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*Research article*

# $2D$ approximately reciprocal $\rho$-convex functions and associated integral inequalities

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**Abstract:** The main objective of this article is to introduce the notion of $2D$ approximately reciprocal $\rho$-convex functions, show that this class of functions unifies several other unrelated classes of reciprocal convex functions, obtain several new refinements of the Hermite-Hadamard type inequalities involving $2D$ approximately reciprocal $\rho$-convex functions, provide some bounds pertaining to the trapezium like inequalities by using partial differentiable $2D$ approximately reciprocal $\rho$-convex functions, and discuss the special cases of the obtained results.

**Keywords:** convex; reciprocal; approximately; Hermite-Hadamard inequality
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## 1. Introduction and preliminaries

Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $\mathcal{X} : I \to \mathbb{R}$ is said to be convex if the inequality

$$\mathcal{X}((1-\mu)x + \mu y) \leq (1-\mu)\mathcal{X}(x) + \mu\mathcal{X}(y)$$

holds for all $x, y \in I$ and $\mu \in [0, 1]$.

It is well-known that convexity has wild applications in pure and applied mathematics [1–8]. In particular, many remarkable inequalities can be found in the literature [9–20] via the convexity theory.

Recently, the generalizations, extensions and variants for the convexity have attracted the attentions of many researchers [21–25].

İşcan [26] introduced the class of reciprocal convex functions as follows.

A real-vauled function $\mathcal{X} : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be reciprocal convex if the inequality

$$\mathcal{X}\left(\frac{xy}{(1-\mu)x + \mu y}\right) \leq \mu\mathcal{X}(x) + (1-\mu)\mathcal{X}(y)$$

holds for all $x, y \in I$ and $\mu \in [0, 1]$.

In [27], Noor et al. introduced and discussed the class of reciprocal $\rho$-convex functions. Later, Noor et al. [28] extended the class of reciprocal convex functions on coordinates and introduced the class of $2D$ reciprocal convex functions.

Let $\Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty)$. Then a real-valued function $\mathcal{X} : \Omega \to \mathbb{R}$ is said to be $2D$ reciprocal convex if the inequality

$$\mathcal{X}\left(\frac{xy}{\mu x + (1-\mu)y}, \frac{uw}{ru + (1-\lambda)w}\right)$$
$$\leq \mu\lambda\mathcal{X}(y, w) + \mu(1-\lambda)\mathcal{X}(y, u) + (1-\mu)\lambda\mathcal{X}(x, w) + (1-\mu)(1-\lambda)\mathcal{X}(x, u)$$

holds for all $x, y \in [a, b]$, $u, w \in [c, d]$ and $\mu, \lambda \in [0, 1]$.

Very recently, Awan et al. [29] gave the definition of approximately reciprocal $\rho$-convex functions depending on a metric function.

It is well-known that the classical Hermite-Hadamard inequality [30–35] is one of the most famous and important inequalities in convexity theory, which can be stated as follows.

The double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}$$

holds for all $a, b \in I$ with $a \neq b$ if $f : I \to \mathbb{R}$ is a convex function.

In the past half century, many researchers have devoted themselves to the generalizations, improvements and variants of the Hermite-Hadamard inequality. For example, Dragomir [36] obtained a two dimensional version of the Hermite-Hadamard inequality using the coordinated convex functions, Budak et al. [37] provided a two dimensional extension of the Hermite-Hadamard inequality by use of coordinated trigonometrically $\rho$-convex functions, İşcan [26] derived a new variant of the Hermite-Hadamard inequality by using the class of reciprocal convex functions, Noor et al. [27] obtained a generalized version of the Hermite-Hadamard inequality via the reciprocal $\rho$-convex functions, and Noor et al. [28] establshed a $2D$ version of the Hermite-Hadamard inequality using $2D$ reciprocal convex functions.

The main purpose of the article is to introduce the $2D$ approximately reciprocal $\rho$-convex functions, discuss how this class of functions unifies several other unrelated classes of reciprocal convex functions by considering some suitable choices of the given function $\Delta(\cdot, \cdot)$ and the real function $\rho(\cdot)$, derive several new refinements of the Hermite-Hadamard like inequalities involving $2D$ approximately reciprocal $\rho$-convex functions, and discuss the special cases of the main obtained results.

## 2. Definition of $2D$ approximately reciprocal $\rho$-convex function and its special cases

In this section, we provide the definition of the class of $2D$ approximately reciprocal $\rho$-convex functions, and discuss its special cases.

**Definition 2.1.** Let $\Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty)$. Then a real-valued function $\mathcal{X} : \Omega \to \mathbb{R}$ is said to be a $2D$ approximately reciprocal $\rho$-convex function if the inequality

$$\mathcal{X}\left(\frac{xy}{\mu x + (1-\mu)y}, \frac{uw}{ru + (1-\lambda)w}\right)$$
$$\leq \rho(\mu)\rho(\lambda)\mathcal{X}(y, w) + \rho(\mu)\rho(1-\lambda)\mathcal{X}(y, u)$$
$$+\rho(1-\mu)\rho(\lambda)\mathcal{X}(x, w) + \rho(1-\mu)\rho(1-\lambda)\mathcal{X}(x, u) + \Delta(x, y) + \Delta(u, w),$$

holds for $x, y \in [a, b]$, $u, w \in [c, d]$ and $\mu, \lambda \in [0, 1]$.

Next, We discuss some special cases of Definition 2.1.

**I**. If we take $\Delta(x, y) = \epsilon(\|x^{-1} - y^{-1}\|)^{\gamma}$ and $\Delta(u, w) = \epsilon(\|u^{-1} - w^{-1}\|)^{\gamma}$ for some $\epsilon \in \mathbb{R}$ and $\gamma > 1$ in Definition 2.1, then we have a new definition of "$\gamma$-paraharmonic $\rho$-convex function of higher order".

**Definition 2.2.** Let $\Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty)$. Then a real-valued function $\mathcal{X} : \Omega \to \mathbb{R}$ is said to be a $2D$ $\gamma$-paraharmonic $\rho$-convex function of higher order if the inequality

$$\mathcal{X}\left(\frac{xy}{\mu x + (1-\mu)y}, \frac{uw}{ru + (1-\lambda)w}\right)$$
$$\leq \rho(\mu)\rho(\lambda)\mathcal{X}(y, w) + \rho(\mu)\rho(1-\lambda)\mathcal{X}(y, u)$$
$$+\rho(1-\mu)\rho(\lambda)\mathcal{X}(x, w) + \rho(1-\mu)\rho(1-\lambda)\mathcal{X}(x, u) + \epsilon(\|x^{-1} - y^{-1}\|)^{\gamma} + \epsilon(\|u^{-1} - w^{-1}\|)^{\gamma}$$

takes place for all $x, y \in [a, b]$, $u, w \in [c, d]$ and $\mu, \lambda \in [0, 1]$.

**II**. If we take $\Delta(x, y) = \epsilon(\|x^{-1} - y^{-1}\|)$ and $\Delta(u, w) = \epsilon(\|u^{-1} - w^{-1}\|)$ for some $\epsilon \in \mathbb{R}$ in Definition 2.1, then we obtain a new definition of "$\epsilon$-paraharmonic $\rho$-convex function".

**Definition 2.3.** Let $\Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty)$. Then a function $\mathcal{X} : \Omega \to \mathbb{R}$ is said to be a $2D$ $\epsilon$-paraharmonic $\rho$-convex function if

$$\mathcal{X}\left(\frac{xy}{\mu x + (1-\mu)y}, \frac{uw}{ru + (1-\lambda)w}\right)$$
$$\leq \rho(\mu)\rho(\lambda)\mathcal{X}(y, w) + \rho(\mu)\rho(1-\lambda)\mathcal{X}(y, u) + \rho(1-\mu)\rho(\lambda)\mathcal{X}(x, w) + \rho(1-\mu)\rho(1-\lambda)\mathcal{X}(x, u)$$
$$+\epsilon(\|x^{-1} - y^{-1}\|) + \epsilon(\|u^{-1} - w^{-1}\|)$$

whenever $x, y \in [a, b]$, $u, w \in [c, d]$ and $\mu, \lambda \in [0, 1]$.

**III**. If we take

$$\Delta(x, y) = -\mu\left(\mu^{\sigma}(1-\mu) + \mu(1-\mu)^{\sigma}\right)\left(\left\|\frac{1}{x} - \frac{1}{y}\right\|\right)^{\sigma}$$

and

$$\Delta(u, w) = -\mu\left(\lambda^{\sigma}(1-\lambda) + \lambda(1-\lambda)^{\sigma}\right)\left(\left\|\frac{1}{u} - \frac{1}{w}\right\|\right)^{\sigma}$$

in Definition 2.1, then we get a new definition of $2D$ reciprocal strong $\rho$-convex function of higher order.

**Definition 2.4.** Let $\Omega = [a,b]\times[c,d] \subseteq (0,\infty)\times(0,\infty)$. Then a real-valued function $\mathcal{X}:\Omega\to\mathbb{R}$ is said to be a $2D$ reciprocal strong $\rho$-convex function of higher order if the inequality

$$\mathcal{X}\left(\frac{xy}{\mu x+(1-\mu)y},\frac{uw}{ru+(1-\lambda)w}\right)$$

$$\leq \rho(\mu)\rho(\lambda)\mathcal{X}(y,w)+\rho(\mu)\rho(1-\lambda)\mathcal{X}(y,u)+\rho(1-\mu)\rho(\lambda)\mathcal{X}(x,w)+\rho(1-\mu)\rho(1-\lambda)\mathcal{X}(x,u)$$
$$-\mu\left(\mu^{\sigma}(1-\mu)+\mu(1-\mu)^{\sigma}\right)\left(\left\|\frac{1}{x}-\frac{1}{y}\right\|\right)^{\sigma}-\mu\left(\lambda^{\sigma}(1-\lambda)+\lambda(1-\lambda)^{\sigma}\right)\left(\left\|\frac{1}{u}-\frac{1}{w}\right\|\right)^{\sigma},$$

is valid for all $x,y\in[a,b]$, $u,w\in[c,d]$ and $\mu,\lambda\in[0,1]$.

**IV**. If we take $\sigma=2$ in Definition 2.4, then

$$\Delta(x,y) = -\mu\mu(1-\mu)\left(\left\|\frac{1}{x}-\frac{1}{y}\right\|\right)^{2}$$

$$\Delta(u,w) = -\mu\lambda(1-\lambda)\left(\left\|\frac{1}{u}-\frac{1}{w}\right\|\right)^{2}$$

and we have the definition of $2D$ reciprocal strong $\rho$-convex function.

**Definition 2.5.** Let $\Omega = [a,b]\times[c,d] \subseteq (0,\infty)\times(0,\infty)$. Then a real-valued function $\mathcal{X}:\Omega\to\mathbb{R}$ is said to be a $2D$ reciprocal strong $\rho$-convex function if the inequality

$$\mathcal{X}\left(\frac{xy}{\mu x+(1-\mu)y},\frac{uw}{ru+(1-\lambda)w}\right)$$

$$\leq \rho(\mu)\rho(\lambda)\mathcal{X}(y,w)+\rho(\mu)\rho(1-\lambda)\mathcal{X}(y,u)+\rho(1-\mu)\rho(\lambda)\mathcal{X}(x,w)+\rho(1-\mu)\rho(1-\lambda)\mathcal{X}(x,u)$$
$$-\mu\mu(1-\mu)\left(\left\|\frac{1}{x}-\frac{1}{y}\right\|\right)^{2}-\mu\lambda(1-\lambda)\left(\left\|\frac{1}{u}-\frac{1}{w}\right\|\right)^{2},$$

holds for $x,y\in[a,b]$, $u,w\in[c,d]$ and $\mu,\lambda\in[0,1]$.

**V**. If we take $\Delta(x,y)=\mu\mu(1-\mu)\left(\frac{1}{x}-\frac{1}{y}\right)^{2}$ and $\Delta(u,w)=\mu\lambda(1-\lambda)\left(\frac{1}{u}-\frac{1}{w}\right)^{2}$ for some $\mu>0$ in Definition 2.1, then we obtain the definition of $2D$ reciprocal relaxed $\rho$-convex function.

**Definition 2.6.** Let $\Omega = [a,b]\times[c,d] \subseteq (0,\infty)\times(0,\infty)$. Then a real-valued function $\mathcal{X}:\Omega\to\mathbb{R}$ is said to be a $2D$ reciprocal relaxed $\rho$-convex function if

$$\mathcal{X}\left(\frac{xy}{\mu x+(1-\mu)y},\frac{uw}{ru+(1-\lambda)w}\right)$$

$$\leq \rho(\mu)\rho(\lambda)\mathcal{X}(y,w)+\rho(\mu)\rho(1-\lambda)\mathcal{X}(y,u)+\rho(1-\mu)\rho(\lambda)\mathcal{X}(x,w)+\rho(1-\mu)\rho(1-\lambda)\mathcal{X}(x,u)$$
$$+\mu\mu(1-\mu)\left(\frac{1}{x}-\frac{1}{y}\right)^{2}+\mu\lambda(1-\lambda)\left(\frac{1}{u}-\frac{1}{w}\right)^{2}$$

whenever $x, y \in [a, b]$, $u, w \in [c, d]$ and $\mu, \lambda \in [0, 1]$.

**VI**. If we take $\Delta(x, y) = -\mu(1-\mu)\left(\frac{xy}{x-y}\right)^2$ and $\Delta(u, w) = -\lambda(1-\lambda)\left(\frac{uw}{u-w}\right)^2$ in Definition 2.1, then we have a new definition of $2D$ strongly $F$ reciprocal $\rho$-convex function.

**Definition 2.7.** Let $\Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty)$. Then a real-valued function $\mathcal{X} : \Omega \to \mathbb{R}$ is said to be a $2D$ strongly $F$ reciprocal $\rho$-convex function if the inequality

$$\mathcal{X}\left(\frac{xy}{\mu x + (1-\mu)y}, \frac{uw}{ru + (1-\lambda)w}\right)$$
$$\leq \rho(\mu)\rho(\lambda)\mathcal{X}(y, w) + \rho(\mu)\rho(1-\lambda)\mathcal{X}(y, u) + \rho(1-\mu)\rho(\lambda)\mathcal{X}(x, w) + \rho(1-\mu)\rho(1-\lambda)\mathcal{X}(x, u)$$
$$-\mu(1-\mu)\left(\frac{xy}{x-y}\right)^2 - \lambda(1-\lambda)\left(\frac{uw}{u-w}\right)^2$$

holds for all $x, y \in [a, b]$, $u, w \in [c, d]$ and $\mu, \lambda \in [0, 1]$.

**Remark 2.8.** It is pertinent to mention here that we can recapture other new classes of reciprocal convexity from Definition 2.1 by considering suitable choices of function $\rho(\cdot)$. For example, if we take $\rho(\mu) = \mu^s$ and $\rho(\lambda) = \lambda^s$ in Definition 2.1, then we have the class of Breckner type $2D$ approximately reciprocal $s$-convex functions; if we take $\rho(\mu) = \mu^{-s}$ and $\rho(\lambda) = \lambda^{-s}$ in Definition 2.1, then we get the class of Godunova-Levin type $2D$ approximately reciprocal $s$-convex functions; if we take $\rho(\mu) = 1$ and $\rho(\lambda) = 1$ in Definition 2.1, then we obtain the class of $2D$ approximately reciprocal $P$-convex functions. Moreover, if we choose suitable function $\Delta(\cdot, \cdot)$ in these discussed classes, then we also can get new refinements of reciprocal convexity, we left the details to the interested readers.

## 3. New variant of the Hermite-Hadamard inequality

In this section, we derive a new variant of the Hermite-Hadamard inequality using the class of $2D$ approximately reciprocal $\rho$-convex functions.

**Theorem 3.1.** Let $\mathcal{X} : \Omega = [a, b] \times [c, d] \to \mathbb{R}$ be an integrable $2D$ approximately reciprocal $\rho$-convex function. Then we have the Hermite-Hadamard type inequality as follows

$$\frac{1}{4\rho^2(\frac{1}{2})}\left[\mathcal{X}\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right) - \frac{ab}{b-a}\int_a^b \frac{\Delta(x, (a^{-1}+b^{-1}-x^{-1})^{-1})}{x^2}\mathrm{d}x\right.$$
$$\left.-\frac{cd}{d-c}\int_c^d \frac{\Delta(u, (c^{-1}+d^{-1}-u^{-1})^{-1})}{u^2}\mathrm{d}u\right]$$
$$\leq \left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d \frac{\mathcal{X}(x, u)}{x^2u^2}\mathrm{d}u\mathrm{d}x$$
$$\leq [\mathcal{X}(a, c) + \mathcal{X}(a, d) + \mathcal{X}(b, c) + \mathcal{X}(b, d)]\int_0^1\int_0^1 \rho(\mu)\rho(\lambda)\mathrm{d}\mu\mathrm{d}\lambda + \Delta(a, b) + \Delta(c, d).$$

*Proof.* It follows from the $2D$ approximately reciprocal $\rho$-convexity of $\mathcal{X}$ that

$$\mathcal{X}\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)$$

$$\leq \rho^2\Big(\frac{1}{2}\Big)\Bigg[\mathcal{X}\left(\frac{ab}{ta+(1-\mu)b},\frac{cd}{rc+(1-\lambda)d}\right)+\mathcal{X}\left(\frac{ab}{ta+(1-\mu)b},\frac{cd}{rd+(1-\lambda)c}\right)$$

$$+\mathcal{X}\left(\frac{ab}{tb+(1-\mu)a},\frac{cd}{rc+(1-\lambda)d}\right)+\mathcal{X}\left(\frac{ab}{tb+(1-\mu)a},\frac{cd}{rd+(1-\lambda)c}\right)\Bigg]$$

$$+\Delta\left(\frac{ab}{ta+(1-\mu)b},\frac{ab}{(1-\mu)a+tb}\right)+\Delta\left(\frac{cd}{rc+(1-\lambda)d},\frac{cd}{(1-\lambda)c+rd}\right).$$

Integrating above inequality with respect to $(\mu,\lambda)$ on $[0,1]\times[0,1]$ leads to

$$\frac{1}{4\rho^2\left(\frac{1}{2}\right)}\Bigg[\mathcal{X}\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)-\frac{ab}{b-a}\int_a^b\frac{\Delta(x,(a^{-1}+b^{-1}-x^{-1})^{-1})}{x^2}\mathrm{d}x$$

$$-\frac{cd}{d-c}\int_c^d\frac{\Delta(u,(c^{-1}+d^{-1}-u^{-1})^{-1})}{u^2}\mathrm{d}u\Bigg]$$

$$\leq\left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d\frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x.$$

Similarly, we have

$$\mathcal{X}\left(\frac{ab}{ta+(1-\mu)b},\frac{cd}{rc+(1-\lambda)d}\right)$$

$$\leq\rho(\mu)\rho(\lambda)\mathcal{X}(b,d)+\rho(\mu)\rho(1-\lambda)\mathcal{X}(b,c)+\rho(1-\mu)\rho(\lambda)\mathcal{X}(a,d)$$

$$+\rho(1-\mu)\rho(1-\lambda)\mathcal{X}(a,c)+\Delta(a,b)+\Delta(c,d).$$

Integrating both sides of the above inequality with respect to $(\mu,\lambda)$ on $[0,1]\times[0,1]$, we get

$$\left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d\frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x$$

$$\leq(\mathcal{X}(a,c)+\mathcal{X}(a,d)+\mathcal{X}(b,c)+\mathcal{X}(b,d))\int_0^1\int_0^1\rho(\mu)\rho(\lambda)\mathrm{d}\mu\mathrm{d}\lambda+\Delta(a,b)+\Delta(c,d).$$

This completes the proof. □

## 4. Applications of Theorem 3.1

In this section, we present some applications of Theorem 3.1.

**I**. If $\rho(\mu) = \mu$ and $\rho(\lambda) = \lambda$, then Theorem 3.1 leads to Corollary 4.1.

**Corollary 4.1.** Let $\mathcal{X} : \Omega = [a, b] \times [c, d] \to \mathbb{R}$ be an integrable $2D$ approximately reciprocal convex function. Then one has

$$\mathcal{X}\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right) - \frac{ab}{b-a}\int_a^b \frac{\Delta(x, (a^{-1}+b^{-1}-x^{-1})^{-1})}{x^2}\mathrm{d}x$$

$$-\frac{cd}{d-c}\int_c^d \frac{\Delta(u, (c^{-1}+d^{-1}-u^{-1})^{-1})}{u^2}\mathrm{d}u$$

$$\leq \left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d \frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x$$

$$\leq \frac{[\mathcal{X}(a,c)+\mathcal{X}(a,d)+\mathcal{X}(b,c)+\mathcal{X}(b,d)]}{4} + \Delta(a,b) + \Delta(c,d).$$

**II**. If $\rho(\mu) = \mu^s$ and $\rho(\lambda) = \lambda^s$, then Theorem 3.1 becomes Corollary 4.2.

**Corollary 4.2.** Let $\mathcal{X} : \Omega = [a, b] \times [c, d] \to \mathbb{R}$ be an integrable Breckner type $2D$ approximately reciprocal $s$-convex function. Then

$$\frac{1}{4^{1-s}}\left[\mathcal{X}\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right) - \frac{ab}{b-a}\int_a^b \frac{\Delta(x, (a^{-1}+b^{-1}-x^{-1})^{-1})}{x^2}\mathrm{d}x\right.$$

$$\left.-\frac{cd}{d-c}\int_c^d \frac{\Delta(u, (c^{-1}+d^{-1}-u^{-1})^{-1})}{u^2}\mathrm{d}u\right]$$

$$\leq \left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d \frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x$$

$$\leq \frac{\mathcal{X}(a,c)+\mathcal{X}(a,d)+\mathcal{X}(b,c)+\mathcal{X}(b,d)}{(s+1)^2} + \Delta(a,b) + \Delta(c,d).$$

**III**. If $\rho(\mu) = \mu^{-s}$ and $\rho(\lambda) = \lambda^{-s}$, then Theorem 3.1 reduces to Corollary 4.3.

**Corollary 4.3.** Let $\mathcal{X} : \Omega = [a, b] \times [c, d] \to \mathbb{R}$ be an integrable Godunova-Levin type $2D$ approximately reciprocal $s$-convex function. Then we get

$$\frac{1}{4^{s+1}}\left[\mathcal{X}\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right) - \frac{ab}{b-a}\int_a^b \frac{\Delta(x, (a^{-1}+b^{-1}-x^{-1})^{-1})}{x^2}\mathrm{d}x\right.$$

$$-\frac{cd}{d-c}\int_c^d \frac{\Delta(u,(c^{-1}+d^{-1}-u^{-1})^{-1})}{u^2}\mathrm{d}u\Bigg]$$

$$\leq \left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d \frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x$$

$$\leq \frac{\mathcal{X}(a,c)+\mathcal{X}(a,d)+\mathcal{X}(b,c)+\mathcal{X}(b,d)}{(1-s)^2}+\Delta(a,b)+\Delta(c,d).$$

**IV**. If $\rho(\mu)=\rho(\lambda)=1$, then Theorem 3.1 leads to Corollary 4.4.

**Corollary 4.4.** Let $\mathcal{X}:\Omega=[a,b]\times[c,d]\to\mathbb{R}$ be an integrable $2D$ approximately reciprocal $P$-convex function. Then one has

$$\frac{1}{4}\Bigg[\mathcal{X}\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)-\frac{ab}{b-a}\int_a^b \frac{\Delta(x,(a^{-1}+b^{-1}-x^{-1})^{-1})}{x^2}\mathrm{d}x$$

$$-\frac{cd}{d-c}\int_c^d \frac{\Delta(u,(c^{-1}+d^{-1}-u^{-1})^{-1})}{u^2}\mathrm{d}u\Bigg]$$

$$\leq \left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d \frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x$$

$$\leq [\mathcal{X}(a,c)+\mathcal{X}(a,d)+\mathcal{X}(b,c)+\mathcal{X}(b,d)]+\Delta(a,b)+\Delta(c,d).$$

**V**. If we take

$$\Delta(a,b)=-\mu(\mu^{\sigma}(1-\mu)+\mu(1-\mu)^{\sigma})\left(\left\|\frac{1}{a}-\frac{1}{b}\right\|\right)^{\sigma}$$

and

$$Delta(c,d)=-\mu(\lambda^{\sigma}(1-\lambda)+\lambda(1-\lambda)^{\sigma}\left(\left\|\frac{1}{c}-\frac{1}{d}\right\|\right)^{\sigma}$$

for some $\mu>0$, then Theorem 3.1 reduces to Corollary 4.5.

**Corollary 4.5.** Let $\mathcal{X}:\Omega=[a,b]\times[c,d]\to\mathbb{R}$ be an integrable $2D$ reciprocal strong $\rho$-convex function of higher order. Then we obtain the inequality

$$\frac{1}{4\rho^2(\frac{1}{2})}\left[\mathcal{X}\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)+\frac{\mu}{2^{\sigma}(\sigma+1)}\left[\left\|\frac{b-a}{ab}\right\|^{\sigma}+\left\|\frac{d-c}{dc}\right\|^{\sigma}\right]\right]$$

$$\leq \left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d \frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x$$

$$\leq (\mathcal{X}(a,c)+\mathcal{X}(a,d)+\mathcal{X}(b,c)+\mathcal{X}(b,d))\int_0^1\int_0^1 \rho(\mu)\rho(\lambda)\mathrm{d}\mu\mathrm{d}\lambda$$
$$-\frac{2\mu}{(\sigma+1)(\sigma+2)}\left[\left\|\frac{1}{a}-\frac{1}{b}\right\|^{\sigma}+\left\|\frac{1}{c}-\frac{1}{d}\right\|^{\sigma}\right].$$

**VI**. If we take $\sigma = 2$. Then Corollary 4.5 becomes Corollary 4.6.

**Corollary 4.6.** Let $\mathcal{X}:\Omega=[a,b]\times[c,d]\to\mathbb{R}$ be an integrable $2D$ reciprocal strong $\rho$-convex function. Then one has

$$\frac{1}{4\rho^2(\frac{1}{2})}\left[\mathcal{X}\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)+\frac{\mu}{12}\left[\left\|\frac{b-a}{ab}\right\|^2+\left\|\frac{d-c}{dc}\right\|^2\right]\right]$$
$$\leq\left(\frac{ab}{b-a}\right)\left(\frac{cd}{d-c}\right)\int_a^b\int_c^d\frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x$$
$$\leq (\mathcal{X}(a,c)+\mathcal{X}(a,d)+\mathcal{X}(b,c)+\mathcal{X}(b,d))\int_0^1\int_0^1 \rho(\mu)\rho(\lambda)\mathrm{d}\mu\mathrm{d}\lambda$$
$$-\frac{\mu}{6}\left[\left\|\frac{1}{a}-\frac{1}{b}\right\|^2+\left\|\frac{1}{c}-\frac{1}{d}\right\|^2\right].$$

## 5. Bounds pertaining to trapezium like inequality using partial differentiable $2D$ approximately reciprocal $\rho$-convex functions

In this section, we present some bounds pertaining to trapezium like inequality using partial differentiable $2D$ approximately reciprocal $\rho$-convex functions. The following auxiliary result will play significant role in our Theorem 5.2.

**Lemma 5.1.** (See [28]) Let $\mathcal{X}:\Omega=[a,b]\times[c,d]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a partial differential function on $\Omega$ such that $\frac{\partial^2\mathcal{X}}{\partial\mu\partial\lambda}\in L_1(\Omega)$. Then

$$\mathcal{X}(a,b,c,d,x,y:\Omega)$$
$$=\frac{ab(b-a)cd(d-c)}{4}\int_0^1\int_0^1\left(\frac{1-2\mu}{(tb+(1-\mu)a)^2}\right)\left(\frac{1-2\lambda}{(rd+(1-\lambda)c)^2}\right)$$
$$\times\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}\left(\frac{ab}{tb+(1-\mu)a},\frac{cd}{rd+(1-\lambda)c}\right)\mathrm{d}\lambda\mathrm{d}\mu,$$

where

$$\mathcal{X}(a,b,c,d,x,y:\Omega)$$
$$=\frac{\mathcal{X}(a,c)+\mathcal{X}(b,c)+\mathcal{X}(a,d)+\mathcal{X}(b,d)}{4}-\frac{1}{2}\left[\frac{ab}{b-a}\left[\int_a^b\frac{\mathcal{X}(x,c)}{x^2}\mathrm{d}x+\int_a^b\frac{\mathcal{X}(x,d)}{x^2}\mathrm{d}x\right]\right.$$

$$+\left[\frac{cd}{d-c}\left[\int_c^d \frac{\mathcal{X}(a,u)}{u^2}\mathrm{d}u + \int_c^d \frac{\mathcal{X}(b,u)}{u^2}\mathrm{d}u\right]\right] + \frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d \frac{\mathcal{X}(x,u)}{x^2u^2}\mathrm{d}u\mathrm{d}x.$$

In order to obtain our results we need the gamma function $\Gamma$ [38,39], beta function $B$ [40] and Gaussian hypergeometric functions ${}_2\mathscr{F}_1$ [41,42], which are defined by

$$\Gamma(x) = \int_0^\infty e^{-x}\mu^{x-1}\mathrm{d}\mu,$$

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 \mu^{x-1}(1-\mu)^{y-1}\,\mathrm{d}\mu$$

and

$${}_2\mathscr{F}_1(x,y;c;z) = \frac{1}{\mathrm{B}(y,c-y)}\int_0^1 \mu^{y-1}(1-\mu)^{c-y-1}(1-zt)^{-x}\mathrm{d}\mu,$$

respectively.

**Theorem 5.2** Let $p,q>1$ with $1/p+1/q=1$, and $\mathcal{X}:\Omega=[a,b]\times[c,d]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a partial differentiable function on $\Omega$ such that $\frac{\partial^2\mathcal{X}}{\partial\mu\partial\lambda}\in L_1(\Omega)$ and $\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}\right|^q$ is a $2D$ approximately reciprocal $\rho$-convex function. Then we have

$$|\mathcal{X}(a,b,c,d,x,y:\Omega)|$$

$$\leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\varphi_1(a,b,c,d:\Omega)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(b,d)\right|^q\right.$$

$$+\varphi_2(a,b,c,d:\Omega)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(a,d)\right|^q + \varphi_3(a,b,c,d:\Omega)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(b,c)\right|^q$$

$$\left.+\varphi_4(a,b,c,d:\Omega)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(a,c)\right|^q + \varphi_5(a,b,c,d:\Omega) + \varphi_6(a,b,c,d:\Omega)\right]^{\frac{1}{q}},$$

where

$$\varphi_1(a,b,c,d;\Omega) = \int_0^1\int_0^1\left[\frac{\rho(\mu)}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{\rho(\lambda)}{(rd+(1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda,$$

$$\varphi_2(a,b,c,d;\Omega) = \int_0^1\int_0^1\left[\frac{\rho(1-\mu)}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{\rho(\lambda)}{(rd+(1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda,$$

$$\varphi_3(a,b,c,d;\Omega) = \int_0^1\int_0^1\left[\frac{\rho(\mu)}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{\rho(1-\lambda)}{(rd+(1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda,$$

$$\varphi_4(a,b,c,d;\Omega) = \int_0^1\int_0^1\left[\frac{\rho(1-\mu)}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{\rho(1-\lambda)}{(rd+(1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda,$$

$$\varphi_5(a,b,c,d;\Omega)=\Delta(a,b)\left(\int_0^1\int_0^1\left[\frac{1}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{1}{(rd+(1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda\right)$$

$$=\Delta(a,b)\left(\left[a^{-2q}\,{}_2\mathscr{F}_1\left(2q,1,2,1-\frac{b}{a}\right)\right]\left[c^{-2q}\,{}_2\mathscr{F}_1\left(2q,1,2,1-\frac{d}{c}\right)\right]\right)$$

and

$$\varphi_6(a,b,c,d;\Omega)=\Delta(c,d)\left(\int_0^1\int_0^1\left[\frac{1}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{1}{(rd+(1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda\right)$$

$$=\Delta(c,d)\left(\left[a^{-2q}\,{}_2\mathscr{F}_1\left(2q,1,2,1-\frac{b}{a}\right)\right]\left[a^{-2q}\,{}_2\mathscr{F}_1\left(2q,1,2,1-\frac{b}{a}\right)\right]\right).$$

*Proof.* It follows from Lemma 5.1, Hölder inequality and the $2D$ approximately reciprocal $\rho$-convexity of $\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}\right|^q$ that

$$|\mathcal{X}(a,b,c,d,x,y:\Omega)|$$

$$=\left|\frac{ab(b-a)cd(d-c)}{4}\int_0^1\int_0^1\left(\frac{1-2\mu}{(tb+(1-\mu)a)^2}\right)\left(\frac{1-2\lambda}{(rd+(1-\lambda)c)^2}\right)\right.$$

$$\left.\times\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}\left[\frac{ab}{tb+(1-\mu)a},\frac{cd}{rd+(1-\lambda)c)}\right]\mathrm{d}\lambda\mathrm{d}\mu\right|$$

$$\leq\frac{ab(b-a)cd(d-c)}{4}\int_0^1\int_0^1[|(1-2\mu)(1-2\lambda)|^p\mathrm{d}\lambda\mathrm{d}\mu]^{\frac{1}{p}}$$

$$\times\Big[\int_0^1\int_0^1\left[\frac{1}{(tb+(1-\mu)a)^{2q}}\frac{1}{(rd+(1-\lambda)c)^{2q}}\right]$$

$$\times\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}\left[\frac{ab}{tb+(1-\mu)a},\frac{cd}{rd+(1-\lambda)c}\right]\right|^q\mathrm{d}\lambda\mathrm{d}\mu\Big]^{\frac{1}{q}}$$

$$\leq\frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\Bigg(\int_0^1\int_0^1\left[\frac{1}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{1}{(rd+(1-\lambda)c)^{2q}}\right]$$

$$\times\Bigg[\rho(\mu)\rho(\lambda)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(b,d)\right|^q+\rho(1-\mu)\rho(\lambda)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(a,d)\right|^q+\rho(\mu)\rho(1-\lambda)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(b,c)\right|^q$$

$$+\rho(1-\mu)\rho(1-\lambda)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(a,c)\right|^q+\Delta(a,b)+\Delta(c,d)\Bigg]\mathrm{d}\lambda\mathrm{d}\mu\Bigg)^{\frac{1}{q}}.$$

This completes the proof. □

**I**. If we take $\rho(\mu) = \mu$ and $\rho(\lambda) = \lambda$, then Theorem 5.2 leads to Corollary 5.3.

**Corollary 5.3**. Let $p, q > 1$ with $1/p + 1/q = 1$, and $\mathcal{X} : \Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty) \to \mathbb{R}$ be a partial differentiable function on $\Omega$ such that $\frac{\partial^2 \mathcal{X}}{\partial\mu\partial\lambda} \in L_1(\Omega)$ and $\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}\right|^q$ is a $2D$ approximately reciprocal convex function. Then one has

$$|\mathcal{X}(a, b, c, d, x, y : \Omega)|$$

$$\leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\varphi_1^*(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b, d)\right|^q\right.$$

$$+\varphi_2^*(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a, d)\right|^q + \varphi_3^*(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b, c)\right|^q$$

$$\left.+\varphi_4^*(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a, c)\right|^q + \varphi_5(a, b, c, d : \Omega) + \varphi_6(a, b, c, d : \Omega)\right]^{\frac{1}{q}},$$

where

$$\varphi_1^*(a, b, c, d; \Omega) = \int_0^1\int_0^1\left[\frac{\mu}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{\lambda}{(rd + (1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{2}\ {}_2\mathscr{F}_1\left(2q, 2, 3, 1 - \frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{2}\ {}_2\mathscr{F}_1\left(2q, 2, 3, 1 - \frac{d}{c}\right)\right],$$

$$\varphi_2^*(a, b, c, d; \Omega) = \int_0^1\int_0^1\left[\frac{(1-\mu)}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{\lambda}{(rd + (1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{2}\ {}_2\mathscr{F}_1\left(2q, 1, 3, 1 - \frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{2}\ {}_2\mathscr{F}_1\left(2q, 2, 3, 1 - \frac{d}{c}\right)\right],$$

$$\varphi_3^*(a, b, c, d; \Omega) = \int_0^1\int_0^1\left[\frac{\mu}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{(1-\lambda)}{(rd + (1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{2}\ {}_2\mathscr{F}_1\left(2q, 2, 3, 1 - \frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{2}\ {}_2\mathscr{F}_1\left(2q, 1, 3, 1 - \frac{d}{c}\right)\right],$$

$$\varphi_4^*(a, b, c, d; \Omega) = \int_0^1\int_0^1\left[\frac{(1-\mu)}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{(1-\lambda)}{(rd + (1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{2}\ {}_2\mathscr{F}_1\left(2q, 1, 3, 1 - \frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{2}\ {}_2\mathscr{F}_1\left(2q, 1, 3, 1 - \frac{d}{c}\right)\right],$$

and $\varphi_5(a, b, c, d; \Omega)$ and $\varphi_6(a, b, c, d; \Omega)$ are given in Theorem 5.2.

**II**. Let $\rho(\mu) = \mu^s$ and $\rho(\lambda) = \lambda^s$. Then Theorem 5.2 reduces to Corollary 5.4.

**Corollary 5.4**. Let $p, q > 1$ with $1/p + 1/q = 1$, and $\mathcal{X} : \Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty) \to \mathbb{R}$ be a partial differentiable function on $\Omega$ such that $\frac{\partial^2 \mathcal{X}}{\partial\mu\partial\lambda} \in L_1(\Omega)$ and $\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}\right|^q$ is a Breckner type $2D$ approximately reciprocal $s$-convex function. Then

$$|\mathcal{X}(a, b, c, d, x, y : \Omega)|$$

$$\leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\varphi_1^{**}(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b, d)\right|^q\right.$$

$$+\varphi_2^{**}(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a, d)\right|^q + \varphi_3^{**}(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b, c)\right|^q$$

$$\left.+\varphi_4^{**}(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a, c)\right|^q + \varphi_5(a, b, c, d : \Omega) + \varphi_6(a, b, c, d : \Omega)\right]^{\frac{1}{q}},$$

where

$$\varphi_1^{**}(a, b, c, d; \Omega) = \int_0^1\int_0^1 \left[\frac{\mu^s}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{\lambda^s}{(rd + (1-\lambda)c)^{2q}}\right] \mathrm{d}\mu \mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{s+1}\, {}_2\mathscr{F}_1\left(2q, s+1, s+2, 1-\frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{s+1}\, {}_2\mathscr{F}_1\left(2q, s+1, s+2, 1-\frac{d}{c}\right)\right],$$

$$\varphi_2^{**}(a, b, c, d; \Omega) = \int_0^1\int_0^1 \left[\frac{(1-\mu)^s}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{\lambda^s}{(rd + (1-\lambda)c)^{2q}}\right] \mathrm{d}\mu \mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{s+1}\, {}_2\mathscr{F}_1\left(2q, 1, s+2, 1-\frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{s+1}\, {}_2\mathscr{F}_1\left(2q, s+1, s+2, 1-\frac{d}{c}\right)\right],$$

$$\varphi_3^{**}(a, b, c, d; \Omega) = \int_0^1\int_0^1 \left[\frac{\mu^s}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{(1-\lambda)^s}{(rd + (1-\lambda)c)^{2q}}\right] \mathrm{d}\mu \mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{s+1}\, {}_2\mathscr{F}_1\left(2q, s+1, s+2, 1-\frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{s+1}\, {}_2\mathscr{F}_1\left(2q, 1, s+2, 1-\frac{d}{c}\right)\right],$$

$$\varphi_4^{**}(a, b, c, d; \Omega) = \int_0^1\int_0^1 \left[\frac{(1-\mu)^s}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{(1-\lambda)^s}{(rd + (1-\lambda)c)^{2q}}\right] \mathrm{d}\mu \mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{s+1}\, {}_2\mathscr{F}_1\left(2q, 1, s+2, 1-\frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{s+1}\, {}_2\mathscr{F}_1\left(2q, 1, s+2, 1-\frac{d}{c}\right)\right],$$

and $\varphi_5(a, b, c, d; \Omega)$ and $\varphi_6(a, b, c, d; \Omega)$ are given in Theorem 5.2.

**III**. If we take $\rho(\mu) = \mu^{-s}$ and $\rho(\lambda) = \lambda^{-s}$, then Theorem 5.2 becomes Corollary 5.5.

**Corollary 5.5**. Let $p, q > 1$ with $1/p + 1/q = 1$, and $\mathcal{X} : \Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty) \to \mathbb{R}$ be a partial differentiable function on $\Omega$ such that $\frac{\partial^2 \mathcal{X}}{\partial\mu\partial\lambda} \in L_1(\Omega)$ and $\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}\right|^q$ is a Godunova-Levin type $2D$ approximately reciprocal $s$-convex function. Then we obtain

$$|\mathcal{X}(a, b, c, d, x, y : \Omega)| \leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\Bigg[\varphi_1^{***}(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b, d)\right|^q$$

$$+\varphi_2^{***}(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a, d)\right|^q + \varphi_3^{***}(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b, c)\right|^q$$

$$+\varphi_4^{***}(a, b, c, d : \Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a, c)\right|^q + \varphi_5(a, b, c, d : \Omega) + \varphi_6(a, b, c, d : \Omega)\Bigg]^{\frac{1}{q}},$$

where

$$\varphi_1^{***}(a, b, c, d; \Omega) = \int_0^1\int_0^1 \left[\frac{\mu^{-s}}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{\lambda^{-s}}{(rd + (1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{1-s}\,{}_2\mathscr{F}_1\left(2q, 1-s, 2-s, 1-\frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{1-s}\,{}_2\mathscr{F}_1\left(2q, 1-s, 2-s, 1-\frac{d}{c}\right)\right],$$

$$\varphi_2^{***}(a, b, c, d; \Omega) = \int_0^1\int_0^1 \left[\frac{(1-\mu)^{-s}}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{\lambda^{-s}}{(rd + (1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{1-s}\,{}_2\mathscr{F}_1\left(2q, 1, 2-s, 1-\frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{1-s}\,{}_2\mathscr{F}_1\left(2q, 1-s, 2-s, 1-\frac{d}{c}\right)\right],$$

$$\varphi_3^{***}(a, b, c, d; \Omega) = \int_0^1\int_0^1 \left[\frac{\mu^{-s}}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{(1-\lambda)^{-s}}{(rd + (1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{1-s}\,{}_2\mathscr{F}_1\left(2q, 1-s, 2-s, 1-\frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{1-s}\,{}_2\mathscr{F}_1\left(2q, 1, 2-s, 1-\frac{d}{c}\right)\right],$$

$$\varphi_4^{***}(a, b, c, d; \Omega) = \int_0^1\int_0^1 \left[\frac{(1-\mu)^{-s}}{(tb + (1-\mu)a)^{2q}}\right]\left[\frac{(1-\lambda)^{-s}}{(rd + (1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[\frac{a^{-2q}}{1-s}\,{}_2\mathscr{F}_1\left(2q, 1, 2-s, 1-\frac{b}{a}\right)\right]\left[\frac{c^{-2q}}{1-s}\,{}_2\mathscr{F}_1\left(2q, 1, 2-s, 1-\frac{d}{c}\right)\right],$$

and $\varphi_5(a, b, c, d; \Omega)$ and $\varphi_6(a, b, c, d; \Omega)$ are given in Theorem 5.2.

**IV**. Let $\rho(\mu) = \rho(\lambda) = 1$. Then Theorem 5.2 leads to Corollary 5.6.

**Corollary 5.6**. Let $p, q > 1$ with $1/p + 1/q = 1$, and $\mathcal{X} : \Omega = [a, b] \times [c, d] \subseteq (0, \infty) \times (0, \infty) \to \mathbb{R}$ be a partial differentiable function on $\Omega$ such that $\frac{\partial^2 \mathcal{X}}{\partial\mu\partial\lambda} \in L_1(\Omega)$ and $\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}\right|^q$ is a $2D$ approximately reciprocal $P$-convex function. Then we have

$$|\mathcal{X}(a, b, c, d, x, y : \Omega)|$$

$$\leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}[\varphi(a,b,c,d:\Omega)]^{\frac{1}{q}}\Bigg[\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b,d)\right|^q$$

$$+\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a,d)\right|^q + \left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b,c)\right|^q + \left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a,c)\right|^q + \Delta(a,b) + \Delta(c,d)\Bigg]^{\frac{1}{q}},$$

where

$$\varphi(a,b,c,d;\Omega) = \int_0^1\int_0^1 \left[\frac{1}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{1}{(rd+(1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda$$

$$= \left[a^{-2q}\ {}_2\mathscr{F}_1\left(2q,1,2,1-\frac{b}{a}\right)\right]\left[c^{-2q}\ {}_2\mathscr{F}_1\left(2q,1,2,1-\frac{d}{c}\right)\right].$$

**V**. Let

$$\Delta(a,b) = -\mu\left(\mu^{\sigma}(1-\mu)+\mu(1-\mu)^{\sigma}\right)\left(\|\ \frac{1}{a}-\frac{1}{b}\ \|\right)^{\sigma}$$

and

$$\Delta(c,d) = -\mu\left(\lambda^{\sigma}(1-\lambda)+\lambda(1-\lambda)^{\sigma}\right)\left(\|\ \frac{1}{c}-\frac{1}{d}\ \|\right)^{\sigma}$$

for some $\mu > 0$. Then Theorem 5.2 reduces to Corollary 5.7.

**Corollary 5.7**. Let $p,q > 1$ with $1/p + 1/q = 1$, and $\mathcal{X} : \Omega = [a,b]\times[c,d] \subseteq (0,\infty)\times(0,\infty) \to \mathbb{R}$ be a partial differentiable function on $\Omega$ such that $\frac{\partial^2 \mathcal{X}}{\partial\mu\partial\lambda} \in L_1(\Omega)$ and $\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}\right|^q$ is a $2D$ reciprocal strong $\rho$-convex function of higher order. Then one has

$$|\mathcal{X}(a,b,c,d,x,y:\Omega)|$$

$$\leq \frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\Bigg[\varphi_1(a,b,c,d:\Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b,d)\right|^q$$

$$+\varphi_2(a,b,c,d:\Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a,d)\right|^q + \varphi_3(a,b,c,d:\Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(b,c)\right|^q$$

$$+\varphi_4(a,b,c,d:\Omega)\left|\frac{\partial^2 \mathcal{X}}{\partial\lambda\partial\mu}(a,c)\right|^q + \varphi_5^*(a,b,c,d:\Omega) + \varphi_6^*(a,b,c,d:\Omega)\Bigg]^{\frac{1}{q}},$$

where $\varphi_1(a,b,c,d:\Omega)$, $\varphi_2(a,b,c,d:\Omega)$, $\varphi_3(a,b,c,d:\Omega)$, $\varphi_4(a,b,c,d:\Omega)$ are given in Theorem 5.2, and

$$\varphi_5^*(a,b,c,d;\Omega)$$

$$= -\mu\left(\left\|\frac{1}{a}-\frac{1}{b}\right\|\right)^{\sigma}\left(\int_0^1\int_0^1\left[\frac{(\mu^{\sigma}(1-\mu)+\mu(1-\mu)^{\sigma}}{(tb+(1-\mu)a)^{2q}}\right]\left[\frac{1}{(rd+(1-\lambda)c)^{2q}}\right]\mathrm{d}\mu\mathrm{d}\lambda\right)$$

$$= -\mu\left(\left\|\frac{1}{a}-\frac{1}{b}\right\|\right)^{\sigma}\left(\left[\frac{a^{-2q}}{(\sigma+1)(\sigma+2)}\ {}_2\mathscr{F}_1\left(2q,\sigma+1,\sigma+3,1-\frac{b}{a}\right)\right.\right.$$

$$+\frac{a^{-2q}}{(\sigma+2)(\sigma+1)}\ {}_2\mathscr{F}_1\left(2q,2,\sigma+3,1-\frac{b}{a}\right)\Big]\Big[c^{-2q}\ {}_2\mathscr{F}_1\left(2q,1,2,1-\frac{d}{c}\right)\Big]\Big),$$

$$\varphi_6^*(a,b,c,d;\Omega)$$

$$=-\mu\left(\left\|\frac{1}{c}-\frac{1}{d}\right\|\right)^{\sigma}\Big(\int_0^1\int_0^1\Big[\frac{1}{(tb+(1-\mu)a)^{2q}}\Big]\Big[\frac{(\lambda^{\sigma}(1-\lambda)+\lambda(1-\lambda)^{\sigma})}{(rd+(1-\lambda)c)^{2q}}\Big]\mathrm{d}\mu\mathrm{d}\lambda\Big)$$

$$=-\mu\left(\left\|\frac{1}{c}-\frac{1}{d}\right\|\right)^{\sigma}\left(\Big[a^{-2q}\ {}_2\mathscr{F}_1\left(2q,1,2,1-\frac{b}{a}\right)\Big]\Big[\frac{c^{-2q}}{(\sigma+1)(\sigma+2)}\ {}_2\mathscr{F}_1\left(2q,\sigma+1,\sigma+3,1-\frac{d}{c}\right)\right.$$

$$\left.+\frac{c^{-2q}}{(\sigma+2)(\sigma+1)}\ {}_2\mathscr{F}_1\left(2q,2,\sigma+3,1-\frac{d}{c}\right)\Big]\right).$$

**VI**. If we take $\sigma=2$, then Corollary 5.7 becomes Corollary 5.8.

**Corollary 5.8**. Let $p,q>1$ with $1/p+1/q=1$, and $\mathcal{X}:\Omega=[a,b]\times[c,d]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a partial differentiable function on $\Omega$ such that $\frac{\partial^2\mathcal{X}}{\partial\mu\partial\lambda}\in L_1(\Omega)$ and $\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}\right|^q$ is a $2D$ reciprocal strong $\rho$-convex function. Then

$$|\mathcal{X}(a,b,c,d,x,y:\Omega)|$$

$$\leq\frac{ab(b-a)cd(d-c)}{4(p+1)^{\frac{2}{p}}}\Big[\varphi_1(a,b,c,d:\Omega)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(b,d)\right|^q$$

$$+\varphi_2(a,b,c,d:\Omega)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(a,d)\right|^q+\varphi_3(a,b,c,d:\Omega)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(b,c)\right|^q$$

$$+\varphi_4(a,b,c,d:\Omega)\left|\frac{\partial^2\mathcal{X}}{\partial\lambda\partial\mu}(a,c)\right|^q+\varphi_5^{**}(a,b,c,d:\Omega)+\varphi_6^{**}(a,b,c,d:\Omega)\Big]^{\frac{1}{q}},$$

where $\varphi_1(a,b,c,d:\Omega)$, $\varphi_2(a,b,c,d:\Omega)$, $\varphi_3(a,b,c,d:\Omega)$, $\varphi_4(a,b,c,d:\Omega)$ are given in Theorem 5.2, and

$$\varphi_5^{**}(a,b,c,d;\Omega)$$

$$=-\mu\left(\left\|\frac{1}{a}-\frac{1}{b}\right\|\right)^{2}\Big(\int_0^1\int_0^1\Big[\frac{\mu(1-\mu)}{(tb+(1-\mu)a)^{2q}}\Big]\Big[\frac{1}{(rd+(1-\lambda)c)^{2q}}\Big]\mathrm{d}\mu\mathrm{d}\lambda\Big)$$

$$=-\mu\left(\left\|\frac{1}{a}-\frac{1}{b}\right\|\right)^{2}\left(\Big[\frac{a^{-2q}}{6}\ {}_2\mathscr{F}_1\left(2q,2,4,1-\frac{b}{a}\right)\Big]\Big[c^{-2q}\ {}_2\mathscr{F}_1\left(2q,1,2,1-\frac{d}{c}\right)\Big]\right),$$

$$\varphi_6^{**}(a,b,c,d;\Omega)$$

$$=-\mu\left(\left\|\frac{1}{c}-\frac{1}{d}\right\|\right)^{2}\Big(\int_0^1\int_0^1\Big[\frac{1}{(tb+(1-\mu)a)^{2q}}\Big]\Big[\frac{\lambda(1-\lambda)}{(rd+(1-\lambda)c)^{2q}}\Big]\mathrm{d}\mu\mathrm{d}\lambda\Big)$$

$$=-\mu\left(\left\|\frac{1}{c}-\frac{1}{d}\right\|\right)^{2}\left(\Big[a^{-2q}\ {}_2\mathscr{F}_1\left(2q,1,2,1-\frac{b}{a}\right)\Big]\Big[\frac{c^{-2q}}{6}\ {}_2\mathscr{F}_1\left(2q,2,4,1-\frac{d}{c}\right)\Big]\right).$$

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## Conflict of interest

The authors declare that they have no competing interests.

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