(a)) \tag{3.5}$$
$$\leq \frac{1}{2k}(f(a)+f(b)).$$

Remark 4. *(i) If we put* $k = 1$ *in (3.5), then [22, Corollary 6] can be obtained.*

Corollary 6. *If we put* $\phi(t) = \Gamma(\tau)t^{\frac{\tau}{k}+1}$, $\tau \geq 1$, $g(x) = I(x) = x$ *and* $p = \omega = 0$ *in (2.30), then following inequality holds:*

$$\left|\Gamma_k(\tau+k)(({}^\tau I^k_{a^+}f)(x) + ({}^\tau I^k_{b^-}f)(x)) - \left((x-a)^{\frac{\tau}{k}}f(a) + (b-x)^{\frac{\tau}{k}}f(b)\right)\right| \tag{3.6}$$
$$\leq \frac{1}{2}\left((x-a)^{\frac{\tau}{k}+1}\frac{|f'(x)|+|f'(a)|}{2} + (b-x)^{\frac{\tau}{k}+1}\frac{|f'(b)|+|f'(x)|}{2}\right).$$

Remark 5. *(i) If we put* $k = 1$ *in (3.6), then [22, Corollary 4] can be obtained.*
(ii) If we put $\tau = 1$ *and* $x = \dfrac{a+b}{2}$ *in (3.6), then [22, Corollary 5] can be obtained.*

4. Concluding remarks

This work elaborates bounds of several kinds of fractional and conformable integral operators in a unified form. The bounds of some generalized fractional and conformable integral operators have been deduced. It will be interesting for readers to verify that these results can produce bounds of fractional and conformable integral operators defined in [6, 7, 10–13, 15–20].

Conflict of interest

The authors declare no conflict of interest.

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