Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4681–4701.
DOI:10.3934/math.2020300
Received: 27 November 2019
Accepted: 21 May 2020
Published: 27 May 2020

Research article

On Fejer type inequalities for co-ordinated hyperbolic ρ-convex functions

Hasan Kara[1,*], **Hüseyin Budak**[1] **and Mehmet Eyüp Kiriş**[2]

[1] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
[2] Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyon, Turkey

* **Correspondence:** Email: hasan64kara@gmail.com.

Abstract: In this study, we first establish some Hermite-Hadamard-Fejer type inequalities for co-ordinated hyperbolic ρ-convex functions. Then, by utilizing these inequalities, we also give some fractional Fejer type inequalities for co-ordinated hyperbolic ρ-convex functions. The inequalities obtained in this study provide generalizations of some result given in earlier works.

Keywords: convex function; hyperbolic ρ-convex functions; Fejer inequality; fractional integrals
Mathematics Subject Classification: 26D07, 26D10, 26D15, 26A33

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [9], [18], [27, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

The Hermite-Hadamard inequality,which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. A number of mathematicians have devoted their efforts.

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities. In [17], Fejer gave a weighted generalization of the inequalities (1.1) as the following:

Theorem 1. *$f : [a,b] \to \mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$ (i.e. $g(x) = g(a+b-x)$).

In this paper we will establish some new Fejér type inequalities for the new concept of co-ordinated hyperbolic ρ-convex functions.

The overall structure of the paper takes the form of four sections including introduction. The paper is organized as follows: we first give the definition of co-ordinated convex functions, the definition of fractional integrals and related Hermite-Hadamard inequality in Section 1. We also recall the concept of hyperbolic ρ-convex functions and co-ordinated hyperbolic ρ-convex functions introduced by Özçelik et. al in [23]. Moreover, we give a lemma and a theorem which will be frequently used in the next section. Some Hermite-Hadamard-Fejer type inequalities for co-ordinated hyperbolic ρ-convex functions are obtained and some special cases of the results are also given in Section 2. Then, we also apply the inequalities obtained in Section 2 to establish some fractional Fejer type inequalities in Section 3. Finally, in Section 4, some conclusions and further directions of research are discussed.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x,u),(y,v) \in \Delta$ and $t,s \in [0,1]$, if it satisfies the following inequality:*

$$\begin{aligned} & f(tx+(1-t)\,y, su+(1-s)\,v) \\ \leq \quad & ts\, f(x,u) + t(1-s)f(x,v) + s(1-t)f(y,u) + (1-t)(1-s)f(y,v). \end{aligned} \tag{1.3}$$

The mapping f is a co-ordinated concave on Δ if the inequality (1.3) holds in reversed direction for all $t,s \in [0,1]$ and $(x,u),(y,v) \in \Delta$.

In [11], Dragomir proved the following inequalities which is Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 2. *Suppose that $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) & \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\ & \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ & \leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \end{aligned} \tag{1.4}$$

$$+ \quad \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\Bigg]$$

$$\leq \quad \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

The above inequalities are sharp. The inequalities in (1.4) hold in reverse direction if the mapping f is a co-ordinated concave mapping.

Over the years, the numerous studies have focused on to establish generalization of the inequality (1.1) and (1.4). For some of them, please see ([1–8], [19–26], [28–36]).

Definition 2. *[29] Let $f \in L_1(\Delta)$ .The Riemann-Lioville integrals $J^{\alpha,\beta}_{a+,c+}, J^{\alpha,\beta}_{a+,d-}, +J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$of order $\alpha,\beta > 0$ with $a,c \geq 0$ are defined by*

$$J^{\alpha,\beta}_{a+,c+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1} f(t,s)\,dsdt, \quad x>a,\ y>c,$$

$$J^{\alpha,\beta}_{a+,d-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1} f(t,s)\,dsdt, \quad x>a,\ y>d,$$

$$J^{\alpha,\beta}_{b-,c+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1} f(t,s)\,dsdt, \quad x<b,\ y>c,$$

$$J^{\alpha,\beta}_{b-,d-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1} f(t,s)\,dsdt, \quad x<b,\ y<d,$$

respectively. Here, Γ is the Gamma funtion,

$$J^{0,0}_{a+,c+}f(x,y) = J^{0,0}_{a+,d-}f(x,y) = J^{0,0}_{b-,c+}f(x,y) = J^{0,0}_{b-,d-}f(x,y)$$

and

$$J^{1,1}_{a+,c+}f(x,y) = \int_a^x\int_c^y f(t,s)\,dsdt.$$

First, we give the definition of hyperbolic ρ-convex functions and some related inequalities. Then we define the co-ordinated hyperbolic ρ-convex functions.

Definition 3. *[10] A function $f : I \to \mathbb{R}$ is said to be hyperbolic ρ-convex, if for any arbitrary closed subinterval $[a,b]$ of I such that we have*

$$f(x) \leq \frac{\sinh[\rho(b-x)]}{\sinh[\rho(b-a)]}f(a) + \frac{\sinh[\rho(x-a)]}{\sinh[\rho(b-a)]}f(b) \tag{1.5}$$

for all $x \in [a,b]$. If we take $x = (1-t)a + tb$, $t \in [0,1]$ in (1.5), then the condition (1.5) becomes

$$f((1-t)a+tb) \leq \frac{\sinh[\rho(1-t)(b-a)]}{\sinh[\rho(b-a)]}f(a) + \frac{\sinh[\rho t(b-a)]}{\sinh[\rho(b-a)]}f(b). \tag{1.6}$$

If the inequality (1.5) holds with "≥", then the function will be called hyperbolic ρ-concave on I.

The following Hermite-Hadamard inequality for hyperbolic ρ-convex function is proved by Dragomir in [10].

Theorem 3. *Suppose that $f : I \to \mathbb{R}$ is hyperbolic ρ-convex on I. Then for any $a, b \in I$, we have*

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sinh\left[\frac{\rho(b-a)}{2}\right] \leq \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{\rho} \tanh\left[\frac{\rho(b-a)}{2}\right]. \tag{1.7}$$

Moreover in [12], Dragomir prove the following Hermite Hadamard-Fejer type inequalities for hyperbolic ρ-convex functions.

Theorem 4. *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic ρ-convex on I and $a, b \in I$. Assume also that $p : [a, b] \longrightarrow \mathbb{R}$ is a positive, symmetric and integrable function on $[a, b]$, then we have*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \int_a^b \cosh\left[\rho\left(x - \frac{a+b}{2}\right)\right] p(x)dx \\ \leq & \int_a^b f(x)p(x)dx \\ \leq & \frac{f(a)+f(b)}{2} \sec h\left[\frac{\rho(b-a)}{2}\right] \int_a^b \cosh\left[\rho\left(x - \frac{a+b}{2}\right)\right] p(x)dx. \end{aligned} \tag{1.8}$$

For the other inequalities for hyperbolic ρ-convex functions, please refer to ([12–15]).

Now we give the definition of co-ordinated hyperbolic ρ-convex functions.

Definition 4. *[23] A function $f : \Delta \to \mathbb{R}$ is said to co-ordinated hyperbolic ρ-convex on Δ, if the inequality*

$$\begin{aligned} f(x,y) \leq & \frac{\sinh[\rho_1(b-x)]}{\sinh[\rho_1(b-a)]} \frac{\sinh[\rho_2(d-y)]}{\sinh[\rho_2(d-c)]} f(a,c) + \frac{\sinh[\rho_1(b-x)]}{\sinh[\rho_1(b-a)]} \frac{\sinh[\rho_2(y-c)]}{\sinh[\rho_2(d-c)]} f(a,d) \\ & + \frac{\sinh[\rho_1(x-a)]}{\sinh[\rho_1(b-a)]} \frac{\sinh[\rho_2(d-y)]}{\sinh[\rho_2(d-c)]} f(b,c) + \frac{\sinh[\rho_1(x-a)]}{\sinh[\rho_1(b-a)]} \frac{\sinh[\rho_2(y-c)]}{\sinh[\rho_2(d-c)]} f(b,d). \end{aligned} \tag{1.9}$$

holds.

If the inequality (1.9) holds with "≥", then the function will be called co-ordinated hyperbolic ρ-concave on Δ.

If we take $x = (1-t)a + tb$ and $y = (1-s)c + sd$ for $t, s, \in [0, 1]$, then the inequality (1.9) can be written as

$$f((1-t)a + tb, (1-s)c + sd)$$

$$\begin{aligned}
&\leq \frac{\sinh\left[\rho_1(1-t)(b-a)\right]}{\sinh\left[\rho_1(b-a)\right]}\frac{\sinh\left[\rho_2(1-s)(d-y)\right]}{\sinh\left[\rho_2(d-c)\right]}f(a,c) \qquad (1.10)\\
&+\frac{\sinh\left[\rho_1(1-t)(b-a)\right]}{\sinh\left[\rho_1(b-a)\right]}\frac{\sinh\left[\rho_2 s(d-y)\right]}{\sinh\left[\rho_2(d-c)\right]}f(a,d)\\
&+\frac{\sinh\left[\rho_1 t(b-a)\right]}{\sinh\left[\rho_1(b-a)\right]}\frac{\sinh\left[\rho_2(1-s)(d-y)\right]}{\sinh\left[\rho_2(d-c)\right]}f(b,c)\\
&+\frac{\sinh\left[\rho_1(b-a)\right]}{\sinh\left[\rho_1(b-a)\right]}\frac{\sinh\left[\rho_2 s(d-y)\right]}{\sinh\left[\rho_2(d-c)\right]}f(b,d).
\end{aligned}$$

Now we give the following useful lemma:

Lemma 1. *[23] If $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is co-ordinated ρ-convex function on Δ, then we have the following inequality*

$$\begin{aligned}
&\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq \frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right] \qquad (1.11)\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\frac{\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]}{\cosh\left[\frac{\rho_1(b-a)}{2}\right]}\frac{\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]}{\cosh\left[\frac{\rho_2(d-c)}{2}\right]}
\end{aligned}$$

for all $(x,y)\in\Delta$.

2. Fejer type inequalities for co-ordinated hyperbolic ρ-convex functions

Theorem 5. *Let $p:\Delta\to\mathbb{R}$ be a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. Let, $f:\Delta\to\mathbb{R}$ be a co-ordinated hyperbolic ρ-convex functions on Δ. We have the following Hermite-Hadamard-Fejer type inequalities:*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)\,dydx\\
&\leq \int_a^b\int_c^d f(x,y)p(x,y)dydx \qquad (2.1)\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cosh\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\frac{\rho_2(d-c)}{2}\right]}\\
&\times\int_a^b\int_c^d\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)\,dydx.
\end{aligned}$$

Proof. Multiplying the inequality (1.11) by $p(x,y)>0$ and then integrating with respect to (x,y) on Δ, we obtain

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p\left(x,y\right)dydx \\
\leq\ & \frac{1}{4}\int_a^b\int_c^d \left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]p(x,y)dydx \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cosh\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\frac{\rho_2(d-c)}{2}\right]} \\
& \times\int_a^b\int_c^d \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)dydx
\end{aligned}
\tag{2.2}
$$

Since p is symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$, one can show that

$$
\begin{aligned}
\int_a^b\int_c^d f(x,c+d-y)p(x,y)dydx &= \int_a^b\int_c^d f(a+b-x,y)p(x,y)dydx \\
&= \int_a^b\int_c^d f(a+b-x,c+d-y)p(x,y)dydx \\
&= \int_a^b\int_c^d f(x,y)p(x,y)dydx.
\end{aligned}
$$

This completes the proof. □

Remark 1. *If we choose $p(x,y)=1$ in Theorem 5 , then we have the following the inequality*

$$
\begin{aligned}
& \frac{4}{\rho_1\rho_2}\sinh\left[\frac{\rho_1\left(b-a\right)}{2}\right]\sinh\left[\frac{\rho_2\left(d-c\right)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \int_a^b\int_c^d f(x,y)dydx \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}\tanh\left[\frac{\rho_1\left(b-a\right)}{2}\right]\tanh\left[\frac{\rho_2\left(d-c\right)}{2}\right]
\end{aligned}
$$

which is proved by Özçelik et. al in [23].

Corollary 1. *Suppose that all assumptions of Theorem 5 are satisfied. Then we have the following inequality,*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d w(x,y)\, dydx \\
\leq\ & \int_a^b \int_c^d f(x,y) w(x,y) \sec h\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \sec h\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydx \qquad (2.3) \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \sec h\left[\frac{\rho_1 (b-a)}{2}\right] \sec h\left[\frac{\rho_2 (d-c)}{2}\right] \int_a^b \int_c^d w(x,y)\, dydx.
\end{aligned}
$$

Proof. Let us define the function $p(x,y)$ by

$$
w(x,y) = \frac{p(x,y)}{\cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right]}.
$$

Clearly, $w(x.y)$ is a a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. If we apply Theorem 5 for the function $w(x,y)$ then we establish the desired inequality (2.3). □

Remark 2. *If we choose $w(x,y) = 1$ for all $(x,y) \epsilon \Delta$ in Corollary 1, then we have the following the inequality*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \sec h\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \sec h\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] dydx \qquad (2.4) \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \sec h\left[\frac{\rho_1 (b-a)}{2}\right] \sec h\left[\frac{\rho_2 (d-c)}{2}\right].
\end{aligned}
$$

which is proved by Özçelik et. al in [23].

Theorem 6. *Let $p : \Delta \to \mathbb{R}$ be a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. Let $f : \Delta \to \mathbb{R}$ be a co-ordinated hyperbolic ρ-convex on Δ, then we have the following Hermite-Hadamard-Fejer type inequalities*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d \cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] p(x,y) dydx \\
\leq\ & \frac{1}{2}\left[\int_a^b \int_c^d f\left(x, \frac{c+d}{2}\right) \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] p(x,y) dydx\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_a^b\int_c^d f\left(\frac{a+b}{2},y\right)\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]p(x,y)dydx\Bigg]\\
\leq\;& \int_a^b\int_c^d f(x,y)p(x,y)dydx \qquad (2.5)\\
\leq\;& \frac{1}{4}\Bigg[\sec h\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b\int_c^d [f(x,c)+f(x,d)]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)dydx\\
&+\sec h\left[\frac{\rho_1(b-a)}{2}\right]\int_a^b\int_c^d [f(a,y)+f(b,y)]\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]p(x,y)dydx\Bigg]\\
\leq\;& \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right]\\
&\times\int_a^b\int_c^d \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)dydx.
\end{aligned}
$$

Proof. Since f is co-ordinated hyperbolic ρ-convex on Δ, if we define the mappings $f_x:[c,d]\to\mathbb{R}$, $f_x(y)=f(x,y)$ and $p_x:[c,d]\to\mathbb{R}$, $p_x(y)=p(x,y)$, then $f_x(y)$ is hyperbolic ρ-convex on $[c,d]$ and $p_x(y)$ is positive, integrable and symmetric about $\frac{c+d}{2}$ for all $x\in[a,b]$. If we apply the inequality (1.8) for the hyperbolic ρ-convex function $f_x(y)$, then we have

$$
\begin{aligned}
&f_x\left(\frac{c+d}{2}\right)\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p_x(y)dy\\
\leq\;& \int_c^d f_x(y)p_x(y)dy \qquad (2.6)\\
\leq\;& \frac{f_x(c)+f_x(d)}{2}\sec h\left[\frac{\rho_2(d-c)}{2}\right]\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p_x(y)dy.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&f\left(x,\frac{c+d}{2}\right)\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)dy\\
\leq\;& \int_c^d f(x,y)p(x,y)dy \qquad (2.7)\\
\leq\;& \frac{f(x,c)+f(x,d)}{2}\sec h\left[\frac{\rho_2(d-c)}{2}\right]\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)dy.
\end{aligned}
$$

Integrating the inequality (2.7) with respect to x from a to b, we obtain

$$\begin{aligned} & \int_a^b \int_c^d f\left(x, \frac{c+d}{2}\right) \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] p(x,y)dydx \\ \leq & \int_a^b \int_c^d f(x,y)p(x,y)dydx \\ \leq & \frac{1}{2}\int_a^b \int_c^d [f(x,c)+f(x,d)] \sec h\left[\frac{\rho_2(d-c)}{2}\right] \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] p(x,y)dydx. \end{aligned} \tag{2.8}$$

Similarly, as f is co-ordinated hyperbolic ρ-convex on Δ, if we define the mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(x) = f(x,y)$ and $p_y : [a,b] \to \mathbb{R}$, $p_y(x) = p(x,y)$, then $f_y(x)$ is hyperbolic ρ-convex on $[a,b]$ and $p_y(x)$ is positive, integrable and symmetric about $\frac{a+b}{2}$ for all $y \in [c,d]$. Utilizing the inequality (1.8) for the hyperbolic ρ-convex function $f_y(x)$, then we obtain the inequality

$$\begin{aligned} & f_y\left(\frac{a+b}{2}\right) \int_a^b \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p_y(x)dx \\ \leq & \int_a^b f_y(x)p_y(x)dx \\ \leq & \frac{f_y(a)+f_y(b)}{2} \sec h\left[\frac{\rho_1(b-a)}{2}\right] \int_a^b \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p_y(x)dx \end{aligned} \tag{2.9}$$

i.e.

$$\begin{aligned} & f\left(\frac{a+b}{2}, y\right) \int_a^b \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p(x,y)dx \\ \leq & \int_a^b f(x,y)p(x,y)dx \\ \leq & \frac{f(a,y)+f(b,y)}{2} \sec h\left[\frac{\rho_1(b-a)}{2}\right] \int_a^b \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p(x,y)dx. \end{aligned} \tag{2.10}$$

Integrating the inequality (2.10) with respect to y on $[c,d]$, we get

$$\int_a^b \int_c^d f\left(\frac{a+b}{2}, y\right) \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p(x,y)dydx$$

$$\leq \int_a^b \int_c^d f(x,y)p(x,y)dydx \tag{2.11}$$
$$\leq \frac{1}{2}\int_a^b \int_c^d [f(a,y)+f(b,y)] \sec h\left[\frac{\rho_1(b-a)}{2}\right] \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p(x,y)dydx.$$

Summing the inequalities (2.8) and (2.11), we obtain the second and third inequalities in (2.5).

Since $f\left(\frac{a+b}{2},y\right)$ is hyperbolic ρ-convex on $[c,d]$ and $p_x(y)$ is positive, integrable and symmetric about $\frac{c+d}{2}$, using the first inequality in (1.8), we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] p(x,y)dy$$
$$\leq \int_c^d f\left(\frac{a+b}{2},y\right)p(x,y)dy. \tag{2.12}$$

Multiplying the inequality (2.12) by $\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]$ and integrating resulting inequality with respect to x on $[a,b]$, we get

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b \int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p(x,y)dydx$$
$$\leq \int_a^b \int_c^d f\left(\frac{a+b}{2},y\right) \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p(x,y)dydx. \tag{2.13}$$

Since $f\left(x,\frac{c+d}{2}\right)$ is hyperbolic ρ-convex on $[a,b]$ and $p_y(x)$ is positive, integrable and symmetric about $\frac{a+b}{2}$, utilizing the first inequality in (1.8), we have the following inequality

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] p(x,y)dx \tag{2.14}$$
$$\leq \int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)dx.$$

Multiplying the inequality (2.14) by $\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]$ and integrating resulting inequality with respect to y on $[c,d]$, we get

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b \int_c^d \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right] \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] p(x,y)dydx \tag{2.15}$$

$$\leq \int_a^b \int_c^d f\left(x, \frac{c+d}{2}\right) \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] p(x,y)dydx.$$

From the inequalities (2.13) and (2.15), we obtain the first inequality in (2.5).

For the proof of last inequality in (2.5), using the second inequality in (1.8) for the hyperbolic ρ-convex functions $f(x,c)$ and $f(x,d)$ on $[a,b]$ and for the symmetric function $p_y(x)$, we obtain the inequalities

$$\begin{aligned} &\int_a^b f(x,c)p(x,y)dx \qquad (2.16)\\ \leq\ &\frac{f(a,c)+f(b,c)}{2} \sec h\left[\frac{\rho_1(b-a)}{2}\right] \int_a^b \cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] p(x,y)dx \end{aligned}$$

and

$$\begin{aligned} &\int_a^b f(x,d)p(x,y)dx \qquad (2.17)\\ \leq\ &\frac{f(a,d)+f(b,d)}{2} \sec h\left[\frac{\rho_1(b-a)}{2}\right] \int_a^b \cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] p(x,y)dx. \end{aligned}$$

If we multiply the inequalities (2.16) and (2.17) by $\sec h\left[\frac{\rho_2(d-c)}{2}\right] \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right]$ and integrating the resulting inequalities on $[c,d]$, then we have

$$\begin{aligned} &\int_a^b \int_c^d f(x,c) \sec h\left[\frac{\rho_2(d-c)}{2}\right] \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] p(x,y)dydx \\ \leq\ &\frac{f(a,c)+f(b,c)}{2} \sec h\left[\frac{\rho_1(b-a)}{2}\right] \sec h\left[\frac{\rho_2(d-c)}{2}\right] \qquad (2.18)\\ &\times \int_a^b \int_c^d \cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] p(x,y)dydx \end{aligned}$$

and

$$\begin{aligned} &\int_a^b \int_c^d f(x,d) \sec h\left[\frac{\rho_2(d-c)}{2}\right] \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] p(x,y)dydx \\ \leq\ &\frac{f(a,d)+f(b,d)}{2} \sec h\left[\frac{\rho_1(b-a)}{2}\right] \sec h\left[\frac{\rho_2(d-c)}{2}\right] \qquad (2.19)\\ &\times \int_a^b \int_c^d \cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] p(x,y)dydx. \end{aligned}$$

Similarly, applying the second inequality in (1.8) for the hyperbolic ρ-convex functions $f(a, y)$ and $f(b, y)$ on $[c, d]$ and for the symmetric function $p_x(y)$, we have

$$\int_c^d f(a,y)p(x,y)dy \tag{2.20}$$
$$\leq \frac{f(a,c)+f(a,d)}{2}\sec h\left[\frac{\rho_2(d-c)}{2}\right]\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)dy$$

and

$$\int_c^d f(b,y)p(x,y)dy \tag{2.21}$$
$$\leq \frac{f(b,c)+f(b,d)}{2}\sec h\left[\frac{\rho_2(d-c)}{2}\right]\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]p(x,y)dy.$$

Multiplying the inequalities (2.20) and (2.21) by $\sec h\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]$ and integrating the resulting inequalities on $[a, b]$, then we have

$$\int_a^b\int_c^d f(a,y)\sec h\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]p(x,y)dydx$$
$$\leq \frac{f(a,c)+f(a,d)}{2}\sec h\left[\frac{\rho_2(d-c)}{2}\right]\sec h\left[\frac{\rho_1(b-a)}{2}\right] \tag{2.22}$$
$$\times\int_a^b\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]p(x,y)dydx$$

and

$$\int_a^b\int_c^d f(b,y)\sec h\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]p(x,y)dydx$$
$$\leq \frac{f(b,c)+f(b,d)}{2}\sec h\left[\frac{\rho_2(d-c)}{2}\right]\sec h\left[\frac{\rho_1(b-a)}{2}\right] \tag{2.23}$$
$$\times\int_a^b\int_c^d \cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]\cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]p(x,y)dydx.$$

Summing the inequalities (2.18), (2.19), (2.22) and (2.23), we establish the last inequality in (2.5). This completes the proof. □

Remark 3. *If we choose* $p(x,y)=1$ *in Theorem 6, then we have*

$$\begin{aligned}
&\frac{4}{\rho_1\rho_2}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\sinh\left[\frac{\rho_2(d-c)}{2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\;& \frac{1}{\rho_1}\sinh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d f\left(\frac{a+b}{2},y\right)dy+\frac{1}{\rho_2}\sinh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\
\leq\;& \int_a^b\int_c^d f(x,y)\,dydx\\
\leq\;& \frac{1}{2}\left[\frac{1}{\rho_2}\tanh\left[\frac{\rho_2(d-c)}{2}\right]\int_a^b[f(x,c)+f(x,d)]\,dx\right.\\
&\left.+\frac{1}{\rho_1}\tanh\left[\frac{\rho_1(b-a)}{2}\right]\int_c^d[f(a,y)+f(b,y)]\,dy\right]\\
\leq\;& \tanh\left[\frac{\rho_1(b-a)}{2}\right]\tanh\left[\frac{\rho_2(d-c)}{2}\right]\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}
\end{aligned} \tag{2.24}$$

which is proved by Özçelik et. al in [23].

Remark 4. *Choosing* $\rho_1=\rho_2=0$ *in Theorem 6, we obtain*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx\\
\leq\;& \frac{1}{2}\int_a^b\int_c^d\left[f\left(x,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},y\right)\right]p(x,y)dydx\\
\leq\;& \int_a^b\int_c^d f(x,y)p(x,y)dydx\\
\leq\;& \frac{1}{4}\int_a^b\int_c^d[f(x,c)+f(x,d)+f(a,y)+f(b,y)]\,p(x,y)dydx\\
\leq\;& \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx.
\end{aligned}$$

which is proved by Budak and Sarikaya in [5].

Corollary 2. *Let $g_1 : [a,b] \to \mathbb{R}$ and $g_1 : [c,d] \to \mathbb{R}$ be two positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$, respectively. If we choose $p(x,y) = \frac{g_1(x)g_2(y)}{G_1G_2}$ for all $(x,y) \in \Delta$ in Theorem 6, then we have*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x, \frac{c+d}{2}\right) g_1(x)dx + \frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2}, y\right) g_2(y)dy\right] \\
\leq\ & \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx \\
\leq\ & \frac{1}{4}\left[\sec h\left[\frac{\rho_2(d-c)}{2}\right]\frac{1}{G_1}\int_a^b [f(x,c)+f(x,d)]\, g_1(x)dx\right. \\
& \left. + \sec h\left[\frac{\rho_1(b-a)}{2}\right]\frac{1}{G_2}\int_c^d [f(a,y)+f(b,y)]\, g_2(y)dy\right] \\
\leq\ & \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right]
\end{aligned} \tag{2.25}$$

where

$$G_1 = \int_a^b \cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] g_1(x)dx \text{ and } G_2 = \int_c^d \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] g_2(y)dy.$$

Remark 5. *If we choose $\rho_1 = \rho_2 = 0$ in Corollary 2, then we have*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x, \frac{c+d}{2}\right) g_1(x)dx + \frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2}, y\right) g_2(y)dy\right] \\
\leq\ & \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx \\
\leq\ & \frac{1}{4}\left[\frac{1}{G_1}\int_a^b [f(x,c)+f(x,d)]\, g_1(x)dx + \frac{1}{G_2}\int_c^d [f(a,y)+f(b,y)]\, g_2(y)dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}
\end{aligned}$$

which is proved by Farid et al. in [16].

3. Fractional inequalities for co-ordinated hyperbolic ρ-convex functions

In this section we obtain some fractional Hermite-Hadamard an Fejer type inequalities for co-ordinated hyperbolic ρ-convex functions.

Theorem 7. *If $f : \Delta \to \mathbb{R}$ is a co-ordinated hyperbolic ρ-convex functions on Δ, then we have the following Hermite-Hadamard and Fejer type inequalities,*

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)H(\alpha,\beta) \\ \leq\ & \left[J^{\alpha,\beta}_{a+,c+}f(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)\right] \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right]H(\alpha,\beta) \end{aligned}$$

where

$$\begin{aligned} H(\alpha,\beta) =\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right] \\ & \times\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}\right. \\ & \left.+(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\right]dydx. \end{aligned}$$

Proof. If we apply Theorem 5 for the symmetric function

$$\begin{aligned} p(x,y) =\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}\right. \\ & \left.+(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\right], \end{aligned}$$

then we get the following inequality

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)H(\alpha,\beta) \\ \leq\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d f(x,y)\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}\right. \\ & \left.+(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\right]dydx \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right]H(\alpha,\beta). \end{aligned}$$

From the definition of the double fractional integrals we have

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d f(x,y)\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}\right.$$

$$+ (x-a)^{\alpha-1}(d-y)^{\beta-1} + (x-a)^{\alpha-1}(y-c)^{\beta-1}\Big] dydx$$
$$= \Big[J^{\alpha,\beta}_{a+,c+}f(b,d) + J^{\alpha,\beta}_{a+,d-}f(b,c) + J^{\alpha,\beta}_{b-,c+}f(a,d) + J^{\alpha,\beta}_{b-,d-}f(a,c)\Big]$$

which completes the proof. □

Remark 6. *If we choose $\rho_1 = \rho_2 = 0$ in Theorem 7, then we have the following fractional Hermite-Hadamard inequality,*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\Big[J^{\alpha,\beta}_{a+,c+}f(b,d) + J^{\alpha,\beta}_{a+,d-}f(b,c) + J^{\alpha,\beta}_{b-,c+}f(a,d) + J^{\alpha,\beta}_{b-,d-}f(a,c)\Big] \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \end{aligned}$$

which was proved by Sarikaya in [29, Theorem 4].

Remark 7. *If we choose $\alpha = \beta = 1$ in Theorem 7, then we have*

$$H(1,1) = \frac{16}{\rho_1\rho_2}\sinh\left(\frac{\rho_1(b-a)}{2}\right)\sinh\left(\frac{\rho_2(d-c)}{2}\right).$$

Thus, we get the following Hermite-Hadamard inequality,

$$\begin{aligned} & \frac{4}{\rho_1\rho_2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\sinh\left(\frac{\rho_1(b-a)}{2}\right)\sinh\left(\frac{\rho_2(d-c)}{2}\right) \\ \leq\ & \int_a^b\int_c^d f(x,y)dydx \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{\rho_1\rho_2}\tanh\left[\frac{\rho_1(b-a)}{2}\right]\tanh\left[\frac{\rho_2(d-c)}{2}\right] \end{aligned}$$

which is proved by Özçelik et al. in [23].

Theorem 8. *Let $p : \Delta \to \mathbb{R}$ be a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. If $f : \Delta \to \mathbb{R}$ is a co-ordinated hyperbolic ρ-convex functions on Δ, then we have the following Hermite-Hadamard-Fejer type inequalities,*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)H_p(\alpha,\beta) \\ \leq\ & \Big[J^{\alpha,\beta}_{a+,c+}(fp)(b,d) + J^{\alpha,\beta}_{a+,d-}(fp)(b,c) + J^{\alpha,\beta}_{b-,c+}(fp)(a,d) + J^{\alpha,\beta}_{b-,d-}(fp)(a,c)\Big] \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right]H_p(\alpha,\beta) \end{aligned}$$

where

$$H_p(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d \cosh\left[\rho_1\left(x-\frac{a+b}{2}\right)\right]\cosh\left[\rho_2\left(y-\frac{c+d}{2}\right)\right]$$
$$\times\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}\right.$$
$$\left.+(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\right]p(x,y)dydx.$$

Proof. Let us define the function $k(x,y)$ by

$$k(x,y) = \frac{p(x,y)}{\Gamma(\alpha)\Gamma(\beta)}\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}\right.$$
$$\left.+(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\right],$$

Clearly, $k(x.y)$ is a a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. If we apply Theorem 5 for the function $k(x,y)$ then we obtain,

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)H_p(\alpha,\beta)$$
$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d f(x,y)p(x,y)\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}\right.$$
$$\left.+(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\right]dydx$$
$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4\cosh\left[\frac{\rho_1(b-a)}{2}\right]\cosh\left[\frac{\rho_2(d-c)}{2}\right]}H_p(\alpha,\beta).$$

From the definition of the double fractional integrals we have

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d f(x,y)\left[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1}\right.$$
$$\left.+(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\right]p(x,y)dydx$$
$$= \left[J^{\alpha,\beta}_{a+,c+}(fp)(b,d)+J^{\alpha,\beta}_{a+,d-}(fp)(b,c)+J^{\alpha,\beta}_{b-,c+}(fp)(a,d)+J^{\alpha,\beta}_{b-,d-}(fp)(a,c)\right].$$

This completes the proof. □

Remark 8. *If we choose $\rho_1 = \rho_2 = 0$ in Theorem 8, then we have the following fractional Hermite-Hadamard inequality,*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\left[J^{\alpha,\beta}_{a+,c+}p(b,d)+J^{\alpha,\beta}_{a+,d-}p(b,c)+J^{\alpha,\beta}_{b-,c+}p(a,d)+J^{\alpha,\beta}_{b-,d-}p(a,c)\right]$$

$$\leq \left[J^{\alpha,\beta}_{a+,c+}(fp)(b,d) + J^{\alpha,\beta}_{a+,d-}(fp)(b,c) + J^{\alpha,\beta}_{b-,c+}(fp)(a,d) + J^{\alpha,\beta}_{b-,d-}(fp)(a,c)\right]$$

$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\left[J^{\alpha,\beta}_{a+,c+}p(b,d) + J^{\alpha,\beta}_{a+,d-}p(b,c) + J^{\alpha,\beta}_{b-,c+}p(a,d) + J^{\alpha,\beta}_{b-,d-}p(a,c)\right]$$

which is proved by Yaldız et all in [34].

Remark 9. *If we choose $\alpha = \beta = 1$ in Theorem 8, then we have Theorem 8 reduces to Theorem 5.*

Theorem 9. *If $f : \Delta \to \mathbb{R}$ is a co-ordinated hyperbolic ρ-convex functions on Δ. Then we have the following Hermite-Hadamard type inequalities for fractional integrals,*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) H_1(\alpha,\beta) \\
\leq\ & \frac{1}{2}\left[\left(J^{\alpha}_{a+}f\left(b, \frac{c+d}{2}\right) + J^{\alpha}_{b-}f\left(a, \frac{c+d}{2}\right)\right) H_2(\beta)\right. \\
& \left. + J^{\beta}_{c+}f\left(d, \frac{a+b}{2}\right) + J^{\beta}_{d-}f\left(c, \frac{a+b}{2}\right) H_3(\alpha)\right] \\
\leq\ & \left[J^{\alpha,\beta}_{a+,c+}f(b,d) + J^{\alpha,\beta}_{a+,d-}f(b,c) + J^{\alpha,\beta}_{b-,c+}f(a,d) + J^{\alpha,\beta}_{b-,d-}f(a,c)\right] \qquad (3.1) \\
\leq\ & \frac{1}{4}\left[\sec h\left[\frac{\rho_2(d-c)}{2}\right]\left(J^{\alpha}_{a+}f(b,c) + J^{\alpha}_{a+}f(b,d) + J^{\alpha}_{b-}f(a,c) + J^{\alpha}_{b-}f(a,d)\right) H_2(\beta)\right. \\
& \left. + \sec h\left[\frac{\rho_1(b-a)}{2}\right]\left(J^{\beta}_{c+}f(a,d) + J^{\beta}_{c+}f(b,d) + J^{\beta}_{d-}f(a,c) + J^{\beta}_{d-}f(b,c)\right) H_3(\alpha)\right] \\
\leq\ & \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} \sec h\left[\frac{\rho_1(b-a)}{2}\right] \sec h\left[\frac{\rho_2(d-c)}{2}\right] H_1(\alpha,\beta)
\end{aligned}$$

where

$$\begin{aligned}
H_1(\alpha,\beta) = & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d \cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right] \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right] \\
& \times\left[(b-x)^{\alpha-1}(d-y)^{\beta-1} + (b-x)^{\alpha-1}(y-c)^{\beta-1}\right. \\
& \left. + (x-a)^{\alpha-1}(d-y)^{\beta-1} + (x-a)^{\alpha-1}(y-c)^{\beta-1}\right] dydx,
\end{aligned}$$

$$H_2(\beta) = \frac{1}{\Gamma(\beta)} \int_c^d \cosh\left[\rho_2\left(y - \frac{c+d}{2}\right)\right]\left[(d-y)^{\beta-1} + (y-c)^{\beta-1}\right] dy$$

and

$$H_3(\alpha,\beta) = \frac{1}{\Gamma(\alpha)} \int_a^b \cosh\left[\rho_1\left(x - \frac{a+b}{2}\right)\right]\left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right] dx.$$

Proof. If we apply Theorem 6 for the symmetric function

$$\begin{aligned} p(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Big[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1} \\ &\quad +(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\Big], \end{aligned}$$

then we get the following inequality

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)H_1(\alpha,\beta) \\ \leq\; & \frac{1}{2}\left[\left(\frac{1}{\Gamma(\alpha)}\int_a^b f\left(x,\frac{c+d}{2}\right)\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx\right)H_2(\beta)\right. \\ & \left.+\left(\frac{1}{\Gamma(\beta)}\int_c^d f\left(\frac{a+b}{2},y\right)\left[(d-y)^{\beta-1}+(y-c)^{\beta-1}\right]dy\right)H_3(\alpha)\right] \\ \leq\; & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d f(x,y)\Big[(b-x)^{\alpha-1}(d-y)^{\beta-1}+(b-x)^{\alpha-1}(y-c)^{\beta-1} \\ & +(x-a)^{\alpha-1}(d-y)^{\beta-1}+(x-a)^{\alpha-1}(y-c)^{\beta-1}\Big]dydx \\ \leq\; & \frac{1}{4}\left[\sec h\left[\frac{\rho_2(d-c)}{2}\right]\left(\frac{1}{\Gamma(\alpha)}\int_a^b[f(x,c)+f(x,d)]\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx\right)H_2(\beta)\right. \\ & \left.+\sec h\left[\frac{\rho_1(b-a)}{2}\right]\left(\frac{1}{\Gamma(\beta)}\int_a^b[f(a,y)+f(b,y)]\left[(d-y)^{\beta-1}+(y-c)^{\beta-1}\right]dx\right)H_3(\alpha)\right] \\ \leq\; & \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\sec h\left[\frac{\rho_1(b-a)}{2}\right]\sec h\left[\frac{\rho_2(d-c)}{2}\right]H_1(\alpha,\beta). \end{aligned}$$

This completes the proof. □

Remark 10. *Under assumptions of Theorem 9 with* $\alpha=\beta=1$, *the inequalities (3.1) reduce to inequalities (2.24) proved by Özçelik et. al in [23].*

Remark 11. *Under assumptions of Theorem 9 with* $\rho_1=\rho_2=0$, *the inequalities (3.1) reduce to inequalities proved by Sarikaya in [29, Theorem 4]*

4. Conclusions

In this paper, we establish some Fejer type inequalities for co-ordinated hyperbolic ρ-convex functions. By using these inequalities we present some inequalities for Riemann-Liouville fractional integrals. In the future works, authors can prove similar inequalities for other fractional integrals.

Conflict of interest

All authors declare no conflicts of interest.

References

1. A. Akkurt, M. Z. Sarikaya, H. Budak, et al. *On the Hadamard's type inequalities for co-ordinated convex functions via fractional integrals*, Journal of King Saud University - Science, **29** (2017), 380–387.
2. T. Ali, M. A. Khan, A. Kilicman, et al. *On the refined Hermite-Hadamard inequalities*, Mathematical Sciences & Applications E-Notes, **6** (2018), 85–92.
3. A. G. Azpeitia, *Convex functions and the Hadamard inequality*, Rev. Colombiana Math., **28** (1994), 7–12.
4. M. K. Bakula, *An improvement of the Hermite-Hadamard inequality for functions convex on the coordinates*, Australian journal of mathematical analysis and applications, **11** (2014), 1–7.
5. H. Budak and M. Z. Sarikaya, *Hermite-Hadamard-Fejer inequalities for double integrals*, submitted, 2018.
6. F. Chen, *A note on the Hermite-Hadamard inequality for convex functions on the co-ordinates*, J. Math. Inequal., **8** (2014), 915–923.
7. F. Chen, *A note on Hermite-Hadamard inequalities for products of convex functions*, J. Appl. Math., **2013** (2013), 935020.
8. F. Chen, S. Wu, *Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions*, J. Nonlinear Sci. Appl., **9** (2016), 705–716.
9. S. S. Dragomir and C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000.
10. S. S. Dragomir, *Some inequalities of Hermite-Hadamard type for hyperbolic ρ-convex functions*, Preprint, 2018.
11. S. S. Dragomir, *On Hadamards inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwan J. Math., **4** (2001), 775–788.
12. S. S. Dragomir, *Some inequalities of Fejer type for hyperbolic ρ-convex functions*, Preprint, 2018.
13. S. S. Dragomir, *Some inequalities of Ostrowski and trapezoid type for hyperbolic ρ-convex functions*, Preprint, 2018.
14. S. S. Dragomir, *Some inequalities of Jensen type for hyperbolic ρ-convex functions*, Preprint, 2018.
15. S. S. Dragomir and B. T. Torebek, *Some Hermite-Hadamard type inequalities in the class of hyperbolic ρ-convex functions*, arXiv preprint, arXiv:1901.06634, 2019.
16. G. Farid, M. Marwan and A. U. Rehman, *Fejer-Hadamard inequlality for convex functions on the co-ordinates in a rectangle from the plane*, International Journal of Analysis and Applications, **10** (2016), 40–47.
17. L. Fejer, *Über die Fourierreihen,* II. Math. Naturwiss. Anz Ungar. Akad. Wiss., **24** (1906), 369–390. (Hungarian)
18. J. Hadamard, *Etude sur les proprietes des fonctions entieres en particulier d'une fonction consideree par Riemann*, J. Math. Pures Appl., **58** (1893), 171–215.

19. U. S. Kırmacı, M. K. Bakula, M. E. Özdemir, et al. *Hadamard-tpye inequalities for s-convex functions*, Appl. Math. Comput., **193** (2007), 26–35.
20. M. A. Latif and M. Alomari, *Hadamard-type inequalities for product two convex functions on the co-ordinetes*, Int. Math. Forum, **4** (2009), 2327–2338.
21. M. A. Latif, S. S. Dragomir, E. Momoniat, *Generalization of some Inequalities for differentiable co-ordinated convex functions with applications*, Moroccan J. Pure Appl. Anal., **2** (2016), 12–32.
22. M. A. Latif and S. S. Dragomir, *On some new inequalities for differentiable co-ordinated convex functions*, J. Inequal. Appl., **2012** (2012), 28.
23. K. Ozcelik, H. Budak, S. S. Dragomir, *On Hermite-Hadamard type inequalities for co-ordinated hyperbolic ρ-convex functions*, Submitted, 2019.
24. M. E. Ozdemir, C. Yildiz and A. O. Akdemir, *On the co-ordinated convex functions*, Appl. Math. Inf. Sci., **8** (2014), 1085–1091.
25. B. G. Pachpatte, *On some inequalities for convex functions*, RGMIA Res. Rep. Coll, **6** (2003), 1–9.
26. Z. Pavic, *Improvements of the Hermite-Hadamard inequality*, J. Inequal. Appl., **2015** (2015), 1–11.
27. J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
28. M. Z. Sarikaya, E. Set, M. E. Ozdemir, et al.*New some Hadamard's type inequalities for co-ordinated convex functions*, Tamsui Oxford Journal of Information and Mathematical Sciences, **28** (2012), 137–152.
29. M. Z. Sarikaya, *On the Hermite-Hadamard-type inequalities for co-ordinated convex functions via fractional integrals*, Integ. Transf. Spec. F., **25** (2014), 134–147.
30. E. Set, M. E. Özdemir, S. S. Dragomir, *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl., **2010** (2010), 148102.
31. K. L. Tseng and S. R. Hwang, *New Hermite-Hadamard inequalities and their applications*, Filomat, **30** (2016), 3667–3680.
32. D. Y. Wang, K. L. Tseng and G. S. Yang, *Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane*, Taiwan. J. Math., **11** (2007), 63–73.
33. R. Xiang and F. Chen, *On some integral inequalities related to Hermite-Hadamard-Fejér inequalities for coordinated convex functions*, Chinese Journal of Mathematics, **2014** (2014), 796132.
34. H. Yaldız, M. Z. Sarikaya, Z. Dahmani, *On the Hermite-Hadamard-Fejer-type inequalities for co-ordinated convex functions via fractional integrals*, An International Journal of Optimization and Control: Theories & Applications, **7** (2017), 205–215.
35. G. S. Yang and K. L. Tseng, *On certain integral inequalities related to Hermite-Hadamard inequalities*, J. Math. Anal. Appl., **239** (1999), 180–187.
36. G. S. Yang and M. C. Hong, *A note on Hadamard's inequality*, Tamkang J. Math., **28** (1997), 33–37.

AIMS Press

© 2020 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)