AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4770–4780.
DOI: 10.3934/math.2020305
Received: 24 December 2019
Accepted: 11 May 2020
Published: 01 June 2020

Research article

Existence of solutions to a class of nonlinear boundary value problems with right and left fractional derivarives

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Abstract: We discuss the existence of solutions for a boundary value problem involving both left Riemann-Liouville and right Caputo types derivatives. For this, we convert the posed problem to a sum of two integral operators then we apply Krasnoselskii's fixed point theorem to conclude the existence of nontrivial solutions.

Keywords: boundary value problem; fractional derivative; fixed point theorem; existence of solution; integral equation
Mathematics Subject Classification: 34A08, 34B15

1. Introduction

Differential equations of noninteger order can represent the dynamics of various memory systems and arise from a variety of applications, including several fields of science and engineering such as geology, physics, optics, chemistry, biology, economics, signal and image processing,... Although the literature on fractional differential equations is now vast, more studies are needed. Recently, the investigation of the qualitative properties of solutions to fractional initial and boundary value problems has attracted the attention of many authors [1–17], and different tools are used in these researches, such as the method of upper and lower solutions, the variational method, the coincidence degree theory, the fixed point theorems ...

The aim of this work is the study of the existence of solutions, for the following nonlinear boundary value problem (P) involving both the right Caputo and the left Riemann-Liouville fractional derivatives:

$$-^{C}D_{1^-}^{\alpha}D_{0^+}^{\beta}u(t)+\omega^2u(t)+f(t,u(t))=0,\ t\in J=[0,1]. \tag{1.1}$$

$$u(0)=0, D_{0^+}^{\beta}u(1)=0, \tag{1.2}$$

where $0 < \alpha, \beta < 1,\ \alpha + \beta > 1$, $\omega \in \mathbb{R}$, ${}^{C}D^{\alpha}_{1^-}$ and $D^{\beta}_{0^+}$ denote respectively the right Caputo derivative and the left Riemann Liouville derivative, u is the unknown function and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function. Let us mention that if α and β tend to one, then problem (P) is a classical oscillator boundary value problem that is investigated in [2]. Note that problem (P) is studied in [13] by lower and upper solutions method, the authors proved the existence of solution under some specific conditions on the nonlinear term f. In the present study, we prove the existence of solution for problem (P) under Lipschitz type condition on the nonlinear term f and by using Krasnoselskii's fixed point theorem.

Different methods are used in the study of differential equations involving mixed type fractional derivatives. By the help of operational method and the successive approximations, some linear differential equations containing left and right fractional derivatives that may appear in fractional variational calculus, are studied in [8,9].

Recently, the method of upper and lower solutions is applied in [13,15,16] to solve nonlinear differential equations containing mixed fractional derivatives.

In [3], the authors considered a coupled system of nonlinear differential equations involving mixed type fractional derivatives

$$\begin{aligned} -{}^{C}D^{\alpha}_{1^-}D^{\beta}_{0^+}x(t) &= f(t, x(t), y(t)) = 0, \\ -{}^{C}D^{p}_{1^-}D^{p}_{0^+}y(t) &= f(t, x(t), y(t)) = 0,\ 0 < t < 1, \end{aligned}$$

with nonlocal boundary conditions

$$\begin{aligned} x(0) &= x'(0) = 0,\ \ x(1) = \gamma y(\eta), 0 < \eta < 1, \\ y(0) &= y'(0) = 0,\ \ y(1) = \delta x(\theta), 0 < \theta < 1, \end{aligned}$$

here $1 < \alpha, p < 2$, $0 < \beta, p < 1$, $\gamma, \delta \in \mathbb{R}$. The existence and uniqueness of solution is proved by the help of Leray-Schauder alternative and Banach fixed point theorem.

By Krasnoselskii's fixed point theorem, the authors in [12,14], investigated some boundary value problems involving mixed type fractional derivatives. In particular in [12], proved, under Lipschitz type condition on the nonlinear term, the existence of solution in a weighted space, for the following boundary value problem

$$\begin{aligned} -{}^{C}D^{\alpha}_{1^-}D^{\beta}_{0^+}u(t) + f(t, u(t)) &= 0,\ 0 < t < 1, \\ \lim_{t \to 0^+} t^{1-\beta}u(t) &= u(1) = u(\eta) \end{aligned}$$

where $0 < \alpha, \beta < 1,\ 1 < \alpha + \beta < 2$.

In [14], the authors studied by the help of Krasnoselskii's fixed point theorem and Arzela Ascoli's theorem, the existence of solution for the problem

$$-{}^{C}D^{\alpha}_{1^-}D^{\beta}_{0^+}u(t) + f(t, u(t)) = 0, 0 < t < 1$$

$$u(0) = u'(0) = u(1) = 0$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$, ${}^{C}D^{\alpha}_{1^-}$ denotes right Caputo derivative, $D^{\beta}_{0^+}$ denotes the left Riemann–Liouville and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfies Lipschitz type condition.

Motivated by the above papers, we study the existence of solutions for problem (P). For this, we convert the problem (P) into an integral equation which we write as a sum of two integral operators, including a contraction and a completely continuous operator, then we apply Riesz compactness criteria and Krasnoselskii fixed point theorem to prove the existence of solution.

2. Preliminaries

We give some background on fractional calculus that can be found in [18,19,20]. Let g be a real function defined on $[0,1]$ and $\alpha > 0$. Then the left and right Riemann-Liouville fractional integrals of order α of g are defined respectively by

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds,$$

$$I_{1^-}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 \frac{g(s)}{(s-t)^{1-\alpha}} ds.$$

The left Riemann-Liouville fractional derivative of order $\alpha > 0$, of a function g is

$$D_{0^+}^{\alpha} g(t) = \frac{d^n}{dt^n} (I_{0^+}^{n-\alpha} g(t))$$

where $n = [\alpha] + 1$.

The left and right Caputo fractional derivatives of order $\alpha > 0$ of a function g are defined respectively as

$${}^C D_{0^+}^{\alpha} g(t) = I_{0^+}^{n-\alpha} \left(\frac{d^n}{dt^n} g(t) \right),$$

$${}^C D_{1^-}^{\alpha} g(t) = (-1)^n I_{1^-}^{n-\alpha} \left(\frac{d^n}{dt^n} g(t) \right).$$

Proposition 2.1. *Let $f \in C^n([0,1])$. Then*

$$I_{0^+}^{\alpha} {}^C D_{0^+}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k,$$

$$I_{1^-}^{\alpha} {}^C D_{1^-}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(1)}{k!} (1-t)^k.$$

Theorem 2.1 (**Riesz compactness criteria**). *[11]. Let F be a bounded set in $L^p(0,1)$, $1 \le p < \infty$. Assume that*
(i) $\lim_{h\to 0} \|\tau_h f - f\|_{L^p} = 0$ uniformly on F, where $\tau_h f(t) = f(t+h)$.
(ii) $\lim_{\varepsilon\to 0} \int_{1-\varepsilon}^1 |f(t)|^p dt = 0$ uniformly on F.
Then F is relatively compact in $L^p(0,1)$.

Theorem 2.2 (Krasnoselskii fixed point Theorem). *[21]. Let Ω be a closed bounded convex nonempty subset of a Banach space E. Suppose that A and B map Ω into E such that*
(i) A is completely continuous,
(ii) B is a contraction mapping,
(iii) $x, y \in M$ implies $Ax + By \in \Omega$.
Then there exists $z \in \Omega$ with $z = Az + Bz$.

3. Main results

To study the nonlinear problem (P), we consider first, the associated linear problem

$$-{}^{C}D_{1^-}^{\alpha}D_{0^+}^{\beta}u(t)+y(t)=0,\ 0<t<1, \tag{3.1}$$

$$u(0)=0, D_{0^+}^{\beta}u(1)=0.$$

Lemma 3.1. *Assume that $y\in L^p(0,1)$, $p>1$, then u is a solution for the linear boundary value problem (3.1)–(1.2) if and only if u satisfies the integral equation*

$$u(t)=\int_0^1 G(t,\tau)y(\tau)d\tau,$$

where

$$G(t,\tau)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\begin{cases}\int_0^{\tau}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds, & 0\le\tau\le t\le 1,\\ \int_0^{t}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}ds, & 0\le t\le\tau\le 1.\end{cases}$$

Proof. Applying the right-hand side fractional integral $I_{1^-}^{\alpha}$ to equation (3.1), we get

$$D_{0^+}^{\beta}u(t)=I_{1^-}^{\alpha}y(t)+a,\quad a\in\mathbb{R}$$

The boundary condition $D_{0^+}^{\beta}u(1)=0$, gives $a=0$, then applying the fractional integral $I_{0^+}^{\beta}$ to the obtained equation, it yields

$$u(t)=I_{0^+}^{\beta}I_{1^-}^{\alpha}y(t)+ct^{\beta-1},\quad c\in\mathbb{R} \tag{3.2}$$

Multiplying the equation (3.2) by $t^{1-\beta}$, then using the condition $u(0)=0$, we obtain $c=0$, thus

$$\begin{aligned}u(t) &= I_{0^+}^{\beta}I_{1^-}^{\alpha}y(t)\\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\int_s^1(\tau-s)^{\alpha-1}y(\tau)d\tau ds.\end{aligned}$$

Finally, by Fubini theorem, we get

$$\begin{aligned}u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^{\tau}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}y(\tau)dsd\tau\\ &\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_t^1\int_0^{t}(t-s)^{\beta-1}(\tau-s)^{\alpha-1}y(\tau)dsd\tau.\end{aligned}$$

□

Lemma 3.2. *The function G is continuous, nonnegative and*

$$G(t,\tau)\le\frac{1}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)},\ \textit{for all } t,\tau\in J.$$

Remark 3.1. *Let us mention the case $\alpha+\beta\to 1^+$. Since $\alpha+\beta>1$ and $0<\alpha,\beta<1$, then $\alpha>\frac{1}{2}$ or $\beta>\frac{1}{2}$. If $\alpha>\frac{1}{2}$, then $\alpha+\beta\to 1^+$implies ($\alpha\to 1^-$ and $\beta\to 0$) or ($\alpha\to\frac{1}{2}^+$ and $\beta\to\frac{1}{2}^-$), then the problem (P) is reduced respectively to*

$$\begin{aligned} u'(t)+\omega^2u(t)+f(t,u(t)) &= 0,\ t\in J=[0,1]. \qquad \textit{P1}\\ u(0) &= 0 \end{aligned}$$

and

$$\begin{aligned} -{}^{C}D_{1^-}^{\frac{1}{2}}D_{0^+}^{\frac{1}{2}}u(t)+\omega^2u(t)+f(t,u(t)) &= 0,\ t\in J=[0,1]. \qquad \textit{P2}\\ u(0) &= 0, D_{0^+}^{\frac{1}{2}}u(1)=0, \end{aligned}$$

For problem (P2), let us fix $\alpha=1/2$, then we have,

$$\begin{aligned} G(1,1) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1(1-s)^{\beta+\alpha-2}ds=\frac{1}{\Gamma(\alpha)\Gamma(\beta)\,(\beta+\alpha-1)}\\ &= \frac{1}{\Gamma(1/2)\Gamma(\beta)}\int_0^1(1-s)^{\beta-\frac{3}{2}}ds=\frac{1}{\Gamma(1/2)\Gamma(\beta)\left(\beta-\frac{1}{2}\right)}\to+\infty,\\ \text{as}\,\beta &\to \frac{1}{2}^- \end{aligned}$$

thus the Green function is not bounded.

Lemma 3.3. *The function $u\in L^p(0,1)$ is a solution of the integral equation*

$$u(t)=\int_0^1 G(t,\tau)f(\tau,u(\tau))d\tau+\omega^2\int_0^1 G(t,\tau)u(\tau)d\tau.$$

if and only if u is a solution of the fractional boundary value problem (P).

Now we define the operators A and B on $L^p(0,1)$ as

$$\begin{aligned} Au(t) &= \int_0^1 G(t,\tau)f(\tau,u(\tau))d\tau,\\ Bu(t) &= \omega^2\int_0^1 G(t,\tau)u(\tau)d\tau. \end{aligned}$$

Obviously, the problem (P) has a solution if and only if the operator $A+B$ has a fixed point in $L^p(0,1)$. Before stating and proving the main results, we introduce the following hypotheses.
(H1) $M=\sup_{0\le t\le 1}|f(t,0)|<\infty$, and there exists a constant k,
$0<\frac{k}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}\le\frac{1}{2}$, such that

$$|f(t,u)-f(t,v)|\le k\,|u-v|\,,0\le t\le 1,u,v\in\mathbb{R}.$$

(H2) $\frac{\omega^2}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}<\frac{1}{2}$.

Theorem 3.1. *Assume that (H1)–(H2) hold, then the fractional boundary value problem (P) has a nontrivial solution in* $L^p(0,1)$.

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.4. *Under the hypotheses (H1)–(H2), the operator A is completely continuous on* $L^p(0,1)$.

Proof. Let
$$\Omega = \{u \in L^p(0,1), \|u\|_{L^p} \leq R\}$$
such that
$$R \geq \frac{M}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta) - (k+\omega^2)} \tag{3.3}$$

Clearly, Ω is a nonempty, bounded and convex subset of the Banach space $L^p(0,1)$. We should prove that A is continuous and relatively compact on $L^p(0,1)$.

Claim 1. The mapping A is continuous on Ω. In fact, consider the sequence $(u_n)_n \in \Omega$, such that $u_n \longrightarrow u$ in $L^p(0,1)$, then from Lemma 3.2, hypothesis (H1) and Hölder inequality, we get

$$\begin{aligned}
|Au_n(t) - Au(t)| &\leq \int_0^1 G(t,\tau)\,|f(\tau,u_n(\tau)) - f(\tau,u(\tau))|\,d\tau \\
&\leq \frac{k}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)} \int_0^1 |u_n(\tau) - u(\tau)|\,d\tau \\
&\leq \frac{k}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)} \|u_n(.) - u(.)\|_{L^p(0,1)}.
\end{aligned}$$

Hence
$$\|Au_n - Au\|_{L^p(0,1)} \leq \frac{k}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)} \|u_n(.) - u(.)\|_{L^p(0,1)} \to 0, \text{as } n \to \infty.$$

Claim 2. (Au) is bounded in $L^p(0,1)$. Indeed, let $u \in \Omega$, then by condition (H1) and Hölder inequality, it yields

$$\begin{aligned}
|Au(t)| &\leq \frac{1}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)} \int_0^1 |f(\tau,u(\tau))|\,d\tau \\
&\leq \frac{1}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)} \left(k\left(\int_0^1 |u(\tau)|\,d\tau\right) + \int_0^1 |f(\tau,0)|\,d\tau \right) \\
&\quad \frac{1}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)} \left(k\left(\int_0^1 |u(\tau)|^p\,d\tau\right)^{\frac{1}{p}} + \int_0^1 |f(\tau,0)|\,d\tau \right) \\
&\leq \frac{kR+M}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)},
\end{aligned}$$

thus
$$\|Au\|_{L^p} \leq \frac{kR+M}{(\alpha+\beta-1)\,\Gamma(\alpha)\Gamma(\beta)}.$$

Claim 3. (Au) is relatively compact. In fact, let $u \in \Omega$, and $p > 1$, we have

$$|Au(t+h) - Au(t)| \leq \int_0^1 |G(t+h,\tau) - G(t,\tau)|\,|f(\tau,u(\tau))|\,d\tau$$

$$\leq \int_0^1 |G(t+h,\tau)-G(t,\tau)|\,(k\,|u(\tau)|+|f(\tau,0)|)d\tau$$
$$\leq (kR+M)\left(\int_0^1 |G(t+h,\tau)-G(t,\tau)|^p\,d\tau\right)^{\frac{1}{p}}$$

$$\leq (kR+M)\left(\int_0^t |G(t+h,\tau)-G(t,\tau)|^p\,d\tau+\int_t^{t+h} |G(t+h,\tau)-G(t,\tau)|^p\,d\tau\right.$$
$$\left.+\int_{t+h}^1 |G(t+h,\tau)-G(t,\tau)|^p\,d\tau\right)^{\frac{1}{p}}$$
$$\leq \frac{kR+M}{\Gamma(\alpha)\Gamma(\beta)}\left(\int_0^t\left(\int_0^\tau((t-s)^{\beta-1}-(t+h-s)^{\beta-1})(\tau-s)^{\alpha-1}ds\right)^p d\tau\right.$$
$$+\int_t^1\left(\int_0^t((t-s)^{\beta-1}-(t+h-s)^{\beta-1})(\tau-s)^{\alpha-1}ds\right)^p d\tau$$
$$\left.+\int_t^{t+h}\left(\int_t^\tau(t+h-s)^{\beta-1}ds\right)^p d\tau\right)^{\frac{1}{p}}$$
$$=\frac{(kR+M)}{\Gamma(\alpha)\Gamma(\beta)}\,(I_1+I_2+I_3)^{\frac{1}{p}}\,,$$

hence

$$|Au(t+h)-Au(t)|\leq \frac{(kR+M)}{\Gamma(\alpha)\Gamma(\beta)}\,(I_1+I_2+I_3)^{\frac{1}{p}}\,. \tag{3.4}$$

Let us calculate I_i, $i=1,2,3$.

$$\begin{aligned} I_1 &= \int_0^t\left(\int_0^\tau((t-s)^{\beta-1}-(t+h-s)^{\beta-1})(\tau-s)^{\alpha-1}ds\right)^p d\tau \\ &\leq (h\,(1-\beta))^p\int_0^t\left(\int_0^\tau(\tau-s)^{\alpha-1}ds\right)^p d\tau\leq\left(\frac{h\,(1-\beta)}{\alpha\,(\alpha+1)}\right)^p. \end{aligned}$$

$$\begin{aligned} I_2 &= \int_t^1\left(\int_0^t((t-s)^{\beta-1}-(t+h-s)^{\beta-1})(\tau-s)^{\alpha-1}ds\right)^p d\tau \\ &\leq (h\,(1-\beta))^p\int_0^t((1-s)^\alpha-(t-s)^\alpha)^p\,ds\leq\frac{(h\,(1-\beta))^p}{\alpha p+1}. \end{aligned}$$

$$\begin{aligned} I_3 &= \int_t^{t+h}\left(\int_t^\tau(t+h-s)^{\beta-1}ds\right)^p d\tau \\ &\leq \frac{1}{\beta^p}\int_t^{t+h}\left(h^\beta-(t+h-\tau)^\beta\right)^p d\tau\leq\frac{h^{\beta p+1}}{\beta^p}. \end{aligned}$$

Finally, we get

$$||Au(.+h)-Au(.)||_{L^p}\leq\frac{(kR+M)}{\Gamma(\alpha)\Gamma(\beta)}\left(\left(\frac{h\,(1-\beta)}{\alpha\,(\alpha+1)}\right)^p+\frac{(h\,(1-\beta))^p}{\alpha p+1}+\frac{h^{\beta p+1}}{\beta^p}\right)^{\frac{1}{p}} \tag{3.5}$$

By taking the limit in (3.5) as $h \to 0$, we obtain that $\|Au(.+h) - Au(.)\|_{L^p} \to 0$ for any $u \in \Omega$.
On the other hand we have by the help of claim 2

$$\int_{1-\varepsilon}^{1} |Au(t)|^p \, dt \leq \varepsilon \left(\frac{kR + M}{(\alpha + \beta - 1)\,\Gamma(\alpha)\Gamma(\beta)} \right)^p \to 0 \text{ as } \varepsilon \to 0.$$

By Theorem 2.1, we conclude that A is relatively compact on Ω. From the above discussion we conclude that A completely continuous on $L^p\,(0,1)$. □

Lemma 3.5. *Under the hypothesis (H2), the mapping B is a contraction on* Ω*.*

Proof. Let $u \in \Omega$ and $t \in J$, we have

$$|Bu(t)| \leq \omega^2 \int_0^1 G(t,\tau)\,|u(\tau)|\, d\tau \leq \frac{\omega^2 R}{(\alpha + \beta - 1)\,\Gamma(\alpha)\Gamma(\beta)} < \frac{R}{2}.$$

thus $\|Bu\|_{L^p(0,1)} < \frac{R}{2}$, and consequently $B(\Omega) \subset \Omega$. Now for $u, v \in \Omega$ and $t \in J$, we have

$$\begin{aligned} |Bu(t) - Bv\,(t)| &\leq \omega^2 \int_0^1 G(t,\tau)\,|u(\tau) - v(\tau)|\, d\tau \\ &\leq \frac{\omega^2}{(\alpha + \beta - 1)\,\Gamma(\alpha)\Gamma(\beta)} \|u - v\|_{L^p}\,, \end{aligned}$$

hence

$$\|Bu - Bv\|_{L^p} \leq \frac{\omega^2}{(\alpha + \beta - 1)\,\Gamma(\alpha)\Gamma(\beta)} \|u - v\|_{L^p}\,,$$

by hypothesis (H2), we conclude that B is a contraction. □

Lemma 3.6. *Assume that hypotheses (H1) and (H2) hold, then* $Au + Bv \in \Omega$ *for all* $u, v \in \Omega$*.*

Proof. Let $u, v \in \Omega$, then taking (3.3) into account, it yields

$$\begin{aligned} \|Au - Bv\|_{L^p} &\leq \|Au\|_{L^p} + \|Bv\|_{L^p} \\ &\leq \frac{R\left(\omega^2 + k\right) + M}{(\alpha + \beta - 1)\,\Gamma(\alpha)\Gamma(\beta)} \leq R, \end{aligned}$$

hence $Au + Bv \in \Omega$. □

Proof of Theorem 3.1. By Lemmas 3.4, 3.5 and 3.6, we conclude respectively that the mapping A is completely continuous, the mapping B is a contraction and $Au + Bv \in \Omega$ for all $u, v \in \Omega$, then all hypotheses of Theorem 2.2 are satisfied. Hence, there exists a nontrivial solution $u \in \Omega$ for problem (P) such that $u = Au + Bu$. The proof is complete.

Now, we give an example to illustrate the usefulness of the obtained results.

Example 1. Consider the problem (P) with

$$\begin{aligned} f(t,x) &= \frac{e^{-t}x}{9 + e^t(1 + x^2)} + e^t, (t,x) \in J \times \mathbb{R}, \\ \omega &= 0.5,\ \alpha = 0.5,\ \beta = 0.8, \end{aligned}$$

$$M = \sup_{0\le t\le 1} |f(t,0)| = e = 2.7183.$$

Let us check hypotheses (H1)-(H2). We have for all $(t,x) \in J \times \mathbb{R}$

$$|f(t,x) - f(t,y)| \le \frac{e^{-t}}{9+e^{t}} |x-y| \le \frac{1}{10} |x-y|,$$

then $k = \frac{1}{10}$, $0 < k = 0.1 \le \frac{1}{2}(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta) = 0.30953$. By Theorem 3.1, we conclude that the problem (P) has a nontrivial solution $u \in L^p(0,1)$, such that $\|u\|_{L^p} \le R$, where $R \ge 10.103$ and $u = Au + Bu$.

Example 2. Consider the problem (P) with

$$\begin{aligned} f(t,x) &= \frac{t^{\frac{1}{3}} \sin x + t^{3}}{15}, (t,x) \in J \times \mathbb{R}, \\ \omega &= \frac{1}{10},\ \alpha = \frac{1}{3},\ \beta = \frac{3}{4}, \\ M &= \sup_{0\le t\le 1} |f(t,0)| = \frac{1}{15} \end{aligned}$$

We have for all $(t,x) \in J \times \mathbb{R}$

$$|f(t,x) - f(t,y)| \le \frac{t^{\frac{1}{3}}}{15} |\sin(x) - \sin(y)| \le \frac{1}{15} |x-y|,$$

and $k = \frac{1}{15}$, $\frac{k}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} = 0.24369 \le \frac{1}{2}$, $\frac{\omega^2}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)} = 3.6554 \times 10^{-2} < \frac{1}{2}$. Thus hypotheses (H1) and (H2) are satisfied.

By Theorem 3.1, we conclude that the problem (P) has a nontrivial solution $u \in L^p(0,1)$, such that $\|u\|_{L^p} \le R$, where $R = 1 \ge \frac{M}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)-\left(k+\omega^2\right)} = 0.33858$ and $u = Au + Bu$.

4. Conclusions

In this article, we have proven the existence of non trivial solutions for a boundary value problem containing different type of fractional derivatives. The existence results are obtained via Krasnoselskii's fixed point theorem. For further investigations we propose to study similar problems with different types of fractional derivatives of higher order, by means of some fixed point theorems.

Acknowledgements

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that improved this paper.

Conflict of interest

All authors declare no conflicts of interest.

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