Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4754–4769.
DOI: 10.3934/math.2020304
Received: 01 Apirl 2020
Accepted: 12 May 2020
Published: 01 June 2020

Research article

Unique positive solution for a p-Laplacian fractional differential boundary value problem involving Riemann-Stieltjes integral

Chengbo Zhai[1,*], **Yuanyuan Ma**[1] **and Hongyu Li**[2]

[1] School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, P.R. China
[2] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, China

* **Correspondence:** Email: cbzhai@sxu.edu.cn.

Abstract: In this article, we study a class of p-Laplacian fractional differential boundary value problem involving Riemann-Stieltjes integral. By two fixed point theorems of a sum operator in partial ordering Banach spaces, we get the existence and uniqueness of positive solutions for addressed problem. Moreover, we can make iterative sequences to approximate the unique positive solution. In addition, two examples are given to illustrate the main results.

Keywords: positive solutions; fractional p-Laplacian equations; Riemann-Stieltjes integral boundary condition
Mathematics Subject Classification: 34B18, 34B15, 26A33

1. Introduction

In this paper, we consider the following p-Laplacian fractional differential equation involving Riemann-Stieltjes integral boundary condition

$$\begin{cases} -\mathcal{D}_t^{\beta}(\varphi_p(-\mathcal{D}_t^{\alpha}z(t)-g(t,z(t),\mathcal{D}_t^{\gamma}z(t)))) = f(t,z(t),\mathcal{D}_t^{\gamma}z(t)),\ 0<t<1, \\ \mathcal{D}_t^{\alpha}z(0)=\mathcal{D}_t^{\alpha+1}z(0)=\mathcal{D}_t^{\gamma}z(0)=0, \\ \mathcal{D}_t^{\alpha}z(1)=0, \mathcal{D}_t^{\gamma}z(1)=\int_0^1 \mathcal{D}_t^{\gamma}z(s)dA(s), \end{cases} \tag{1.1}$$

where $D_t^{\alpha}, D_t^{\beta}, D_t^{\gamma}$ are the Riemann-Liouville fractional derivatives of orders α,β,γ with $0<\gamma\le 1<\alpha\le 2<\beta<3, \alpha-\gamma>1$, $\int_0^1 D_t^{\gamma}z(t)dA(s)$ denotes a Riemann-Stieltjes integral, and A is a function of bounded variation. The p-Laplacian operator is defined as $\varphi_p(s)=|s|^{p-2}s,\ p>2$, $\varphi_p(s)$ is invertible and its inverse operator is $\varphi_q(s)$, where $q=\frac{p}{p-1}$ is the conjugate index of p.

Fractional calculus and fractional differential equations arise in many fields, such as, mathematics, physics, economics, engineering, biology, electroanalytical chemistry, capacitor theory, electrical circuits, control theory, and fluid dynamics, see [1–49]. The problem (1.1) can be regarded as a fractional order model for the turbulent flow in a porous medium, see [8,50]. As we know, integral boundary value problems have different applications in applied fields such as blood flow problems, chemical engineering, underground water flow and population dynamics. For example, in [31], Meng and Cui considered the following fractional differential equation involving integral boundary condition

$$\begin{cases} \mathcal{D}_{\alpha}x(t) = f(t, x(t)),\ 0 < t < 1, \\ x(0) = \int_0^1 x(t)dA(t), \end{cases} \tag{1.2}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\int_0^1 x(t)dA(t)$ denotes the Riemann-Stieltjes integral with positive Stieltjes measure. $\mathcal{D}_{\alpha}$ is the conformable fractional derivative of order $0 < \alpha \leq 1$ at $t > 0$. By topological degree theory, the method of lower and upper solutions and a fixed point theorem, they discussed the existence of at least three solutions to the problem (1.2).

In [8], the authors studied the following fractional differential boundary value problem

$$\begin{cases} -\mathcal{D}_t^{\beta}(\varphi_p(-\mathcal{D}_t^{\alpha}x))(t) = f(x(t), \mathcal{D}_t^{\gamma}x(t)),\ t \in (0,1), \\ \mathcal{D}_t^{\alpha}x(0) = \mathcal{D}_t^{\alpha+1}x(0) = \mathcal{D}_t^{\alpha}x(1) = 0, \\ \mathcal{D}_t^{\gamma}x(0) = 0,\ \mathcal{D}_t^{\gamma}x(1) = \int_0^1 \mathcal{D}_t^{\gamma}x(s)dA(s), \end{cases} \tag{1.3}$$

where $\mathcal{D}_t^{\alpha}$, $\mathcal{D}_t^{\beta}$, $\mathcal{D}_t^{\gamma}$ are the Riemann-Liouville derivatives, $\int_0^1 x(s)dA(s)$ is the Riemann-Stieltjes integral and $0 < \gamma \leq 1 < \alpha \leq 2 < \beta < 3$, $\alpha - \gamma > 1$, A is a function bounded variation, φ_p is the p-Laplacian operator. By employing a fixed point theorem for mixed monotone operator, they obtained the existence and uniqueness of positive solutions for the problem (1.3).

When $g \equiv 0$ in (1.1), the author in [21] gave the existence and uniqueness of positive solutions by using monotone iterative technique. Motivated by the results mentioned above and wide applications of different boundary value conditions, we consider the existence and uniqueness of positive solutions for p-Laplacian fractional order differential equation involving Riemann-Stieltjes integral boundary condition (1.1). In Section 2, we present some preliminaries that can be used to prove our main results. The main theorems are formulated and proved in Section 3. Two simple examples are given to illustrate the main results in Section 4.

2. Preliminaries and known results

In the following, we start with some basic concepts and lemmas.

Definition 2.1. *[1]* For a function $x : (0, +\infty) \to \mathbb{R}$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,$$

provide that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2. *[1]* For a function $x : (0, +\infty) \to \mathbb{R}$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is

$$\mathcal{D}_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1}x(s)ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

To reduce the p-Laplacian fractional order differential equation (1.1) to a convenient form, for $x \in C[0, 1]$, making a change of variable $z(t) = I^\gamma x(t)$. By the definitions of the Riemann-Liouville fractional integral and derivative, we can see that $I^\gamma x(t) \to 0, \mathcal{D}_t^\alpha x(t) \to 0$ as $t \to 0^+$. So we first get $x(0) = 0$. From [8,21], the problem (1.1) reduces to an equivalent boundary value problem as follows:

$$\begin{cases} -\mathcal{D}_t^\beta \varphi_p(-\mathcal{D}_t^{\alpha-\gamma} x(t) - g(t, I^\gamma x(t), x(t))) = f(t, I^\gamma x(t), x(t)) \\ \mathcal{D}_t^{\alpha-\gamma} x(0) = D_t^{\alpha-\gamma+1} x(0) = \mathcal{D}_t^{\alpha-\gamma} x(0) = 0, \\ x(0) = 0, x(1) = \int_0^1 x(s) dA(s). \end{cases} \tag{2.1}$$

So, to get the existence and uniqueness of positive solutions for the problem (1.1), we only need to condiser the equivalent problem (2.1). To do this, we fist give an important function

$$G_\beta(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} [t(1-s)]^{\beta-1}, \ 0 \le t \le s \le 1, \\ 1[t(1-s)]^{\beta-1} - (t-s)^{\beta-1}, \ 0 \le s \le t \le 1. \end{cases} \tag{2.2}$$

From Lemma 2.2 in [8], we have the following conclusion:

Lemma 2.1. Given $f, \ g \in L^1[0, 1]$, $0 < \gamma \le 1 < \alpha \le 2 < \beta < 3$ and $\alpha - \gamma > 1$, the fractional order p-Laplacian differential equation

$$\begin{cases} -\mathcal{D}_t^\beta (\varphi_p(-\mathcal{D}_t^{\alpha-\gamma} x(t) - g(t)) = f(t), \\ \mathcal{D}_t^{\alpha-\gamma} x(0) = \mathcal{D}_t^{\alpha-\gamma+1} x(0) = \mathcal{D}_t^{\alpha-\gamma} x(1) = 0, \\ x(0) = 0, x(1) = \int_0^1 x(s) dA(s) \end{cases} \tag{2.3}$$

has a unique solution

$$x(t) = \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G_\beta(s, \tau) f(\tau) d\tau \right) ds + \int_0^1 H(t, s) g(s) ds,$$

where

$$H(t, s) = \frac{\mathcal{G}_A(s)}{1 - \mathcal{A}} t^{\alpha-\gamma-1} + G_{\alpha-\gamma}(t, s) \tag{2.4}$$

with

$$\mathcal{A} = \int_0^1 t^{\alpha-\gamma-1} dA(t), \ \mathcal{G}_A(s) = \int_0^1 G_{\alpha-\beta}(t, s) dA(t).$$

Lemma 2.2. *[21]* Let $0 \le \mathcal{A} < 1$ and $\mathcal{G}_A(s) \ge 0$ for $s \in [0, 1]$, then the functions $G_\beta(t, s)$ and $H(t, s)$ satisfy:

(1) $G_\beta(t, s) > 0, H(t, s) > 0$, for $t, s \in (0, 1)$;

(2) $\frac{t^{\beta-1}(1-t)s(1-s)^{\beta-1}}{\Gamma(\beta)} \le G_\beta(t, s) \le \frac{\beta-1}{\Gamma(\beta)} t^{\beta-1}(1-t)$ for $t, s \in [0, 1]$;

(3) There exist two positive constants d, e such that

$$d t^{\alpha-\gamma-1} \mathcal{G}_A(s) \le H(t, s) \le e t^{\alpha-\gamma-1}, \ t, s \in [0, 1].$$

Let $(E, \|\cdot\|)$ be a real Banach space and θ be the zero element of E. E is partially ordered by a cone $P \subset E$, i.e., $x \le y$ if and only if $y - x \in P$. A cone P is called normal if there exists a constant $N > 0$

such that, for all $x, y \in E$, $\theta \leq x \leq y \Rightarrow \|x\| \leq N\|y\|$; in this case, N is called the normality constant of P. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay (Ax \geq Ay)$.

For $x, y \in E$, the notation $x \sim y$ denotes that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), define $P_h = \{x \in E : x \sim h\}$. It is clear to see that $P_h \subset P$.

Definition 2.3. *[51]* Let $0 < \delta < 1$. An operator $A : P \to P$ is said to be $\delta-$concave if $A(tx) \geq t^{\delta} Ax$ for $t \in (0, 1)$, $x \in P$. An operator $A : P \to P$ is called to be sub-homogeneous if $A(tx) \geq tAx$ for $t > 0$, $x \in P$.

In papers [52, 53], the authors investigated a sum operator equation

$$Ax + Bx = x, \tag{2.5}$$

where A, B are monotone operators. They gave the existence and uniqueness of positive solutions for (2.5) and obtained some interesting theorems.

Lemma 2.3. *[52]* Let E be a real Banach space. P is a normal cone in E, $A : P \to P$ is an increasing δ-concave operator and $B : P \to P$ is an increasing sub-homogeneous operator. Suppose that
(i) there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;
(ii) there exists a constant $\delta_0 > 0$ such that $Ax \geq \delta_0 Bx$ for all $x \in P$.
Then the operator equation (2.5) has a unique solution x^* in P_h. Further, making the sequence $y_n = Ay_{n-1} + By_{n-1}$, $n = 1, 2 \ldots$ for any initial value $y_0 \in P_h$, one has $y_n \to x^*$ as $n \to \infty$.

Lemma 2.4. *[53]* Let E be a real Banach space. P is a normal cone in E, $A : P \to P$ is an increasing operator, and $B : P \to P$ is a decreasing operator. In addition,
(i) for $x \in P$ and $t \in (0, 1)$, there exist $\phi_i(t) \in (t, 1), i = 1, 2$ such that

$$A(tx) \geq \phi_1(t)Ax, \quad B(tx) \leq \frac{1}{\phi_2(t)} Bx; \tag{2.6}$$

(ii) there is $h_0 \in P_h$ such that $Ah_0 + Bh_0 \in P_h$.

Then the operator equation (2.5) has a unique solution x^* in P_h. Further, for any initial values $x_0, y_0 \in P_h$, making the sequences

$$x_n = Ax_{n-1} + By_{n-1}, y_n = Ay_{n-1} + Bx_{n-1}, n = 1, 2 \ldots,$$

one has $x_n \to x^*, y_n \to x^*$ as $n \to \infty$.

Remark 2.1. If B is a null operator, the conclusions in Lemmas 2.1 and 2.2 are still right.

3. Main results

In this section, we intend to obtain some results on the existence and uniqueness of positive solutions for the problem (1.1) by using Lemmas 2.3 and 2.4.

We work in a Banach space $E = C[0,1]$ with the usual norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Let $P = \{x \in C[0,1] : x(t) \geq 0, t \in [0,1]\}$, then it is a normal cone in $C[0,1]$. Hence this space is equipped with a partial order

$$x \leq y,\ x, y \in C[0,1] \Leftrightarrow x(t) \leq y(t),\ t \in [0,1].$$

Theorem 3.1. Let $0 \leq A < 1$ and $\mathcal{G}_A(s) \geq 0$ for $s \in [0,1]$. Assume
(H_1) $f, g : [0,1) \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous and increasing with respect to the second and third arguments, $g(t,0,0) \not\equiv 0,\ t \in [0,1]$;
(H_2) for $\lambda \in (0,1)$, $f(t, \lambda x, \lambda y) \geq \lambda^{\frac{1}{q-1}} f(t,x,y)$ for $x, y \in [0,+\infty)$ and there exists a constant $\delta \in (0,1)$ such that $g(t, \lambda x, \lambda y) \geq \lambda^{\delta} g(t,x,y)$ for all $t \in [0,1],\ x, y \in [0,+\infty)$;
(H_3) There exists a constant $\delta_0 > 0$ such that $f(t,x,y) \leq \delta_0 \leq g(t,0,0),\ t \in [0,1],\ x \geq 0,\ y \geq 0$.
Then there is a unique $y^* \in P_h$, where $h(t) = t^{\alpha-\gamma-1},\ t \in [0,1]$, such that the problem (1.1) has a unique positive solution $z^*(t) = I^{\gamma} y^*(t)$ in set $\Omega := \{I^{\gamma} y(t) | y \in P_h\}$. And for any initial value $y_0 \in P_h$, making sequences

$$y_n(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau) f(\tau, I^\gamma y_{n-1}(\tau), y_{n-1}(\tau))d\tau\right)ds + \int_0^1 H(t,s) g(s, I^\gamma y_{n-1}(s), y_{n-1}(s))ds$$

and $z_n(t) = I^{\gamma} y_n(t),\ n = 1, 2\ldots,$ we have $y_n(t) \to y^*(t)$ and $z_n(t) \to z^*(t)$ as $n \to \infty$, where $G_\beta(s,\tau), H(t,s)$ are given as in (2.2), (2.4) respectively.
Proof. From Lemma 2.1, we know that the problem (2.1) has an integral formulation give by

$$y(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau) f(\tau, I^\gamma y(\tau), y(\tau))d\tau\right)ds + \int_0^1 H(t,s) g(s, I^\gamma y(s), y(s))ds.$$

Define two operators $A : P \to E$ and $B : P \to E$ by

$$\begin{aligned} Ay(t) &= \int_0^1 H(t,s) g(s, I^\gamma y(s), y(s))ds, \\ By(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau) f(\tau, I^\gamma y(\tau), y(\tau))d\tau\right)ds. \end{aligned}$$

Then we see that y is the solution of the problem (2.1) if and only if $y = Ay + By$. From (H_1), (2.4) and Lemma 2.2, we can easily get $A : P \to P$ and $B : P \to P$. In the following, we show that A, B satisfy all assumptions of Lemma 2.3.

Firstly, we prove that A, B are two increasing operators. For $y_1, y_2 \in P$ with $y_1 \geq y_2$, we have $y_1(t) \geq y_2(t),\ t \in [0,1]$ and thus $I^\gamma y_1(t) \geq I^\gamma y_2(t)$. By (H_1), Lemma 2.2,

$$Ay_1(t) = \int_0^1 H(t,s) g(s, I^\gamma y_1(s), y_1(s))ds \geq \int_0^1 H(t,s) g(s, I^\gamma y_2(s), y_2(s))ds = Ay_2(t).$$

Further, noting that $\varphi_p(t)$ is increasing in t, we obtain

$$\begin{aligned} By_1(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau) f(\tau, I^\gamma y_1(\tau), y_1(\tau))d\tau\right)ds \\ &\geq \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau) f(\tau, I^\gamma y_2(\tau), y_2(\tau))d\tau\right)ds = By_2(t). \end{aligned}$$

That is, $Ay_1 \geq Ay_2$ and $By_1 \geq By_2$.

Secondly, we claim that operator A is $\delta-$concave and operator B is sub-homogeneous. For any $\lambda \in (0,1)$ and $y \in P$, from (H_2),

$$\begin{aligned}A(\lambda y)(t) =& \int_0^1 H(t,s)g(s, I^\gamma(\lambda y)(s), \lambda y(s))ds = \int_0^1 H(t,s)g(s, \lambda I^\gamma y(s), \lambda y(s))ds\\ \geq& \lambda^\delta \int_0^1 H(t,s)g(s, I^\gamma y(s), y(s))ds = \lambda^\delta Ay(t),\end{aligned}$$

that is, $A(\lambda y) \geq \lambda^\delta Ay$ for $\lambda \in (0,1),\ y \in P$. So operator A is $\delta-$concave. Also, for any $\lambda \in (0,1)$ and $y \in P$, by (H_2),

$$\begin{aligned}B(\lambda y)(t) =& \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma(\lambda y)(\tau), \lambda y(\tau))d\tau\right)ds\\ =& \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, \lambda I^\gamma y(\tau), \lambda y(\tau))d\tau\right)ds\\ \geq& \int_0^1 H(t,s)\varphi_q\left(\lambda^{\frac{1}{q-1}}\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma y(\tau), y(\tau))d\tau\right)ds\\ =&\lambda \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma y(\tau), y(\tau))d\tau\right)ds = \lambda By(t),\end{aligned}$$

that is, $B(\lambda y) \geq \lambda By$ for $\lambda \in (0,1),\ y \in P$. So operator B is sub-homogeneous.

Thirdly, we show $Ah \in P_h$ and $Bh \in P_h$. Let

$$m_1 = d\int_0^1 \mathcal{G}_A(s)g(s,0,0)ds,\ m_2 = e\int_0^1 g(s, \frac{1}{\Gamma(\gamma+1)}, 1)ds,$$

$$l_1 = \frac{d}{(\Gamma(\beta))^{q-1}}\int_0^1 \mathcal{G}_A(s)[s^{\beta-1}(1-s)]^{q-1}ds \cdot \left[\int_0^1 \tau^{\beta-1}(1-\tau)f(\tau,0,0)d\tau\right]^{q-1},$$

$$l_2 = e\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\left[\int_0^1 f(\tau, \frac{1}{\Gamma(\gamma+1)}, 1)d\tau\right]^{q-1}.$$

From (H_1) and Lemma 2.2,

$$\begin{aligned}Ah(t) =& \int_0^1 H(t,s)g(s, I^\gamma h(s), h(s))ds \leq e\int_0^1 g(s, I^\gamma 1, 1)ds \cdot t^{\alpha-\gamma-1}\\ =&e\int_0^1 g(s, \frac{t^\gamma}{\Gamma(\gamma+1)}, 1)ds \cdot h(t) \leq e\int_0^1 g(s, \frac{1}{\Gamma(\gamma+1)}, 1)ds \cdot h(t) = m_2 \cdot h(t).\end{aligned}$$

Also,

$$\begin{aligned}Ah(t) =& \int_0^1 H(t,s)g(s, I^\gamma h(s), h(s))ds \geq d\int_0^1 \mathcal{G}_A(s)g(s, I^\gamma 0, 0)ds \cdot t^{\alpha-\gamma-1}\\ =&d\int_0^1 \mathcal{G}_A(s)g(s,0,0)ds \cdot h(t) = m_1 \cdot h(t).\end{aligned}$$

By similar discussion, it follows from (H_1) and Lemma 2.2 that

$$\begin{aligned}Bh(t)=&\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau,I^\gamma h(\tau),h(\tau))d\tau\right)ds\\ \le&\int_0^1 et^{\alpha-\gamma-1}\left(\int_0^1\frac{\beta-1}{\Gamma(\beta)}s^{\beta-1}(1-s)f(\tau,I^\gamma h(\tau),h(\tau))d\tau\right)^{q-1}ds\\ \le& e\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\left(\int_0^1 f(\tau,I^\gamma h(\tau),h(\tau))d\tau\right)^{q-1}\cdot t^{\alpha-\gamma-1}\\ \le& e\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\left(\int_0^1 f(\tau,I^\gamma 1,1)d\tau\right)^{q-1}\cdot t^{\alpha-\gamma-1}\\ =& e\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\left(\int_0^1 f(\tau,\frac{\tau^\gamma}{\Gamma(\gamma+1)},1)d\tau\right)^{q-1}\cdot h(t)\\ \le& e\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\left(\int_0^1 f(\tau,\frac{1}{\Gamma(\gamma+1)},1)d\tau\right)^{q-1}\cdot h(t)=l_2\cdot h(t)\end{aligned}$$

and

$$\begin{aligned}Bh(t)=&\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau,I^\gamma h(\tau),h(\tau))d\tau\right)ds\\ \ge& dt^{\alpha-\gamma-1}\int_0^1\mathcal{G}_A(s)\left(\int_0^1\frac{\tau^{\beta-1}(1-\tau)s(1-s)^{\beta-1}}{\Gamma(\beta)}f(\tau,I^\gamma h(\tau),h(\tau))d\tau\right)^{q-1}ds\\ =&\frac{d}{(\Gamma(\beta))^{q-1}}\int_0^1\mathcal{G}_A(s)[s^{\beta-1}(1-s)]^{q-1}ds\cdot\left(\int_0^1\tau^{\beta-1}(1-\tau)f(\tau,I^\gamma h(\tau),h(\tau))d\tau\right)^{q-1}t^{\alpha-\gamma-1}\\ \ge&\frac{d}{(\Gamma(\beta))^{q-1}}\int_0^1\mathcal{G}_A(s)[s^{\beta-1}(1-s)]^{q-1}ds\cdot\left(\int_0^1\tau^{\beta-1}(1-\tau)f(\tau,I^\gamma 0,0)d\tau\right)^{q-1}\cdot t^{\alpha-\gamma-1}\\ =&\frac{d}{(\Gamma(\beta))^{q-1}}\int_0^1\mathcal{G}_A(s)[s^{\beta-1}(1-s)]^{q-1}ds\cdot\left(\int_0^1\tau^{\beta-1}(1-\tau)f(\tau,0,0)d\tau\right)^{q-1}\cdot h(t)=l_1\cdot h(t).\end{aligned}$$

Note that $g(t,0,0)\not\equiv 0$, $\mathcal{G}_A(s)\ge 0$ and $f(\tau,\frac{1}{\Gamma(\gamma+1)})\ge f(\tau,0,0)$, we can easily prove $0<m_1\le m_2$ and $0<l_1\le l_2$, and thus $m_1h\le Ah\le m_2h,\ l_1h\le Bh\le l_2h$. So we have $Ah,Bh\in P_h$. It means that the first condition of Lemma 2.3 holds.

Next we prove that the second condition of Lemma 2.3 is also satisfied. For $y\in P$, by (H_3),

$$\begin{aligned}By(t)=&\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau,I^\gamma y(\tau),y(\tau))d\tau\right)ds\\ \le&\int_0^1 H(t,s)\varphi_q\left(\int_0^1\frac{\beta-1}{\Gamma(\beta)}s^{q-1}(1-s)f(\tau,I^\gamma y(\tau),y(\tau))d\tau\right)ds\\ \le&\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\int_0^1 H(t,s)ds\cdot\varphi_q\left(\int_0^1 f(\tau,I^\gamma y(\tau),y(\tau))d\tau\right)\\ \le&\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\int_0^1{\delta_0}^{q-1}H(t,s)ds=\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\delta_0^{q-2}\int_0^1\delta_0 H(t,s)ds\end{aligned}$$

$$
\begin{aligned}
&\le\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\delta_0^{q-2}\int_0^1 H(t,s)g(s,0,0)ds\\
&\le\left(\frac{1}{\Gamma(\beta-1)}\right)^{q-1}\delta_0^{q-2}\int_0^1 H(t,s)g(s,I^\gamma y(s),y(s))ds\\
&=\left(\frac{1}{\Gamma(\beta-1)}\right)^{q-1}\delta_0^{q-2}Ay(t).
\end{aligned}
$$

Let $\delta_0{}' = [\Gamma(\beta-1)]^{q-1}\delta_0^{2-q}$, so we obtain $Ay(t) \ge \delta_0{}'By(t),\ t \in [0,1]$. Therefore, $Ay \ge \delta_0{}'By$ for $y \in P$.

By the above discussion and Lemma 2.3, we know that operator equation $Ay + By = y$ has a unique solution y^* in P_h; for any initial value $y_0 \in P_h$, making a sequence $y_n = Ay_{n-1} + By_{n-1},\ n = 1,2,\dots,$ we have $y_n \to y^*$ as $n \to \infty$. Evidently, $z^*(t) := I^\gamma y^*(t)$ is the unique solution of the problem (1.1) in $\Omega = \{I^\gamma y(t)|y \in P_h\}$. And for any initial value $y_0 \in P_h$, the sequences

$$
y_{n+1}(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau,I^\gamma y_n(\tau),y_n(\tau))d\tau\right)ds + \int_0^1 H(t,s)g(s,I^\gamma y_n(s),y_n(s))ds
$$

and $z_n(t) = I^\gamma y_n(t),\ n = 1,2\dots$ satisfy $y_n(t) \to y^*(t)$ and $z_n(t) \to z^*(t)$ as $n \to \infty$. □

Corollary 3.1. Let $0 \le A < 1$ and $\mathcal{G}_A(s) \ge 0$ for $s \in [0,1]$. Assume that f satisfies (H_1) and for $\lambda \in (0,1)$, there exists a constant $\delta \in (0,1)$ such that $f(t,\lambda x,\lambda y) \ge \lambda^\delta f(t,x,y)$ for all $t \in [0,1],\ x,y \in [0,+\infty)$.
Then there is a unique $y^* \in P_h$, where $h(t) = t^{\alpha-\gamma-1},\ t \in [0,1]$, such that the following problem

$$
\begin{cases}
-\mathcal{D}_t^\beta(\varphi_p(-\mathcal{D}_t^\alpha z(t)) = f(t,z(t),\mathcal{D}_t^\gamma z(t)),\ 0<t<1,\\
\mathcal{D}_t^\alpha z(0) = \mathcal{D}_t^{\alpha+1}z(0) = \mathcal{D}_t^\gamma z(0) = 0,\\
\mathcal{D}_t^\alpha z(1) = 0, \mathcal{D}_t^\gamma z(1) = \int_0^1 \mathcal{D}_t^\gamma z(s)dA(s),
\end{cases}
$$

has a unique positive solution $z^* = I^\gamma y^*$ in $\Omega = \{I^\gamma y(t)|y \in P_h\}$. And for any initial value $y_0 \in P_h$, making the sequences

$$
y_{n+1}(t) = \int_0^1 H(t,s)\varphi_q(\int_0^1 G_\beta(s,\tau)g(\tau,I^\gamma y_n(\tau),y_n(\tau))d\tau)ds,\ n = 0,1,2\dots
$$

and $z_n(t) = I^\gamma y_n(t),\ n = 1,2\dots,$ we have $y_n(t) \to y^*(t)$ and $z_n(t) \to z^*(t)$ as $n \to \infty$, where $G_\beta(s,\tau), H(t,s)$ are given as in (2.2), (2.4) respectively.
Proof. From Remark 2.1 and Theorem 3.1, the conclusion holds. □

Theorem 3.2. Let $0 \le A < 1$ and $\mathcal{G}_A(s) \ge 0$ for $s \in [0,1]$. Assume f satisfies (H_1) and
(H_4) $g : [0,1]\times[0,+\infty)\times[0,+\infty)$ is continuous and decreasing with respect to second and third arguments, $g(t,\frac{1}{\Gamma(\gamma+1)},1) \not\equiv 0, t \in [0,1]$;
(H_5) for $\lambda \in (0,1)$, there exist $\phi_i(\lambda) \in (\lambda,1)(i = 1,2)$ such that

$$
f(t,\lambda x,\lambda y) \ge \phi_1{}^{\frac{1}{q-1}}(\lambda)f(t,x,y),\ g(t,\lambda x,\lambda y) \le \frac{1}{\phi_2(\lambda)}g(t,x,y)
$$

for $t \in [0,1], x,y \in [0,+\infty)$.
Then there is a unique $y^* \in P_h$, where $h(t) = t^{\alpha-\gamma-1},\ t \in [0,1]$, such that the problem (1.1) has a unique

positive solution $z^*(t) = I^\gamma y^*(t)$ in set $\Omega = \{I^\gamma y(t) | y \in P_h\}$. And for any initial values $x_0, y_0 \in P_h$, putting the sequences

$$x_{n+1}(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma x_n(\tau), x_n(\tau))d\tau\right)ds + \int_0^1 H(t,s)g(s, I^\gamma y_n(s), y_n(s))ds$$
$$y_{n+1}(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma y_n(\tau), y_n(\tau))d\tau\right)ds + \int_0^1 H(t,s)g(s, I^\gamma x_n(s), x_n(s))ds$$

and $\overline{z}_n(t) = I^\gamma x_n(t),\ z_n(t) = I^\gamma y_n(t),\ n = 0, 1, 2\ldots$, we have $x_n(t) \to y^*(t), y_n(t) \to y^*(t), \overline{z}_n(t) \to z^*(t),\ z_n(t) \to z^*(t)$ as $n \to \infty$, where $G_\beta(s,\tau), H(t,s)$ are given as in (2.2), (2.4) respectively.

Proof. Similar to the proof of Theorem 3.1, we still consider two operators $A : P \to E$ and $B : P \to E$ given by

$$Ay(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma y(\tau), y(\tau))d\tau\right)ds,$$
$$By(t) = \int_0^1 H(t,s)g(s, I^\gamma y(s), y(s))ds.$$

It follows from Lemma 2.2, (H_1) and (H_4) that $A : P \to P$ is increasing and $B : P \to P$ is decreasing.

Further, from (H_5), for $\lambda \in (0,1)$,

$$\begin{aligned}
A(\lambda y)(t) =& \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma(\lambda y)(\tau), \lambda y(\tau))d\tau\right)ds \\
=& \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, \lambda I^\gamma(y)(\tau), \lambda y(\tau))d\tau\right)ds \\
\geq& \int_0^1 H(t,s)\left(\int_0^1 G_\beta(s,\tau)\phi_1^{\frac{1}{q-1}}(\lambda)f(\tau, I^\gamma y(\tau), y(\tau))d\tau\right)^{q-1} ds \\
=&\phi_1(\lambda)\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma y(\tau), y(\tau))d\tau\right)ds = \phi_1(\lambda)Ay(t)
\end{aligned}$$

and

$$\begin{aligned}
B(\lambda y)(t) =& \int_0^1 H(t,s)g(s, I^\gamma(\lambda y(s)), \lambda y(s))ds = \int_0^1 H(t,s)g(s, \lambda I^\gamma(y(s)), \lambda y(s))ds \\
\leq& \int_0^1 H(t,s)\frac{1}{\phi_2(\lambda)}g(s, I^\gamma y(s), y(s))ds \\
=&\frac{1}{\phi_2(\lambda)}\int_0^1 H(t,s)g(s, I^\gamma y(s), y(s))ds = \frac{1}{\phi_2(\lambda)}By(t),
\end{aligned}$$

that is, A, B satisfy (2.6). Next, we prove $Ah + Bh \in P_h$. Let

$$n_1 = d\int_0^1 \mathcal{G}_A(s)g(s, \frac{1}{\Gamma(\gamma+1)}, 1)ds,\ n_2 = e\int_0^1 g(s,0,0)ds.$$

By Lemma 2.2,

$$Ah(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma h(\tau), h(\tau))d\tau\right)ds$$

$$
\begin{aligned}
&\leq \int_0^1 e t^{\alpha-\gamma-1}\left(\int_0^1 \frac{\beta-1}{\Gamma(\beta)} s^{\beta-1}(1-s) f(\tau, I^\gamma h(\tau), h(\tau)) d\tau\right)^{q-1} ds \\
&\leq e\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\left(\int_0^1 f(\tau, I^\gamma h(\tau), h(\tau)) d\tau\right)^{q-1} \cdot t^{\alpha-\gamma-1} \\
&\leq e\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\left(\int_0^1 f(\tau, I^\gamma 1, 1) d\tau\right)^{q-1} \cdot h(t) \\
&\leq e\left(\frac{\beta-1}{\Gamma(\beta)}\right)^{q-1}\left(\int_0^1 f(\tau, \frac{1}{\gamma+1}, 1) d\tau\right)^{q-1} \cdot h(t) = l_2 \cdot h(t),
\end{aligned}
$$

$$
\begin{aligned}
Ah(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau) f(\tau, I^\gamma h(\tau), h(\tau)) d\tau\right) ds \\
&\geq d t^{\alpha-\gamma-1}\int_0^1 \mathcal{G}_A(s)\left(\int_0^1 \frac{\tau^{\beta-1}(1-\tau)s(1-s)^{\beta-1}}{\Gamma(\beta)} f(\tau, I^\gamma h(\tau), h(\tau)) d\tau\right)^{q-1} ds \\
&= \frac{d}{(\Gamma(\beta))^{q-1}}\int_0^1 \mathcal{G}_A(s)[s^{\beta-1}(1-s)]^{q-1} ds \cdot \left(\int_0^1 \tau^{\beta-1}(1-\tau) f(\tau, I^\gamma h(\tau), h(\tau)) d\tau\right)^{q-1} t^{\alpha-\gamma-1} \\
&\geq \frac{d}{(\Gamma(\beta))^{q-1}}\int_0^1 \mathcal{G}_A(s)[s^{\beta-1}(1-s)]^{q-1} ds \cdot \left(\int_0^1 \tau^{\beta-1}(1-\tau) f(\tau, I^\gamma 0, 0) d\tau\right)^{q-1} \cdot h(t) \\
&= \frac{d}{(\Gamma(\beta))^{q-1}}\int_0^1 \mathcal{G}_A(s)[s^{\beta-1}(1-s)]^{q-1} ds \cdot \left(\int_0^1 \tau^{\beta-1}(1-\tau) f(\tau, 0, 0) d\tau\right)^{q-1} \cdot h(t) \\
&= l_1 \cdot h(t)
\end{aligned}
$$

and

$$
\begin{aligned}
Bh(t) = \int_0^1 H(t,s) g(s, I^\gamma h(s), h(s)) ds &\geq d t^{\alpha-\gamma-1}\int_0^1 \mathcal{G}_A(s) g(s, I^\gamma h(s), h(s)) ds \\
&\geq d t^{\alpha-\gamma-1}\int_0^1 \mathcal{G}_A(s) g(s, I^\gamma 1, 1) ds = d t^{\alpha-\gamma-1}\int_0^1 \mathcal{G}_A(s) g(s, \frac{1}{\Gamma(\gamma+1)}, 1) ds \\
&= n_1 \cdot h(t),
\end{aligned}
$$

$$
\begin{aligned}
Bh(t) = \int_0^1 H(t,s) g(s, I^\gamma h(s), h(s)) ds &\leq e t^{\alpha-\gamma-1}\int_0^1 g(s, I^\gamma 0, 0) ds \\
&= e t^{\alpha-\gamma-1}\int_0^1 g(s, 0, 0) ds = n_2 \cdot h(t).
\end{aligned}
$$

Hence, $Ah(t) + Bh(t) \leq l_2 \cdot h(t) + n_2 \cdot h(t) = (l_2 + n_2) \cdot h(t)$ and

$$
Ah(t) + Bh(t) \geq l_1 \cdot h(t) + n_1 \cdot h(t) = (l_1 + n_1) \cdot h(t).
$$

In addition, it is easy to show $l_2 + n_2 \geq l_1 + n_1 > 0$. Therefore, $Ah + Bh \in P_h$.

Consequently, by using Lemma 2.4, operator equation $Ay + By = y$ has a unique solution y^* in P_h; for given initial values $x_0,\ y_0 \in P_h$, putting the sequences

$$
x_n = Ax_{n-1} + By_{n-1},\quad y_n = Ay_{n-1} + Bx_{n-1},\quad n = 1, 2 \ldots,
$$

we have $x_n \to y^*,\ y_n \to y^*$ as $n \to \infty$. Evidently, $z^*(t) = I^\gamma y^*(t)$ is the unique solution of the problem (1.1) in $\Omega = \{I^\gamma y(t)|y \in P_h\}$. And for given initial values $x_0, y_0 \in P_h$, the following sequences

$$x_{n+1} = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma x_n(\tau), x_n(\tau))d\tau\right)ds + \int_0^1 H(t,s)g(s, I^\gamma y_n(s), y_n(s))ds,$$
$$y_{n+1} = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma y_n(\tau), y_n(\tau))d\tau\right)ds + \int_0^1 H(t,s)g(s, I^\gamma y_n(s), y_n(s))ds$$

and $\overline{z}_n(t) = I^\gamma x_n(t),\quad z_n(t) = I^\gamma y_n(t),\quad n = 0,1,2\ldots,$ satisfy $x_n(t) \to y^*(t), y_n(t) \to y^*(t), \overline{z}_n(t) \to z^*(t),\ z_n(t) \to z^*(t)$ as $n \to \infty$. □

Corollary 3.2. Let $0 \le A < 1$ and $\mathcal{G}_A(s) \ge 0$ for $s \in [0,1]$. Assume f satisfies $(H_1), (H_5)$. Then there is a unique $y^* \in P_h$, where $h(t) = t^{\alpha-\gamma-1},\ t \in [0,1]$, such that the following problem

$$\begin{cases} -\mathcal{D}_t^\beta(\varphi_p(-\mathcal{D}_t^\alpha z))(t) = f(t, z(t), \mathcal{D}_t^\gamma z(t)),\ 0 < t < 1, \\ \mathcal{D}_t^\alpha z(0) = \mathcal{D}_t^{\alpha+1} z(0) = \mathcal{D}_t^\gamma z(0) = 0, \\ \mathcal{D}_t^\alpha z(1) = 0, \mathcal{D}_t^\gamma z(1) = \int_0^1 \mathcal{D}_t^\gamma z(s)dA(s), \end{cases}$$

has a unique positive solution $z^* = I^\gamma y^*$ in $\Omega = \{I^\gamma y(t)|y \in P_h\}$, where $h(t) = t^{\alpha-\gamma-1}, t \in [0,1]$. And for any initial value $y_0 \in P_h$, putting the sequences

$$y_{n+1} = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, I^\gamma y_n(\tau), y_n(\tau))d\tau\right)ds, n = 0,1,2\ldots,$$

and $z_n(t) = I^\gamma y_n(t),\ n = 1,2\ldots,$ we have $y_n(t) \to y^*(t)$ and $z_n(t) \to z^*(t)$ as $n \to \infty$, where $G_\beta(s,\tau), H(t,s)$ are given as in (2.2), (2.4) respectively.

Proof. From Remark 2.1 and Theorem 3.2, the conclusions hold. □

4. Examples

In this section, two examples are given to illustrate our main results.

Example 4.1. Consider the following 3-Laplacian fractional differential equation with Riemann-Stieltjes integral boundary conditions

$$\begin{cases} -\mathcal{D}_t^{\frac{9}{4}}\left(\varphi_3\left(-\mathcal{D}_t^{\frac{7}{4}}z(t) - t^{\frac{1}{4}}(z^{\frac{1}{4}}(t) + (\mathcal{D}_t^{\frac{1}{2}}z(t))^{\frac{1}{3}}) - 3\right)\right) = \cos^2 t + \frac{z^{\frac{1}{3}}(t)}{1+z^{\frac{1}{3}}(t)} + \frac{(\mathcal{D}_t^{\frac{1}{2}}z(t))^{\frac{1}{4}}}{1+(\mathcal{D}_t^{\frac{1}{2}}z(t))^{\frac{1}{4}}},\ 0 < t < 1, \\ \mathcal{D}_t^{\frac{7}{4}}z(0) = \mathcal{D}_t^{\frac{11}{4}}z(0) = \mathcal{D}_t^{\frac{1}{2}}z(0) = 0, \\ \mathcal{D}_t^{\frac{7}{4}}z(1) = 0, \mathcal{D}_t^{\frac{1}{2}}z(1) = \int_0^1 \mathcal{D}_t^{\frac{1}{2}}z(s)dA(s), \end{cases} \tag{4.1}$$

where $\alpha = \frac{7}{4},\ \beta = \frac{9}{4},\ \gamma = \frac{1}{2},\ p = 3,\ q = \frac{3}{2}, A$ is a function of bounded variation by

$$A(t) = \begin{cases} 0,\ 0 \le t < \frac{1}{4}, \\ \frac{1}{3},\ \frac{1}{4} \le t < \frac{1}{2}, \\ 2,\ \frac{1}{2} \le t \le 1. \end{cases}$$

Further, $f(t,x,y)=\cos^2 t+\frac{x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}+\frac{y^{\frac{1}{4}}}{1+y^{\frac{1}{4}}}$, $g(t,x,y)=t^{\frac{1}{4}}(x^{\frac{1}{4}}+y^{\frac{1}{3}})+3$, clearly, $f,\ g\in C([0,1]\times[0,+\infty)\times[0,+\infty),\ [0,+\infty))$, $g(t,0,0)\not\equiv 0$. For fixed $t\in(0,1)$, $f(t,x,y)$, $g(t,x,y)$ are increasing in x and y. So, the condition (H_1) is satisfied.

In addition, take $\delta=\frac{1}{2}$, for $t\in[0,1]$, $\lambda\in(0,1)$, $x,\ y\in[0,+\infty)$, we have

$$g(t,\lambda x,\lambda y)=t^{\frac{1}{4}}(\lambda^{\frac{1}{4}}x^{\frac{1}{4}}+\lambda^{\frac{1}{3}}y^{\frac{1}{3}})+3\geq\lambda^{\frac{1}{2}}[t^{\frac{1}{4}}(x^{\frac{1}{4}}+y^{\frac{1}{3}})+3]=\lambda^{\delta}g(t,x,y).$$

On the other hand, for $t\in[0,1]$, $\lambda\in(0,1)$, $x,y\in[0,+\infty)$,

$$f(t,\lambda x,\lambda y)=\cos^2 t+\frac{(\lambda x)^{\frac{1}{3}}}{1+(\lambda x)^{\frac{1}{3}}}+\frac{(\lambda y)^{\frac{1}{4}}}{1+(\lambda y)^{\frac{1}{4}}}\geq\lambda^2\cos^2 t+\lambda^2\frac{x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}+\lambda^2\frac{y^{\frac{1}{4}}}{1+y^{\frac{1}{4}}}=\lambda^{\frac{1}{q-1}}f(t,x,y).$$

Hence, the condition (H_2) is satisfied.

Take $\delta_0{}'=3$, then

$$f(t,x,y)=\cos^2 t+\frac{x^{\frac{1}{3}}}{1+x^{\frac{1}{3}}}+\frac{y^{\frac{1}{4}}}{1+y^{\frac{1}{4}}}\leq\delta_0{}'=3g(t,0,0).$$

The condition (H_3) is also satisfied. So Theorem 3.1 shows that the problem (4.1) has a unique positive solution in $\Omega=\{I^{\gamma}y(t)|y\in P_h\}$, where $h(t)=t^{\frac{1}{4}}$, $t\in[0,1]$.

Example 4.2. Consider the following 3-Laplacian fractional differential equation with Riemann-Stieltjes integral boundary conditions:

$$\begin{cases} -\mathcal{D}_t^{\frac{9}{4}}(\varphi_3(-\mathcal{D}_t^{\frac{7}{4}}z(t)-[t^{\frac{1}{4}}(z^{\frac{1}{4}}(t)+(\mathcal{D}_t^{\frac{1}{2}}z(t))^{\frac{1}{3}})+1]^{-1}=t^{\frac{1}{3}}[z^{\frac{1}{3}}(t)+(\mathcal{D}_t^{\frac{1}{2}}z(t))^{\frac{1}{4}}]+2, t\in(0,1),\\ \mathcal{D}_t^{\frac{7}{4}}z(0)=\mathcal{D}_t^{\frac{11}{4}}z(0)=\mathcal{D}_t^{\frac{1}{2}}z(0)=0,\\ \mathcal{D}_t^{\frac{7}{4}}z(1)=0, \mathcal{D}_t^{\frac{1}{2}}z(1)=\int_0^1\mathcal{D}_t^{\frac{1}{2}}z(s)dA(s),\end{cases} \tag{4.2}$$

where $\alpha=\frac{7}{4}$, $\beta=\frac{9}{4}$, $\gamma=\frac{1}{2}$, $p=3$, $q=\frac{3}{2}$, A is a function of bounded variation by

$$A(t)=\begin{cases}0,\ 0\leq t<\frac{1}{4},\\ \frac{1}{3},\ \frac{1}{4}\leq t<\frac{1}{2},\\ 2,\ \frac{1}{2}\leq t\leq 1.\end{cases}$$

Let $f(t,x,y)=t^{\frac{1}{3}}(x^{\frac{1}{3}}+y^{\frac{1}{4}})+2$, $g(t,x,y)=[t^{\frac{1}{4}}(x^{\frac{1}{4}}+y^{\frac{1}{3}})+1]^{-1}$, clearly, $f,\ g\in C([0,1)\times[0,+\infty)\times[0,+\infty),\ [0,+\infty))$, $f(t,0,0)\not\equiv 0$, $g(t,\frac{1}{\Gamma(\gamma+1)},1)\not\equiv 0$. For fixed $t\in[0,1)$, $f(t,x,y)$ is increasing in x and y, $g(t,x,y)$ is decreasing in x and y. So, the conditions (H_4) and (H_5) are satisfied. Take $\phi_1(\lambda)=\lambda^{\frac{1}{6}}$, $\phi_2(\lambda)=\lambda^{\frac{1}{3}}$, then $\phi_1(\lambda)$, $\phi_2(\lambda)\in(\lambda,1)$ for $\lambda\in(0,1)$. Thus,

$$f(t,\lambda x,\lambda y)=t^{\frac{1}{3}}(\lambda^{\frac{1}{3}}x^{\frac{1}{3}}+\lambda^{\frac{1}{4}}y^{\frac{1}{4}})+2\geq\lambda^{\frac{1}{3}}[t^{\frac{1}{3}}(x^{\frac{1}{3}}+y^{\frac{1}{4}})+2]=(\phi_1(\lambda))^{\frac{1}{q-1}}f(t,x,y),$$

$$g(t,\lambda x,\lambda y)=[t^{\frac{1}{4}}(\lambda^{\frac{1}{4}}x^{\frac{1}{4}}+\lambda^{\frac{1}{3}}y^{\frac{1}{3}})+1]^{-1}\geq\lambda^{-\frac{1}{3}}[t^{\frac{1}{4}}(x^{\frac{1}{4}}+y^{\frac{1}{3}})+1]^{-1}=\frac{1}{\phi_2(\lambda)}g(t,x,y).$$

So Theorem 3.2 implies that the problem (4.2) has a unique positive solution in $\Omega=\{I^{\gamma}y(t)|y\in P_h\}$, where $h(t)=t^{\frac{1}{4}}$, $t\in[0,1]$.

5. Conclusions

Integral boundary value problems have many applications in applied fields such as blood flow problems, chemical engineering, underground water flow and population dynamics. For nonlinear fractional differential equations with p-Laplacian operator subject to different boundary conditions, there are many works reported on the existence or multiplicity of positive solutions. But the unique results are very rare. In this paper, we study a p-Laplacian fractional order differential equation involving Riemann-Stieltjes integral boundary condition (1.1). By means of the properties of Green's function and two fixed point theorems of a sum operator in partial ordering Banach spaces, we establish some new existence and uniqueness criteria for (1.1). Our result shows that the unique positive solution exists in a special set P_h and can be approximated by constructing an iterative sequence for any initial point in P_h. Finally, two interesting examples are given to illustrate the application of our main results.

Acknowledgments

This paper was by the Youth Science Foundation of China (11801322), Shanxi Province Science Foundation (201901D111020) and Graduate Science and Technology Innovation Project of Shanxi (2019BY014).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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