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Research article

Estimates of bounds on the weighted Simpson type inequality and their applications

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Abstract: Based on the established integral identity, some bounds involving the weighted Simpson type inequality are obtained where the first derivative of considered mappings is (m, h)-preinvex, boundedness or Lipschitzian. As applications, certain generalized inequalities in connection with weighted Simpson type quadrature formula, continuous random variables and F-divergence measures are investigated, respectively.

Keywords: Simpson type inequality; (m, h)-preinvex mapping; F-divergence measures
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1. Introduction

The following inequality is well known as Simpson's inequality:

$$\left|\frac{1}{6}\left[F(\varepsilon_1)+4F\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)+F(\varepsilon_2)\right]-\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}F(x)\mathrm{d}x\right|\leq\frac{1}{2880}\left\|F^{(4)}\right\|_\infty(\varepsilon_2-\varepsilon_1)^4, \qquad (1.1)$$

where $F:[\varepsilon_1,\varepsilon_2]\to\mathbb{R}$ is a four times continuously differentiable mapping on $(\varepsilon_1,\varepsilon_2)$ and $\left\|F^{(4)}\right\|_\infty=\sup_{t\in(\varepsilon_1,\varepsilon_2)}\left|F^{(4)}(t)\right|<\infty$.

Because of the extensive applications of the Simpson's inequality (1.1), many authors have extended their studies via different classes of mappings. For example, Noor et al. [17] presented certain Simpson-type inequalities for geometrically relative convex functions. Du et al. [4] gave several properties and inequalities pertaining to Hadamard–Simpson type for the generalized (s, m)-preinvexity. Matłoka [16] established some weighted Simpson type inequalities for h-convexity. Further inequalities of the Simpson type related to other kinds of convex mappings in question with applications to fractional integrals can be found in [2, 3, 6, 7, 11, 18, 20]. More recent results with respect to (1.1), the interested reader is directed to [5, 8, 10, 13, 15, 19] and the references cited therein.

Let us recall that Hudzik and Maligranda [9] introduced a class of mappings, called s-convex mapping, as follows: A mapping $F : \mathbb{R}_0 = (0, \infty) \to \mathbb{R}$ is named s-convex if

$$F(t\varepsilon_1 + (1-t)\varepsilon_2) \le t^s F(\varepsilon_1) + (1-t)^s F(\varepsilon_2)$$

holds for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_0$ and $t \in [0,1]$ with certain fixed $s \in (0,1]$. In [22], Shuang and Qi proved the following result for this mapping.

Theorem 1. *Let $F : \mathcal{H} \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on $\mathcal{H}^\circ$(the interior of $\mathcal{H}$) , $\varepsilon_1, \varepsilon_2 \in \mathcal{H}$ along with $\varepsilon_1 < \varepsilon_2$ and $F' \in L^1([\varepsilon_1, \varepsilon_2])$. If $|F'|^q$ with $q > 1$ is an s-convex mapping on $[\varepsilon_1, \varepsilon_2]$ for certain fixed $s \in (0,1]$, then*

$$\begin{aligned}
&\left|\frac{1}{10}\Big[F(\varepsilon_1) + 8F\Big(\frac{\varepsilon_1+\varepsilon_2}{2}\Big) + F(\varepsilon_2)\Big] - \frac{1}{\varepsilon_2 - \varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2} F(x)\mathrm{d}x\right| \\
&\le \frac{\varepsilon_2-\varepsilon_1}{4}\left(\frac{(q-1)(4^{(2q-1)/(q-1)}+1)}{(2q-1)5^{(2q-1)/(q-1)}}\right)^{1-\frac{1}{q}}\Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\left\{\left[\left|F'(\varepsilon_1)\right|^q + \left|F'\Big(\frac{\varepsilon_1+\varepsilon_2}{2}\Big)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad \left. + \left[\left|F'\Big(\frac{\varepsilon_1+\varepsilon_2}{2}\Big)\right|^q + \left|F'(\varepsilon_2)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

We next recall the concept of the η-path and m-preinvex mappings.

Definition 1. *[1] Let $\mathcal{K} \subset \mathbb{R}^n$ be a nonempty invex set with regard to $\eta : \mathcal{K} \times \mathcal{K} \to \mathbb{R}^n$. For all $x, y \in \mathcal{K}$, the η-path P_{xv} joining the points x and $v = x + \eta(y, x)$ is defined by*

$$P_{xv} := \{z | z = x + t\eta(y, x), t \in [0,1]\}.$$

Definition 2. *[12] Let $\mathcal{K} \subseteq [0, d^*]$ with $d^* > 0$ be an invex set with regard to $\eta : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$, for mapping $F : \mathcal{K} \to \mathbb{R}$, if*

$$F(u + t\eta(v, u)) \le (1-t)F(u) + mtF\Big(\frac{v}{m}\Big)$$

is valid for all $u, v \in \mathcal{K}$, $t \in [0,1]$ and $m \in (0,1]$, then we say that F is an m-preinvex mapping with regard to η.

Recently, in [14], Li et al. introduced the following class of mappings of (s, m)-preinvex.

Definition 3. *[14] Let $\mathcal{K} \subseteq [0, d^*]$ with $d^* > 0$ be an invex set with regard to $\eta : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$, for mapping $F : \mathcal{K} \to \mathbb{R}$, if*

$$F(u + t\eta(v, u)) \le (1-t)^s F(u) + mt^s F\Big(\frac{v}{m}\Big)$$

is valid for all $u, v \in \mathcal{K}$, $t \in [0,1]$ and $(s, m) \in (0,1] \times (0,1]$ with $\frac{v}{m} \le d^*$, then we say that f is a (s, m)-preinvex mapping with regard to η.

Here, our main purpose is to obtain some bounds pertaining to the weighted Simpson type inequality. To this end, we consider the coming three cases: (i) the considered mapping is (m, h)-preinvex; (ii) the derivative of the considered mapping is bounded; (iii) the derivative of the considered mapping satisfies the Lipschitz condition.

2. A definition and a lemma

As one can see, the definitions of the m-preinvex and (s, m)-preinvex mappings have similar configurations. This observation leads us to generalize these convexities and presents the following definition, to be referred to as the (m, h)-perinvex mapping.

Definition 4. *Let $\mathcal{J}$, $\mathcal{K}$ be intervals in $\mathbb{R}$, $(0, 1) \subseteq \mathcal{J}$, and let $h : \mathcal{J} \to \mathbb{R}$ be a non-negative real mapping. A mapping $F : \mathcal{K} \to (0, \infty)$ is said to be (m, h)-perinvex mapping, if the following inequality*

$$F(u + t\eta(v, u)) \le h(1 - t)F(u) + mh(t)F\Big(\frac{v}{m}\Big) \tag{2.1}$$

is valid for all $u, v \in \mathcal{K}$ and $t \in (0, 1)$ with certain fixed $m \in (0, 1]$.

Some particular cases of Definition 4 are discussed as follows.
I. If $h(t) = t$, then Definition 4 reduces to the definition for m-preinvexity.
II. If $h(t) = t^s$ for $s \in (0, 1]$, then Definition 4 reduces to the definition for (s, m)-preinvexity.
IV. If $h(t) = 1$, then Definition 4 reduces to the definition for (m, P)-preinvexity.
V. If $h(t) = t(1 - t)$, then Definition 4 reduces to the definition for (m, tgs)-preinvexity.

Example 1. *Consider the mapping $F(u) = -|u|$ with*

$$\eta(v, u) = \begin{cases} v - u, & uv \ge 0, \\ u - v, & uv < 0. \end{cases}$$

If we take $h(t) = t^s$, $t \in (0, 1)$ with $s \ge 1$, then by Definition 4, F is an (m, h)-perinvex mapping with regard to $\eta : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ and certain fixed $m \in (0, 1]$. However, if we take $h(t) = t^s$, $t \in (0, 1)$ with $0 < s < 1$, by letting $u = 1$, $v = 5$, $t = \frac{1}{3}$ and $s = \frac{1}{4}$, then we have

$$F(u + t\eta(v, u)) = -\frac{7}{3} \approx -2.3333 > -4.7028 \approx -\frac{2^{\frac{1}{4}} + 5}{3^{\frac{1}{4}}} = h(1 - t)F(u) + mh(t)F\Big(\frac{v}{m}\Big),$$

which demonstrates that F is not (m, h)-perinvex on $\mathbb{R}$ with respect to the same η.

In the whole article, let $\mathcal{K} \subseteq [0, d^*]$ with $d^* > 0$ be an invex set with respect to $\eta : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ and $a, b \in \mathcal{K}$ with $\eta(b, a) > 0$. Assume that $F : \mathcal{K} \to \mathbb{R}$ is a differentiable mapping such that F' is integrable on the η-Path $P_{x\tau} : \tau = x + \eta(y, x)$ with $x, y \in [a, b]$. Let $g : [a, a + \eta(b, a)] \to [0, +\infty)$ be differentiable mapping on $\mathcal{K}$ and be symmetric to $a + \frac{1}{2}\eta(b, a)$. We write $\|g\|_{[a,a+\eta(b,a)],\infty} = \sup_{x\in[a,a+\eta(b,a)]} |g(x)|$.

For the sake of simplicity, we define the following notation:

$$\begin{aligned}\Psi_{F,g}(a, b; \eta) :=\ & \frac{1}{10\eta(b, a)}\Big[F(a) + 8F\Big(a + \frac{1}{2}\eta(b, a)\Big) + F(a + \eta(b, a))\Big] \int_a^{a+\eta(b,a)} g(x)\mathrm{d}x \\ & - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} g(x)F(x)\mathrm{d}x.\end{aligned}$$

Now, we present the succeeding lemma.

Lemma 1. *The subsequent integral identity with $F', g \in L^1([a, a+\eta(b,a)])$ exists:*

$$\Psi_{F,g}(a,b;\eta) = \frac{\eta(b,a)}{4}\left\{ \int_0^1 \mathcal{R}_1(t)F'\Big(a + \frac{1-t}{2}\eta(b,a)\Big)\mathrm{d}t + \int_0^1 \mathcal{R}_2(t)F'\Big(a + \frac{2-t}{2}\eta(b,a)\Big)\mathrm{d}t \right\}, \tag{2.2}$$

where

$$\mathcal{R}_1(t) = \frac{4}{5}\int_0^1 g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u - \int_0^t g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u$$

and

$$\mathcal{R}_2(t) = \frac{1}{5}\int_0^1 g\Big(a + \frac{2-u}{2}\eta(b,a)\Big)\mathrm{d}u - \int_0^t g\Big(a + \frac{2-u}{2}\eta(b,a)\Big)\mathrm{d}u.$$

Proof. By integration by parts and changing the variables, one has

$$\begin{aligned}
Q_1 &= \int_0^1 \mathcal{R}_1(t)F'\Big(a + \frac{1-t}{2}\eta(b,a)\Big)\mathrm{d}t \\
&= \int_0^1 \Big[\frac{4}{5}\int_0^1 g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u - \int_0^t g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u\Big]F'\Big(a + \frac{1-t}{2}\eta(b,a)\Big)\mathrm{d}t \\
&= \frac{-2}{\eta(b,a)}\Big[\frac{4}{5}\int_0^1 g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u - \int_0^t g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u\Big]F\Big(a + \frac{1-t}{2}\eta(b,a)\Big)\Big|_0^1 \\
&\quad - \frac{2}{\eta(b,a)}\int_0^1 g\Big(a + \frac{1-t}{2}\eta(b,a)\Big)F\Big(a + \frac{1-t}{2}\eta(b,a)\Big)\mathrm{d}t \\
&= \frac{-2}{\eta(b,a)}\Big[-\frac{1}{5}F(a) - \frac{4}{5}F\Big(a + \frac{1}{2}\eta(b,a)\Big)\Big]\int_0^1 g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u \\
&\quad - \frac{2}{\eta(b,a)}\int_0^1 g\Big(a + \frac{1-t}{2}\eta(b,a)\Big)F\Big(a + \frac{1-t}{2}\eta(b,a)\Big)\mathrm{d}t \\
&= \frac{4}{\eta^2(b,a)}\Big[\frac{1}{5}F(a) + \frac{4}{5}F\Big(a + \frac{1}{2}\eta(b,a)\Big)\Big]\int_a^{a+\frac{1}{2}\eta(b,a)} g(x)\mathrm{d}x - \frac{4}{\eta^2(b,a)}\int_a^{a+\frac{1}{2}\eta(b,a)} g(x)F(x)\mathrm{d}x.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
Q_2 &= \int_0^1 \mathcal{R}_2(t)F'\Big(a + \frac{2-t}{2}\eta(b,a)\Big)\mathrm{d}t \\
&= \frac{1}{\eta^2(b,a)}\Big[\frac{4}{5}F\Big(a + \frac{1}{2}\eta(b,a)\Big) + \frac{1}{5}F(a + \eta(b,a))\Big]\int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} g(x)\mathrm{d}x - \frac{4}{\eta^2(b,a)}\int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} g(x)F(x)\mathrm{d}x.
\end{aligned}$$

Since $g(x)$ is symmetric with relate to $a + \frac{1}{2}\eta(b,a)$, one has

$$\int_a^{a+\frac{1}{2}\eta(b,a)} g(x)\mathrm{d}x = \int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} g(x)\mathrm{d}x = \frac{1}{2}\int_a^{a+\eta(b,a)} g(x)\mathrm{d}x.$$

As a result, one has

$$\frac{\eta(b,a)}{4}\Big(Q_1 + Q_2\Big) = \Psi_{F,g}(a,b;\eta),$$

which completes the proof.

Remark 1. *If we take $\eta(b,a)=b-a$ in Lemma 1, then we have*

$$\frac{1}{10(b-a)}\Big[F(a)+8F\Big(\frac{a+b}{2}\Big)+F(b)\Big]\int_a^b g(x)\mathrm{d}x-\frac{1}{b-a}\int_a^b F(x)g(x)\mathrm{d}x$$
$$=\frac{b-a}{4}\Big\{\int_0^1\Big[\frac{4}{5}\int_0^1 g\Big(ua+(1-u)\frac{a+b}{2}\Big)\mathrm{d}u-\int_0^t g\Big(ua+(1-u)\frac{a+b}{2}\Big)\mathrm{d}u\Big]F'\Big(ta+(1-t)\frac{a+b}{2}\Big)\mathrm{d}t$$
$$+\int_0^1\Big[\frac{1}{5}\int_0^1 g\Big(u\frac{a+b}{2}+(1-u)b\Big)\mathrm{d}u-\int_0^t g\Big(u\frac{a+b}{2}+(1-u)b\Big)\mathrm{d}u\Big]F'\Big(t\frac{a+b}{2}+(1-t)b\Big)\mathrm{d}t\Big\}.$$

Specially, if we choose $g(x)=1$, then we obtain Lemma 2.1 proved by Shuang and Qi in [22].

3. Main results

Our first main result is stated by the coming theorem.

Theorem 2. *If $|F'|^q$ for $q\geq 1$ is (m,h)-preinvex on $\mathcal{K}^\circ$ with certain fixed $m\in(0,1]$, then the following inequality is valid:*

$$\Big|\Psi_{F,g}(a,b;\eta)\Big|\leq\frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}$$
$$\times\Big\{\Big(\frac{13}{10}\Big)^{1-\frac{1}{q}}\Big[\Big|F'(a)\Big|^q\int_0^1\Big|\frac{4}{5}+t\Big|h\Big(\frac{1+t}{2}\Big)\mathrm{d}t+m\Big|F'\Big(\frac{b}{m}\Big)\Big|^q\int_0^1\Big|\frac{4}{5}+t\Big|h\Big(\frac{1-t}{2}\Big)\mathrm{d}t\Big]^{\frac{1}{q}}$$
$$+\Big(\frac{7}{10}\Big)^{1-\frac{1}{q}}\Big[\Big|F'(a)\Big|^q\int_0^1\Big|\frac{1}{5}+t\Big|h\Big(\frac{t}{2}\Big)\mathrm{d}t+m\Big|F'\Big(\frac{b}{m}\Big)\Big|^q\int_0^1\Big|\frac{1}{5}+t\Big|h\Big(1-\frac{t}{2}\Big)\mathrm{d}t\Big]^{\frac{1}{q}}\Big\}. \tag{3.1}$$

Proof. Using Lemma 1 and noticing that $\|g\|_{[a,a+\frac{1}{2}\eta(b,a)],\infty},\|g\|_{[a+\frac{1}{2}\eta(b,a),a+\eta(b,a)],\infty}\leq\|g\|_{[a,a+\eta(b,a)],\infty}$, we have that

$$\Big|\Psi_{F,g}(a,b;\eta)\Big|$$
$$\leq\frac{\eta(b,a)}{4}\Big\{\int_0^1\Big|\mathcal{R}_1(t)F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\Big|\mathrm{d}t+\int_0^1\Big|\mathcal{R}_2(t)F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big)\Big|\mathrm{d}t\Big\}$$
$$=\frac{\eta(b,a)}{4}\Big\{\int_0^1\Big|\frac{4}{5}\int_0^1 g\Big(a+\frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u-\int_0^t g\Big(a+\frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u\Big|\Big|F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\Big|\mathrm{d}t$$
$$+\int_0^1\Big|\frac{1}{5}\int_0^1 g\Big(a+\frac{2-u}{2}\eta(b,a)\Big)\mathrm{d}u-\int_0^t g\Big(a+\frac{2-u}{2}\eta(b,a)\Big)\mathrm{d}u\Big|\Big|F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big)\Big|\mathrm{d}t\Big\}$$
$$\leq\frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\Big\{\int_0^1\Big|\frac{4}{5}\int_0^1\mathrm{d}u+\int_0^t\mathrm{d}u\Big|\Big|F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\Big|\mathrm{d}t$$
$$+\int_0^1\Big|\frac{1}{5}\int_0^1\mathrm{d}u+\int_0^t\mathrm{d}u\Big|\Big|F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big)\Big|\mathrm{d}t\Big\}$$
$$=\frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\Big\{\int_0^1\Big|\frac{4}{5}+t\Big|\Big|F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\Big|\mathrm{d}t+\int_0^1\Big|\frac{1}{5}+t\Big|\Big|F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big)\Big|\mathrm{d}t\Big\}. \tag{3.2}$$

Utilizing the power mean inequality, one has

$$\Big|\Psi_{F,g}(a,b;\eta)\Big| \le \frac{b-a}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\bigg\{\Big(\int_0^1\Big|\frac{4}{5}+t\Big|\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big[\int_0^1\Big|\frac{4}{5}+t\Big|\Big|F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\Big|^q\mathrm{d}t\Big]^{\frac{1}{q}}$$
$$+\Big(\int_0^1\Big|\frac{1}{5}+t\Big|\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big[\int_0^1\Big|\frac{1}{5}+t\Big|\Big|F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big)\Big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\bigg\}.$$

Direct computation yields that

$$\int_0^1\Big|\frac{4}{5}+t\Big|\mathrm{d}t=\frac{13}{10} \tag{3.3}$$

and

$$\int_0^1\Big|\frac{1}{5}+t\Big|\mathrm{d}t=\frac{7}{10}. \tag{3.4}$$

Using the (m,h)-preinvexity of $|F'|^q$ on $\mathcal{K}^\circ$, we have

$$\Big|F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\Big|^q \le h\left(\frac{1+t}{2}\right)\Big|F'(a)\Big|^q+h\left(\frac{1-t}{2}\right)m\Big|F'\Big(\frac{b}{m}\Big)\Big|^q \tag{3.5}$$

and

$$\Big|F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big)\Big|^q \le h\Big(\frac{t}{2}\Big)\Big|F'(a)\Big|^q+h\Big(1-\frac{t}{2}\Big)m\Big|F'\Big(\frac{b}{m}\Big)\Big|^q. \tag{3.6}$$

Thus, the proof is completed.

Direct computation yields the subsequent corollary.

Corollary 1. *Consider Theorem 2.*
(i) Taking $h(t)=t$, we get the following inequality for m-preinvex mappings:

$$\Big|\Psi_{F,g}(a,b;\eta)\Big| \le \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\bigg\{\Big(\frac{13}{10}\Big)^{1-\frac{1}{q}}\Big[\frac{61}{60}\Big|F'(a)\Big|^q+\frac{17}{60}m\Big|F'\Big(\frac{b}{m}\Big)\Big|^q\Big]^{\frac{1}{q}}$$
$$+\Big(\frac{7}{10}\Big)^{1-\frac{1}{q}}\Big[\frac{13}{60}\Big|F'(a)\Big|^q+\frac{29}{60}m\Big|F'\Big(\frac{b}{m}\Big)\Big|^q\Big]^{\frac{1}{q}}\bigg\}.$$

(ii) Taking $h(t)=t^s$ with $s\in(0,1)$, we get the coming inequality for (m,s)-preinvex mappings:

$$\Big|\Psi_{F,g}(a,b;\eta)\Big| \le \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\bigg\{\Big(\frac{13}{10}\Big)^{1-\frac{1}{q}}\Big[\Phi_1\Big|F'(a)\Big|^q+\Phi_2 m\Big|F'\Big(\frac{b}{m}\Big)\Big|^q\Big)\Big]^{\frac{1}{q}}$$
$$+\Big(\frac{7}{10}\Big)^{1-\frac{1}{q}}\Big[\Phi_3\Big|F'(a)\Big|^q+\Phi_4 m\Big|F'\Big(\frac{b}{m}\Big)\Big|^q\Big]^{\frac{1}{q}}\bigg\},$$

where

$$\Phi_1=\frac{2^{s+4}+9s\cdot 2^{s+1}-4s-3}{5\cdot 2^s(s+1)(s+2)},$$

$$\Phi_2 = \frac{4s+13}{5\cdot 2^s(s+1)(s+2)},$$

$$\Phi_3 = \frac{6s+7}{5\cdot 2^s(s+1)(s+2)}$$

and

$$\Phi_4 = \frac{3\cdot 2^{s+3} + s2^{s+1} - 6s - 17}{5\cdot 2^s(s+1)(s+2)}.$$

(iii) Taking $h(t) = 1$, we get the succeeding inequality for (m, P)-preinvex mappings:

$$\left|\Psi_{F,g}(a,b;\eta)\right| \le \frac{\eta(b,a)}{2}\|g\|_{[a,a+\eta(b,a)],\infty}\left[\left|F'(a)\right|^q + m\left|F'\Big(\frac{b}{m}\Big)\right|^q\right]^{\frac{1}{q}}.$$

(iv) Taking $h(t) = t(1-t)$, we get the subsequent inequality for (m, tgs)-preinvex mappings:

$$\left|\Psi_{F,g}(a,b;\eta)\right| \le \frac{\eta(b,a)}{40}\|g\|_{[a,a+\eta(b,a)],\infty}\left[13\left(\frac{47}{312}\right)^{\frac{1}{q}} + 7\left(\frac{11}{14}\right)^{\frac{1}{q}}\right]\left[\left|F'(a)\right|^q + m\left|F'\Big(\frac{b}{m}\Big)\right|^q\right]^{\frac{1}{q}}.$$

Another similar result is obtained in the following theorem.

Theorem 3. *If $|F'|^q$ for $q > 1$ is (m, h)-preinvex on $\mathcal{K}^\circ$ with certain fixed $m \in (0, 1]$, then the succeeding inequality is effective:*

$$\begin{aligned}\left|\Psi_{F,g}(a,b;\eta)\right| \le{}& \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\\ &\times\left\{\Theta_1^{1-\frac{1}{q}}\left[\left|F'(a)\right|^q\int_0^1 h\Big(\frac{1+t}{2}\Big)\mathrm{d}t + m\left|F'\Big(\frac{b}{m}\Big)\right|^q\int_0^1 h\Big(\frac{1-t}{2}\Big)\mathrm{d}t\right]^{\frac{1}{q}}\right.\\ &\left.+\Theta_2^{1-\frac{1}{q}}\left[\left|F'(a)\right|^q\int_0^1 h\Big(\frac{t}{2}\Big)\mathrm{d}t + m\left|F'\Big(\frac{b}{m}\Big)\right|^q\int_0^1 h\Big(1-\frac{t}{2}\Big)\mathrm{d}t\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{3.7}$$

where

$$\Theta_1 = \frac{(q-1)(9^{(2q-1)/(q-1)} - 4^{(2q-1)/(q-1)})}{(2q-1)5^{(2q-1)/(q-1)}}$$

and

$$\Theta_2 = \frac{(q-1)(6^{(2q-1)/(q-1)} - 1)}{(2q-1)5^{(2q-1)/(q-1)}}.$$

Proof. Continuing from inequality (3.2) in the proof of Theorem 2 and using the Hölder's integral inequality, one has

$$\begin{aligned}\left|\Psi_{F,g}(a,b;\eta)\right| \le{}& \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\\ &\times\left\{\left(\int_0^1\left|\frac{4}{5}+t\right|^{\frac{q}{q-1}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\int_0^1\left|F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\right|^q\mathrm{d}t\right]^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1\left|\frac{1}{5}+t\right|^{\frac{q}{q-1}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\int_0^1\left|F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big)\right|^q\mathrm{d}t\right]^{\frac{1}{q}}\right\}.\end{aligned} \tag{3.8}$$

Utilizing (3.5) and (3.6) in (3.8) we obtain the desired inequality in (3.7), since

$$\int_0^1 \left|\frac{4}{5}+t\right|^{\frac{q}{q-1}} \mathrm{d}t = \frac{(q-1)(9^{(2q-1)/(q-1)} - 4^{(2q-1)/(q-1)})}{(2q-1)5^{(2q-1)/(q-1)}}.$$

and

$$\int_0^1 \left|\frac{1}{5}+t\right|^{\frac{q}{q-1}} \mathrm{d}t = \frac{(q-1)(6^{(2q-1)/(q-1)} - 1)}{(2q-1)5^{(2q-1)/(q-1)}}.$$

Thus, the proof is completed.

By elementary calculation, it is easy to obtain the coming result.

Corollary 2. *Considered Theorem 3.*
(i) Taking $h(t) = t$, we get the subsequent inequality for m-preinvex mappings:

$$\begin{aligned}
\left|\Psi_{F,g}(a,b;\eta)\right| &\leq \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty} \\
&\quad \times \left\{\Theta_1^{1-\frac{1}{q}}\left[\frac{3}{4}\left|F'(a)\right|^q + \frac{1}{4}m\left|F'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} + \Theta_2^{1-\frac{1}{q}}\left[\frac{1}{4}\left|F'(a)\right|^q + \frac{3}{4}m\left|F'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\} \\
&\leq \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\Theta_1^{1-\frac{1}{q}} \\
&\quad \times \left\{\left[\frac{3}{4}\left|F'(a)\right|^q + \frac{1}{4}m\left|F'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} + \left[\frac{1}{4}\left|F'(a)\right|^q + \frac{3}{4}m\left|F'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

(ii) Taking $h(t) = t^s$ with $s \in (0,1)$, we get the succeeding inequality for (m,s)-preinvex mappings:

$$\begin{aligned}
\left|\Psi_{F,g}(a,b;\eta)\right| \leq \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}&\left\{\Theta_1^{1-\frac{1}{q}}\left[\frac{2^{s+1}-1}{2^s(s+1)}\left|F'(a)\right|^q + \frac{1}{2^s(s+1)}m\left|F'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left. + \Theta_2^{1-\frac{1}{q}}\left[\frac{1}{2^s(s+1)}\left|F'(a)\right|^q + \frac{2^{s+1}-1}{2^s(s+1)}m\left|F'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

(iii) Taking $h(t) = 1$, we get the coming inequality for (m,P)-preinvex mappings:

$$\left|\Psi_{F,g}(a,b;\eta)\right| \leq \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\left[\Theta_1^{1-\frac{1}{q}} + \Theta_2^{1-\frac{1}{q}}\right]\left[\left|F'(a)\right|^q + m\left|F'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

(iv) Taking $h(t) = t(1-t)$, we get the following inequality for (m,tgs)-preinvex mappings:

$$\left|\Psi_{F,g}(a,b;\eta)\right| \leq \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[\Theta_1^{1-\frac{1}{q}} + \Theta_2^{1-\frac{1}{q}}\right]\left[\left|F'(a)\right|^q + m\left|F'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

A different approach leads to the following result.

Theorem 4. *If $|F'|$ is (m,h)-preinvex on $\mathcal{K}^{\circ}$ with certain fixed $m \in (0,1]$, then the succeeding inequality*

$$\begin{aligned}
&\left|\Psi_{F,g}(a,b;\eta)\right| \\
&\leq \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty} \\
&\times\left\{\left(\frac{13}{10}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\frac{4}{5}+t\right|h^q\left(\frac{1+t}{2}\right)\mathrm{d}t\right)^{\frac{1}{q}}\left|F'(a)\right|+\left(\int_0^1\left|\frac{4}{5}+t\right|h^q\left(\frac{1-t}{2}\right)\mathrm{d}t\right)^{\frac{1}{q}}m\left|F'\left(\frac{b}{m}\right)\right|\right]\right. \\
&\left.+\left(\frac{7}{10}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\frac{1}{5}+t\right|h^q\left(\frac{t}{2}\right)\mathrm{d}t\right)^{\frac{1}{q}}\left|F'(a)\right|+\left(\int_0^1\left|\frac{1}{5}+t\right|h^q\left(1-\frac{t}{2}\right)\mathrm{d}t\right)^{\frac{1}{q}}m\left|F'\left(\frac{b}{m}\right)\right|\right]\right\}
\end{aligned} \tag{3.9}$$

holds for $q>1$.

Proof. Utilizing Lemma 1 and the (m,h)-preinvexity of $|F'|$ on $\mathcal{K}^{\circ}$, one has

$$\begin{aligned}
&\left|\Psi_{F,g}(a,b;\eta)\right| \\
&\leq \frac{\eta(b,a)}{4}\left\{\int_0^1\left|\frac{4}{5}\int_0^1 g\left(a+\frac{1-u}{2}\eta(b,a)\right)\mathrm{d}u-\int_0^t g\left(a+\frac{1-u}{2}\eta(b,a)\right)\mathrm{d}u\right|\left|F'\left(a+\frac{1-t}{2}\eta(b,a)\right)\right|\mathrm{d}t\right. \\
&\left.+\int_0^1\left|\frac{1}{5}\int_0^1 g\left(a+\frac{2-u}{2}\eta(b,a)\right)\mathrm{d}u-\int_0^t g\left(a+\frac{2-u}{2}\eta(b,a)\right)\mathrm{d}u\right|\left|F'\left(a+\frac{2-t}{2}\eta(b,a)\right)\right|\mathrm{d}t\right\} \\
&\leq \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\left\{\int_0^1\left|\frac{4}{5}+t\right|\left|F'\left(a+\frac{1-t}{2}\eta(b,a)\right)\right|\mathrm{d}t+\int_0^1\left|\frac{1}{5}+t\right|\left|F'\left(a+\frac{2-t}{2}\eta(b,a)\right)\right|\mathrm{d}t\right\} \\
&\leq \frac{\eta(b,a)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}\left\{\int_0^1\left|\frac{4}{5}+t\right|\left[h\left(\frac{1+t}{2}\right)\left|F'(a)\right|+h\left(\frac{1-t}{2}\right)m\left|F'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t\right. \\
&\left.+\int_0^1\left|\frac{1}{5}+t\right|\left[h\left(\frac{t}{2}\right)\left|F'(a)\right|+h\left(1-\frac{t}{2}\right)m\left|F'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t\right\}.
\end{aligned} \tag{3.10}$$

Using the power mean inequality, one has

$$\begin{aligned}
&\int_0^1\left|\frac{4}{5}+t\right|\left[h\left(\frac{1+t}{2}\right)\left|F'(a)\right|+h\left(\frac{1-t}{2}\right)m\left|F'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t \\
&\leq\left(\int_0^1\left|\frac{4}{5}+t\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\frac{4}{5}+t\right|h^q\left(\frac{1+t}{2}\right)\mathrm{d}t\right)^{\frac{1}{q}}\left|F'(a)\right|+\left(\int_0^1\left|\frac{4}{5}+t\right|h^q\left(\frac{1-t}{2}\right)\mathrm{d}t\right)^{\frac{1}{q}}m\left|F'\left(\frac{b}{m}\right)\right|\right]
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\int_0^1\left|\frac{1}{5}+t\right|\left[h\left(\frac{t}{2}\right)\left|F'(a)\right|+h\left(1-\frac{t}{2}\right)m\left|F'\left(\frac{b}{m}\right)\right|\right]\mathrm{d}t \\
&\leq\left(\int_0^1\left|\frac{1}{5}+t\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\frac{1}{5}+t\right|h^q\left(\frac{t}{2}\right)\mathrm{d}t\right)^{\frac{1}{q}}\left|F'(a)\right|+\left(\int_0^1\left|\frac{1}{5}+t\right|h^q\left(1-\frac{t}{2}\right)\mathrm{d}t\right)^{\frac{1}{q}}m\left|F'\left(\frac{b}{m}\right)\right|\right].
\end{aligned} \tag{3.12}$$

Making use of (3.3), (3.4), (3.11) and (3.12) in (3.10), we deduce the desired result inequality in (3.9). Thus, the proof is completed.

Remark 2. *By taking different mapping for h in Theorems 4, we also establish a series of Simpson type inequalities similar to Corollary 1 and Corollary 2.*

To prove the following result, we deal with the boundedness of F'.

Theorem 5. *Let F' be integrable on $[a, a+\eta(b,a)]$ and there exist constants $\gamma < \delta$ such that $-\infty < \gamma \le F'(x) \le \delta < +\infty$ for all $x \in [a, a+\eta(b,a)]$. Then*

$$\left|\Psi_{F,g}(a,b;\eta) - \frac{\eta(b,a)(\gamma+\delta)}{8}\Big[\int_0^1 \mathcal{R}_1(t)\mathrm{d}t + \int_0^1 \mathcal{R}_2(t)\mathrm{d}t\Big]\right| \le \frac{\eta(b,a)(\delta-\gamma)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}, \tag{3.13}$$

where $\mathcal{R}_1(t)$ and $\mathcal{R}_2(t)$ are defined in Lemma 1.

Proof. From Lemma 1, one has

$$\begin{aligned}
\Psi_{F,g}(a,b;\eta) = {} & \frac{\eta(b,a)}{4}\bigg\{\int_0^1 \mathcal{R}_1(t)\Big[F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big) - \frac{\gamma+\delta}{2} + \frac{\gamma+\delta}{2}\Big]\mathrm{d}t \\
& + \int_0^1 \mathcal{R}_2(t)\Big[F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big) - \frac{\gamma+\delta}{2} + \frac{\gamma+\delta}{2}\Big]\mathrm{d}t\bigg\} \\
= {} & \frac{\eta(b,a)}{4}\bigg\{\int_0^1 \mathcal{R}_1(t)\Big[F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big) - \frac{\gamma+\delta}{2}\Big]\mathrm{d}t \\
& + \int_0^1 \mathcal{R}_2(t)\Big[F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big) - \frac{\gamma+\delta}{2}\Big]\mathrm{d}t\bigg\} \\
& + \frac{\eta(b,a)(\gamma+\delta)}{8}\Big[\int_0^1 \mathcal{R}_1(t)\mathrm{d}t + \int_0^1 \mathcal{R}_2(t)\mathrm{d}t\Big].
\end{aligned}$$

For simplicity we define the quantity

$$\mathcal{T} := \Psi_{F,g}(a,b;\eta) - \frac{\eta(b,a)(\gamma+\delta)}{8}\Big[\int_0^1 \mathcal{R}_1(t)\mathrm{d}t + \int_0^1 \mathcal{R}_2(t)\mathrm{d}t\Big].$$

Thus

$$\begin{aligned}
\mathcal{T} = {} & \frac{\eta(b,a)}{4}\bigg\{\int_0^1 \mathcal{R}_1(t)\Big[F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big) - \frac{\gamma+\delta}{2}\Big]\mathrm{d}t \\
& + \int_0^1 \mathcal{R}_2(t)\Big[F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big) - \frac{\gamma+\delta}{2}\Big]\mathrm{d}t\bigg\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\mathcal{T}| \le {} & \frac{\eta(b,a)}{4}\bigg\{\int_0^1 |\mathcal{R}_1(t)|\Big|F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big) - \frac{\gamma+\delta}{2}\Big|\mathrm{d}t \\
& + \int_0^1 |\mathcal{R}_2(t)|\Big|F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big) - \frac{\gamma+\delta}{2}\Big|\mathrm{d}t\bigg\} \\
\le {} & \frac{\eta(b,a)(\delta-\gamma)}{8}\bigg\{\int_0^1 |\mathcal{R}_1(t)|\mathrm{d}t + \int_0^1 |\mathcal{R}_2(t)|\mathrm{d}t\bigg\}.
\end{aligned}$$

Since F' satisfies $-\infty < \gamma \leq F'(x) \leq \delta < +\infty$, one has

$$\gamma - \frac{\gamma+\delta}{2} \leq F'(x) - \frac{\gamma+\delta}{2} \leq \delta - \frac{\gamma+\delta}{2},$$

which implies that

$$\left|F'(x) - \frac{\gamma+\delta}{2}\right| \leq \frac{\delta-\gamma}{2}.$$

Also, g is symmetric to $a + \frac{1}{2}\eta(b,a)$, we obtain

$$\begin{aligned}
|\mathcal{T}| \leq& \frac{\eta(b,a)(\delta-\gamma)}{8}\bigg\{\int_0^1 \left|\frac{4}{5}\int_0^1 g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u - \int_0^t g\Big(a + \frac{1-u}{2}\eta(b,a)\Big)\mathrm{d}u\right|\mathrm{d}t \\
&+ \int_0^1 \left|\frac{1}{5}\int_0^1 g\Big(a + \frac{2-u}{2}\eta(b,a)\Big)\mathrm{d}u - \int_0^t g\Big(a + \frac{2-u}{2}\eta(b,a)\Big)\mathrm{d}u\right|\mathrm{d}t\bigg\} \\
\leq& \frac{\eta(b,a)(\delta-\gamma)}{8}\|g\|_{[a,a+\eta(b,a)],\infty}\bigg\{\int_0^1 \left|\frac{4}{5}\int_0^1 \mathrm{d}u + \int_0^t \mathrm{d}u\right|\mathrm{d}t + \int_0^1 \left|\frac{1}{5}\int_0^1 \mathrm{d}u + \int_0^t \mathrm{d}u\right|\mathrm{d}t\bigg\} \\
=& \frac{\eta(b,a)(\delta-\gamma)}{4}\|g\|_{[a,a+\eta(b,a)],\infty}.
\end{aligned}$$

This ends the proof.

If the considered mapping F' satisfies a Lipschitz condition, then one has the coming result.

Theorem 6. *Assume that F' satisfies a Lipschitz condition for some $\mathcal{L} > 0$, then the following inequality holds:*

$$\left|\Psi_{F,g}(a,b;\eta) - \frac{\eta(b,a)}{4}\left[F'(a)\int_0^1 \mathcal{R}_1(t)\mathrm{d}t + F'(a+\eta(b,a))\int_0^1 \mathcal{R}_2(t)\mathrm{d}t\right]\right| \leq \frac{\eta^2(b,a)\mathcal{L}}{8}\|g\|_{[a,a+\eta(b,a)],\infty}, \tag{3.14}$$

where $\mathcal{R}_1(t)$ and $\mathcal{R}_2(t)$ are defined in Lemma 1.

Proof. Using Lemma 1, one has

$$\begin{aligned}
\Psi_{F,g}(a,b;\eta) =& \frac{\eta(b,a)}{4}\bigg\{\int_0^1 \mathcal{R}_1(t)\Big[F'\Big(a + \frac{1-t}{2}\eta(b,a)\Big) - F'(a) + F'(a)\Big]\mathrm{d}t \\
&+ \int_0^1 \mathcal{R}_2(t)\Big[F'\Big(a + \frac{2-t}{2}\eta(b,a)\Big) - F'(a+\eta(b,a)) + F'(a+\eta(b,a))\Big]\mathrm{d}t\bigg\} \\
=& \frac{\eta(b,a)}{4}\bigg\{\int_0^1 \mathcal{R}_1(t)\Big[F'\Big(a + \frac{1-t}{2}\eta(b,a)\Big) - F'(a)\Big]\mathrm{d}t \\
&+ \int_0^1 \mathcal{R}_2(t)\Big[F'\Big(a + \frac{2-t}{2}\eta(b,a)\Big) - F'(a+\eta(b,a))\Big]\mathrm{d}t\bigg\} \\
&+ \frac{\eta(b,a)}{4}\Big[F'(a)\int_0^1 \mathcal{R}_1(t)\mathrm{d}t + F'(a+\eta(b,a))\int_0^1 \mathcal{R}_2(t)\mathrm{d}t\Big].
\end{aligned}$$

For simplicity we define the quantity

$$\mathcal{G} := \Psi_{F,g}(a,b;\eta) - \frac{\eta(b,a)}{4}\bigg[\int_0^1 \mathcal{R}_1(t)\mathrm{d}tF'(a) + \int_0^1 \mathcal{R}_2(t)\mathrm{d}tF'\big(a+\eta(b,a)\big)\bigg].$$

Thus,

$$\mathcal{G} = \frac{\eta(b,a)}{4}\bigg\{\int_0^1 \mathcal{R}_1(t)\Big[F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big) - F'(a)\Big]\mathrm{d}t + \int_0^1 \mathcal{R}_2(t)\Big[F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big) - F'\big(a+\eta(b,a)\big)\Big]\mathrm{d}t\bigg\}.$$

Since F' satisfies Lipschitz conditions for certain $\mathcal{L} > 0$, one has

$$\Big|F'\Big(a+\frac{1-t}{2}\eta(b,a)\Big) - F'(a)\Big| \le \mathcal{L}\Big|a+\frac{1-t}{2}\eta(b,a) - a\Big| = \mathcal{L}|1-t|\frac{\eta(b,a)}{2}$$

and

$$\Big|F'\Big(a+\frac{2-t}{2}\eta(b,a)\Big) - F'\big(a+\eta(b,a)\big)\Big| \le \mathcal{L}\Big|a+\frac{2-t}{2}\eta(b,a) - a - \eta(b,a)\Big| = \mathcal{L}|t|\frac{\eta(b,a)}{2}.$$

As a result, we have

$$\big|\mathcal{G}\big| \le \frac{\eta^2(b,a)\mathcal{L}}{8}\bigg\{\int_0^1 (1-t)\big|\mathcal{R}_1(t)\big|\mathrm{d}t + \int_0^1 t\big|\mathcal{R}_2(t)\big|\mathrm{d}t\bigg\}.$$

Also, since g is symmetric to $a + \frac{1}{2}\eta(b,a)$, we get

$$\begin{aligned}\big|\mathcal{G}\big| &\le \frac{\eta^2(b,a)\mathcal{L}}{8}\|g\|_{[a,a+\eta(b,a)],\infty}\bigg\{\int_0^1 (1-t)\Big|\frac{4}{5}+t\Big|\mathrm{d}t + \int_0^1 t\Big|\frac{1}{5}+t\Big|\mathrm{d}t\bigg\}\\ &= \frac{\eta^2(b,a)\mathcal{L}}{8}\|g\|_{[a,a+\eta(b,a)],\infty}.\end{aligned}$$

This completes the proof.

4. Applications

4.1. Weighted Simpson type quadrature formula

Let X be a partition of the interval $[a,b]$, i.e., $X : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and let $l_j := x_{j+1} - x_j$ for $j = 0, 1, \cdots, n-1$. Let us consider the quadrature formula

$$\int_a^b F(x)g(x)\mathrm{d}x = \mathcal{A}_S(F,g,X) + \mathcal{R}_S(F,g,X),$$

where

$$\mathcal{A}_S(F,g,X) = \sum_{j=0}^{n-1}\frac{1}{5}\bigg[\frac{F(x_j)+F(x_{j+1})}{2} + 4F\Big(\frac{x_j+x_{j+1}}{2}\Big)\bigg]\int_{x_j}^{x_{j+1}} g(x)\mathrm{d}x$$

for the weighted Simpson type formula and $\mathcal{R}_S(F,g,X)$ denotes the related approximation error of the integral $\int_a^b F(x)g(x)\mathrm{d}x$.

Now, we derive an error estimate related to the weighted Simpson type formula.

Proposition 1. *Suppose that all hypotheses of Theorem 2 are satisfied and $q = 1$, $\eta(b, a) = b - a$ with $m = 1$. Then the following weighted Simpson type error estimate satisfies*

$$\left|\mathcal{R}_S(F, g, X)\right| \leq \sum_{j=0}^{n-1} \frac{l_j^2}{4}\left[\frac{37}{30}\left|F'\left(x_j\right)\right| + \frac{23}{30}\left|F'\left(x_{j+1}\right)\right|\right]\|g\|_{[x_j,x_{j+1}],\infty},$$

where $l_j := x_{j+1} - x_j$ for $j = 0, 1, \cdots, n-1$.

Proof. Applying the inequality (i) in Corollary 1 with $q = 1$ on subinterval $[x_j, x_{j+1}]$ $(j = 0, 1, \cdots, n-1)$ of the partition X, we deduce that

$$\begin{aligned}&\left|\frac{1}{10l_j}\left[F(x_j) + 8F\Big(\frac{x_j + x_{j+1}}{2}\Big) + F(x_{j+1})\right]\int_{x_j}^{x_{j+1}} g(x)\mathrm{d}x - \frac{1}{l_j}\int_{x_j}^{x_{j+1}} F(x)g(x)\mathrm{d}x\right|\\&\leq \frac{l_j}{4}\|g\|_{[x_j,x_{j+1}],\infty}\left[\frac{37}{30}\left|F'\left(x_j\right)\right| + \frac{23}{30}\left|F'\left(x_{j+1}\right)\right|\right].\end{aligned}$$

Summing over j from 0 to $n-1$ and using the triangle inequality, we have that

$$\begin{aligned}&\left|\mathcal{R}_S(F, g, X)\right| = \left|\mathcal{A}_S(F, g, X) - \int_a^b F(x)g(x)\mathrm{d}x\right|\\&= \sum_{j=0}^{n-1}\left|\frac{1}{10}\left[F(x_j) + 8F\Big(\frac{x_j + x_{j+1}}{2}\Big) + F(x_{j+1})\right]\int_{x_j}^{x_{j+1}} g(x)\mathrm{d}x - \int_{x_j}^{x_{j+1}} F(x)g(x)\mathrm{d}x\right|\\&\leq \sum_{j=0}^{n-1} \frac{l_j^2}{4}\left[\frac{37}{30}\left|F'\left(x_j\right)\right| + \frac{23}{30}\left|F'\left(x_{j+1}\right)\right|\right]\|g\|_{[x_j,x_{j+1}],\infty}.\end{aligned}$$

Thus, the proof is completed. □

4.2. Random variable

Let $0 < a < b$, $r \in \mathbb{R}$, and let X be a continuous random variable having the continuous probability density mapping $g : [a, b] \to [0, 1]$, which is symmetric with respect to $\frac{a+b}{2}$.

Also, the r-moment is defined by

$$E_r(X) := \int_a^b x^r g(x)\mathrm{d}x,$$

which is assumed to be finite.

Now we have the following results.

Proposition 2. *Suppose that all hypotheses of Theorem 4 are satisfied and $\eta(b, a) = b - a$ with $m = 1$. Then we have the following ineqaulity*

$$\begin{aligned}&\left|\frac{1}{10}\left[a^r + 8\Big(\frac{a + b}{2}\Big)^r + b^r\right] - E_r(X)\right|\\&\leq \frac{|r|(b - a)^2}{4}\|g\|_{[a,b],\infty}\left\{\left(\frac{13}{10}\right)^{1-\frac{1}{q}}\left[\Delta_1^{\frac{1}{q}} a^{r-1} + \Delta_2^{\frac{1}{q}} b^{r-1}\right] + \left(\frac{7}{10}\right)^{1-\frac{1}{q}}\left[\Delta_3^{\frac{1}{q}} a^{r-1} + \Delta_4^{\frac{1}{q}} b^{r-1}\right]\right\},\end{aligned}$$

where

$$\Delta_1 = \frac{2^{q\tau+4} + 9q\tau \cdot 2^{q\tau+1} - 4q\tau - 3}{5 \cdot 2^{q\tau}(q\tau+1)(q\tau+2)},$$

$$\Delta_2 = \frac{4q\tau + 13}{5 \cdot 2^{q\tau}(q\tau+1)(q\tau+2)},$$

$$\Delta_3 = \frac{6q\tau + 7}{5 \cdot 2^{q\tau}(q\tau+1)(q\tau+2)}$$

and

$$\Delta_4 = \frac{3 \cdot 2^{q\tau+3} + q\tau 2^{q\tau+1} - 6q\tau - 17}{2^{q\tau}5^{q\tau+2}(q\tau+1)(q\tau+2)}.$$

Proof. Let $F(x) = \frac{x^r}{r}$, $x \in [a,b]$ with $r \in (-\infty, 0) \cup (0,1] \cup [2, +\infty)$, and let $h(t) = t^\tau$ for $\tau \in (-\infty, -1) \cup (-1, 1]$. Then $|F'|$ is h-convex (see Example 7 in [23]). Applying $F(x) = \frac{x^r}{r}$, $x \in [a,b]$ to the inequality (3.9), we have that

$$\begin{aligned}
&\left|\frac{1}{10}\left[a^r + 8\Big(\frac{a+b}{2}\Big)^r + b^r\right] - E_r(X)\right| \\
&\leq \frac{|r|(b-a)^2}{4}\|g\|_{[a,b],\infty} \\
&\quad \times \left\{\Big(\frac{13}{10}\Big)^{1-\frac{1}{q}}\left[\Big(\int_0^1 \Big|\frac{4}{5}+t\Big|\Big(\frac{1+t}{2}\Big)^{q\tau}\mathrm{d}t\Big)^{\frac{1}{q}} a^{r-1} + \Big(\int_0^1 \Big|\frac{4}{5}+t\Big|\Big(\frac{1-t}{2}\Big)^{q\tau}\mathrm{d}t\Big)^{\frac{1}{q}} b^{r-1}\right]\right. \\
&\quad \left. + \Big(\frac{7}{10}\Big)^{1-\frac{1}{q}}\left[\Big(\int_0^1 \Big|\frac{1}{5}+t\Big|\Big(\frac{t}{2}\Big)^{q\tau}\mathrm{d}t\Big)^{\frac{1}{q}} a^{r-1} + \Big(\int_0^1 \Big|\frac{1}{5}+t\Big|\Big(1-\frac{t}{2}\Big)^{q\tau}\mathrm{d}t\Big)^{\frac{1}{q}} b^{r-1}\right]\right\}.
\end{aligned}$$

The desired inequality follows from the above by noting that

$$\int_0^1 \Big|\frac{4}{5}+t\Big|\Big(\frac{1+t}{2}\Big)^{q\tau}\mathrm{d}t = \frac{2^{q\tau+4} + 9q\tau \cdot 2^{q\tau+1} - 4q\tau - 3}{5 \cdot 2^{q\tau}(q\tau+1)(q\tau+2)},$$

$$\int_0^1 \Big|\frac{4}{5}+t\Big|\Big(\frac{1-t}{2}\Big)^{q\tau}\mathrm{d}t = \frac{4q\tau + 13}{5 \cdot 2^{q\tau}(q\tau+1)(q\tau+2)},$$

$$\int_0^1 \Big|\frac{1}{5}+t\Big|\Big(\frac{t}{2}\Big)^{q\tau}\mathrm{d}t = \frac{6q\tau + 7}{5 \cdot 2^{q\tau}(q\tau+1)(q\tau+2)}$$

and

$$\int_0^1 \Big|\frac{1}{5}+t\Big|\Big(1-\frac{t}{2}\Big)^{q\tau}\mathrm{d}t = \frac{3 \cdot 2^{q\tau+3} + q\tau 2^{q\tau+1} - 6q\tau - 17}{5 \cdot 2^{q\tau}(q\tau+1)(q\tau+2)}.$$

Thus, the proof is completed. □

4.3. F-divergence measures

Among the various applications of probability theory, one of the primary topics is to find an appropriate measure of the distance between any two probability distributions. The set ϕ and the σ-finite measure μ are given, and the set of all probability densities on μ to be defined on $\Lambda := \{\varrho|\varrho : \phi \to \mathbb{R}, \varrho(x) > 0, \int_\phi \varrho(x)\mathrm{d}\mu(x) = 1\}$ are considered.

Let $F : (0, \infty) \to \mathbb{R}$ be given mapping and consider $D_F(\varrho, \omega)$ be defined by

$$D_F(\varrho, \omega) := \int_\phi \varrho(x) F\Big[\frac{\omega(x)}{\varrho(x)}\Big]\mathrm{d}\mu(x), \quad \varrho, \omega \in \Lambda. \tag{4.1}$$

If F is convex, then (4.1) is called as the Csisźar F-divergence.

In [21], Shioya and Da-te proposed the Hermite–Hadamard (HH) divergence

$$D^F_{HH}(\varrho, \omega) = \int_\phi \varrho(x) \frac{\int_1^{\frac{\omega(x)}{\varrho(x)}} F(t)\mathrm{d}t}{\frac{\omega(x)}{\varrho(x)} - 1}\mathrm{d}\mu(x), \quad \varrho, \omega \in \Lambda, \tag{4.2}$$

where F is convex on $(0, \infty)$ with $F(1) = 0$. In the same article, they also introduced the property of HH divergence that $D^F_{HH}(\varrho; \omega) \geq 0$ with the equality holds if and only if $\varrho = \omega$. Now, we have the following results.

Proposition 3. *Let all hypotheses of Theorem 3 and $g(x) = 1$, $m = 1$, $q = 2$ with $F(1) = 0$. If ϱ, $\omega \in \Lambda$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{10}\Big[D_F(\varrho, \omega) + 8\int_\phi \varrho(x) F\Big(\frac{\varrho(x) + \omega(x)}{2\varrho(x)}\Big)\mathrm{d}\mu(x)\Big] - D^F_{HH}(\varrho, \omega)\right| \\
&\leq \Big(\frac{133}{75}\Big)^{\frac{1}{2}} \Bigg\{\Big[\frac{3}{4}\big|F'(1)\big|^2 \int_\Phi \frac{(\omega(x) - \varrho(x))^2}{\varrho(x)}\mathrm{d}\mu(x) \\
&\quad + \frac{1}{4}\int_\Phi \frac{(\omega(x) - \varrho(x))^2}{\varrho(x)}\Big|F'\Big(\frac{\omega(x)}{\varrho(x)}\Big)\Big|^2\mathrm{d}\mu(x)\Big]^{\frac{1}{2}} \\
&\quad + \Big[\frac{1}{4}\big|F'(1)\big|^2 \int_\Phi \frac{(\omega(x) - \varrho(x))^2}{\varrho(x)}\mathrm{d}\mu(x) \\
&\quad + \frac{3}{4}\int_\Phi \frac{(\omega(x) - \varrho(x))^2}{\varrho(x)}\Big|F'\Big(\frac{\omega(x)}{\varrho(x)}\Big)\Big|^2\mathrm{d}\mu(x)\Big]^{\frac{1}{2}}\Bigg\}.
\end{aligned} \tag{4.3}$$

Proof. Let $\Phi_1 = \{x \in \phi : \omega(x) > \varrho(x)\}$, $\Phi_2 = \{x \in \phi : \omega(x) < \varrho(x)\}$ and $\Phi_3 = \{x \in \phi : \omega(x) = \varrho(x)\}$, and let $g(x) = 1$, $\eta(b, a) = b - a$, $m = 1$ with $q = 2$.

Obviously, if $x \in \Phi_3$, then inequality holds in (4.3). Now if $x \in \Phi_1$, applying the second inequality in (i) of Corollary 2 for $a = 1$ and $b = \frac{\omega(x)}{\varrho(x)}$, multiplying both hand sides of the obtained results by $\varrho(x)$

and then integrating on Φ_1, then we get that

$$\begin{aligned}
&\left|\frac{1}{10}\left[\int_{\Phi_1}\varrho(x)F\Big(\frac{\omega(x)}{\varrho(x)}\Big)\mathrm{d}\mu(x)+8\int_{\Phi_1}\varrho(x)F\Big(\frac{\varrho(x)+\omega(x)}{2\varrho(x)}\Big)\mathrm{d}\mu(x)\right]\right.\\
&\quad\left.-\int_{\Phi_1}\varrho(x)\frac{\int_1^{\frac{\omega(x)}{\varrho(x)}}F(t)\mathrm{d}t}{\frac{\omega(x)}{\varrho(x)}-1}\mathrm{d}\mu(x)\right|\\
&\le\Big(\frac{133}{75}\Big)^{\frac{1}{2}}\left\{\left[\frac{3}{4}\left|F'(1)\right|^2\int_{\Phi_1}\frac{(\omega(x)-\varrho(x))^2}{\varrho(x)}\mathrm{d}\mu(x)\right.\right.\\
&\quad\left.+\frac{1}{4}\int_{\Phi_1}\frac{(\omega(x)-\varrho(x))^2}{\varrho(x)}\left|F'\Big(\frac{\omega(x)}{\varrho(x)}\Big)\right|^2\mathrm{d}\mu(x)\right]^{\frac{1}{2}}\\
&\quad+\left[\frac{1}{4}\left|F'(1)\right|^2\int_{\Phi_1}\frac{(\omega(x)-\varrho(x))^2}{\varrho(x)}\mathrm{d}\mu(x)\right.\\
&\quad\left.\left.+\frac{3}{4}\int_{\Phi_1}\frac{(\omega(x)-\varrho(x))^2}{\varrho(x)}\left|F'\Big(\frac{\omega(x)}{\varrho(x)}\Big)\right|^2\mathrm{d}\mu(x)\right]^{\frac{1}{2}}\right\}.
\end{aligned}\tag{4.4}$$

Similarly if $x\in\Phi_2$, applying the second inequality in (i) of Corollary 2 for $a=\frac{\omega(x)}{\varrho(x)}$, $b=1$, multiplying both hand sides of the obtained results by $\varrho(x)$ and then integrating on Φ_2, then we get that

$$\begin{aligned}
&\left|\frac{1}{10}\left[\int_{\Phi_2}\varrho(x)F\Big(\frac{\omega(x)}{\varrho(x)}\Big)\mathrm{d}\mu(x)+8\int_{\Phi_2}\varrho(x)F\Big(\frac{\varrho(x)+\omega(x)}{2\varrho(x)}\Big)\mathrm{d}\mu(x)\right]\right.\\
&\quad\left.-\int_{\Phi_2}\varrho(x)\frac{\int_1^{\frac{\omega(x)}{\varrho(x)}}F(t)\mathrm{d}t}{\frac{\omega(x)}{\varrho(x)}-1}\mathrm{d}\mu(x)\right|\\
&\le\Big(\frac{133}{75}\Big)^{\frac{1}{2}}\left\{\left[\frac{1}{4}\left|F'(1)\right|^2\int_{\Phi_2}\frac{(\omega(x)-\varrho(x))^2}{\varrho(x)}\mathrm{d}\mu(x)\right.\right.\\
&\quad\left.+\frac{3}{4}\int_{\Phi_2}\frac{(\omega(x)-\varrho(x))^2}{\varrho(x)}\left|F'\Big(\frac{\omega(x)}{\varrho(x)}\Big)\right|^2\mathrm{d}\mu(x)\right]^{\frac{1}{2}}\\
&\quad+\left[\frac{3}{4}\left|F'(1)\right|^2\int_{\Phi_2}\frac{(\omega(x)-\varrho(x))^2}{\varrho(x)}\mathrm{d}\mu(x)\right.\\
&\quad\left.\left.+\frac{1}{4}\int_{\Phi_2}\frac{(\omega(x)-\varrho(x))^2}{\varrho(x)}\left|F'\Big(\frac{\omega(x)}{\varrho(x)}\Big)\right|^2\mathrm{d}\mu(x)\right]^{\frac{1}{2}}\right\}.
\end{aligned}\tag{4.5}$$

Combining (4.4) and (4.5) and then using triangular inequality, we obtain the desired result.

5. Conclusion

In this study, we establish a weighted Simpson type identity and obtain some bounds involving the weighted Simpson type inequality for the first-order differentiable mappings. As applications, we apply the investigated results to an error estimate for weighted Simpson type quadrature formula, continuous random variables and F-divergence measures, respectively. With these ideas and techniques stated in this study, it is likely to discover further estimations of other form weighted integral inequalities which involve other related classes of mappings.

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Conflict of interest

No conflict of interest was declared by the authors.

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