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Research article

Generalized inequalities for integral operators via several kinds of convex functions

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Abstract: This paper investigates the bounds of an integral operator for several kinds of convex functions. By applying definition of $(h-m)$-convex function upper bounds of left sided (1.12) and right sided (1.13) integral operators are formulated which particularly provide upper bounds of various known conformable and fractional integrals. Further a modulus inequality is investigated for differentiable functions whose derivative in absolute value are $(h-m)$-convex. Moreover a generalized Hadamard inequality for $(h-m)$-convex functions is proved by utilizing these operators. Also all the results are obtained for (α, m)-convex functions. Finally some applications of proved results are discussed.

Keywords: $(h-m)$-convex function; (α, m)-convex function; integral operators; fractional integral operators; conformable integral operators
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1. Introduction

Convex functions play a vital role in Mathematical analysis, Mathematical statistics and Optimization theory. These functions produce an elegant theory of convex analysis, see ([1–3]).

Definition 1. *A function $f : [a, b] \to \mathbb{R}$ is said to be convex if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$. If inequality (1.1) *holds in revers order, then f will be concave on $[a, b]$.*

Convex functions have been generalized theoretically extensively; these generalizations include m-convex function, n-convex function, r-convex function, h-convex function, $(h-m)$-convex function, (α, m)-convex function, s-convex function and many others. We are interested in using $(h-m)$-convex function [4] and (α, m)-convex function [5].

Definition 2. *Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : [0,b] \to \mathbb{R}$ is called $(h-m)$-convex function, if f is non-negative satisfying*

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + mh(1-\alpha)f(y), \tag{1.2}$$

for all $x, y \in [0,b]$, $m \in [0,1]$, $\alpha \in (0,1)$.

By choosing suitable values for function h and m, the above definition produces the functions defined on non-negative real line which are comprised in the following remark:

Remark 1. *(i) If $m = 1$, then h-convex function is obtained.*
(ii) If $h(x) = x$, then m-convex function is obtained.
(iii) If $h(x) = x$ and $m = 1$, then convex function is obtained.
(iv) If $h(x) = 1$ and $m = 1$, then p-function is obtained.
(v) If $h(x) = x^s$ and $m = 1$, then s-convex function is obtained.
(vi) If $h(x) = \dfrac{1}{x}$ and $m = 1$, then Godunova-Levin function is obtained.
(vii) If $h(x) = \dfrac{1}{x^s}$ and $m = 1$, then s-Godunova-Levin function of second kind is obtained.

For some recent citations and utilizations of $(h-m)$-convex functions one can see [4, 6, 7].

Definition 3. *A function $f : [0,b] \to \mathbb{R}$, $b > 0$ is said to be (α, m)-convex, where $(\alpha, m) \in [0,1]^2$ if*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha})f(y) \tag{1.3}$$

holds for all $x, y \in [0,b]$ and $t \in [0,1]$.

The functions which are deducible from above definition are given in the following remark:

Remark 2. *(i) If (α, m)=$(1, m)$, then (1.3) produces the definition of m-convex function.*
(ii) If (α, m)=$(1,1)$, then (1.3) produces the definition of convex function.
(iii) If (α, m)=$(1,0)$, then (1.3) produces the definition of star-shaped function.
For some recent citations and utilizations of (α, m)-convex functions one can see [8–12].

Recently, many mathematicians have used convex function to obtain fractional and conformable integral inequalities, see ([13–15]). Our objective in this paper is to prove general inequalities for integral operators given in Definition 8 via $(h-m)$-convex and (α, m)-convex functions. These results are interestingly associated with fractional and conformable integral operators. In the following we give some definitions of operators derivable from integral operators given in Definition 8.

Definition 4. *Let $f \in L_1[a,b]$. Then the left-sided and right-sided Riemann-Liouville fractional integral operators of order $\mu \in \mathbb{C}$, $(\mathcal{R}(\mu) > 0)$ are defined by*

$$^{\mu}I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt, \ \ x > a, \tag{1.4}$$

$$ {}^{\mu}I_{b_-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)dt, \;\; x < b. \tag{1.5} $$

A k-analogue of Riemann-Liouville fractional integral operator is given in [16].

Definition 5. *Let $f \in L_1[a,b]$. Then the k-fractional integral operators of f of order $\mu, \mathcal{R}(\mu) > 0$, $k > 0$ are defined by*

$$ {}^{\mu}I^k_{a^+}f(x) = \frac{1}{k\Gamma_\kappa(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)dt, \;\; x > a, \tag{1.6} $$

$$ {}^{\mu}I^k_{b_-}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} f(t)dt, \;\; x < b. \tag{1.7} $$

A more general definition of the Riemann-Liouville fractional integral operators is given in [17].

Definition 6. *Let $f : [a,b] \to \mathbb{R}$ be a integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\mu \in \mathbb{C}$, $(\mathcal{R}(\mu) > 0)$ are defined by*

$$ {}^{\mu}_{g}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(x)-g(t))^{\mu-1} g'(t)f(t)dt, \;\; x > a, \tag{1.8} $$

$$ {}^{\mu}_{g}I_{b_-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t)-g(x))^{\mu-1} g'(t)f(t)dt, \;\; x < b, \tag{1.9} $$

where $\Gamma(.)$ is the gamma function.

A k-analogue of above definition is given in [18].

Definition 7. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided k-fractional integral operators of a function f with respect to another function g on $[a,b]$ of order $\mu, \mathcal{R}(\mu) > 0$ are defined by*

$$ {}^{\mu}_{g}I^k_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1} g'(t)f(t)dt, x > a, \tag{1.10} $$

$$ {}^{\mu}_{g}I^k_{b_-}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t)-g(x))^{\frac{\mu}{k}-1} g'(t)f(t)dt, x < b, \tag{1.11} $$

where $\Gamma_k(.)$ is the k-analogue of gamma function.

The following integral operator is given in [19].

Definition 8. *Let $f, g : [a,b] \longrightarrow \mathbb{R}$, $0 < a < b$ be the functions such that f be positive and $f \in L_1[a,b]$, and g be differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a,\infty)$. Then for $x \in [a,b]$ the left and right integral operators are defined by*

$$ (F^{\phi,g}_{a^+}f)(x) = \int_a^x K_g(x,t;\phi)f(t)d(g(t)), \;\; x > a, \tag{1.12} $$

$$ (F^{\phi,g}_{b_-}f)(x) = \int_x^b K_g(t,x;\phi)f(t)d(g(t)), \;\; x < b, \tag{1.13} $$

where $K_g(x,y;\phi) = \dfrac{\phi(g(x)-g(y))}{g(x)-g(y)}$.

Integral operators defined in (1.12) and (1.13) produce several fractional and conformable integral operators defined in [17, 20–27].

Remark 3. *Integral operators given in (1.12) and (1.13) produce several known fractional and conformable integral operators corresponding to different settings of ϕ and g.*

(i) If we put $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then (1.12) and (1.13) integral operators coincide with (1.10) and (1.11) fractional integral operators.

(ii) If we put $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$, $\mu > 0$, then (1.12) and (1.13) integral operators coincide with (1.8) and (1.9) fractional integral operators.

(iii) If we put $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}, \mu \geq k$ and $g(x) = x$, then (1.12) and (1.13) integral operators coincide with (1.6) and (1.7) fractional integral operators.

(iv) If we put $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$, $\mu > 0$ and $g(x) = x$, then (1.12) and (1.13) integral operators coincide with (1.4) and (1.5) fractional integral operators.

(v) If we put $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$ and $g(x) = \frac{x^\rho}{\rho}, \rho > 0$, then (1.12) and (1.13) produce Katugampola fractional integral operators defined by Chen et al. in [20].

(vi) If we put $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$ and $g(x) = \frac{x^{\tau+s}}{\tau+s}$, $s > 0$, then (1.12) and (1.13) produce generalized conformable integral operators defined by Khan et al. in [24].

(vii) If we put $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, $g(x) = \frac{(x-a)^s}{s}$, $s > 0$, in (1.12), and $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.13) respectively, then conformable (k, s)-fractional integrals are obtained as defined by Habib et al. in [22].

(viii) If we put $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ and $g(x) = \frac{x^{1+s}}{1+s}$, then (1.12) and (1.13) produce conformable fractional integrals defined by Sarıkaya et al. in [26].

(ix) If we put $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$, $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.12) and $\phi(t) = \frac{t^\mu}{\Gamma(\mu)}$, $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.13) respectively, then conformable fractional integrals are obtained as defined by Jarad et al. in [23].

(x) If we put $\phi(t) = t^{\frac{\mu}{k}}\mathcal{F}^{\sigma,k}_{\rho,\lambda}(w(t)^\rho)$, then (1.12) and (1.13) produce generalized k-fractional integral operators defined by Tunc et al. in [27].

(xi) If we put $\phi(t) = \dfrac{exp(-At)}{\mu}, A = \dfrac{1-\mu}{\mu}, \mu > 0$, then following generalized fractional integral operators with exponential kernel defined in [21] are obtained and given as follows:

$$ {}^{\mu}_{g}E_{a^+}f(x) = \frac{1}{\mu}\int_a^x exp\left(-\frac{1-\mu}{\mu}(g(x)-g(t))\right)f(t)dt, x > a, \tag{1.14}$$

$$ {}^{\mu}_{g}E_{b_-}f(x) = \frac{1}{\mu}\int_x^b exp\left(-\frac{1-\mu}{\mu}(g(x)-g(t))\right)f(t)dt, x < b. \tag{1.15}$$

(xii) If we put $\phi(t) = \dfrac{t^\mu}{\Gamma(\mu)}$ and $g(t) = lnt$, then Hadamard fractional integral operators are obtained [17, 25].

(xiii) If we put $\phi(t) = \dfrac{t^\mu}{\Gamma(\mu)}$ and $g(t) = -t^{-1}$, then Harmonic fractional integral operators defined in [17] are obtained and given as follows:

$$ {}^{\mu}R_{a^+}f(x) = \frac{t^\mu}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}\frac{f(t)}{t^{\mu+1}}dt, \ \ x > a, \tag{1.16}$$

$$ {}^{\mu}R_{b_-}f(x)=\frac{t^{\mu}}{\Gamma(\mu)}\int_a^x (t-x)^{\mu-1}\frac{f(t)}{t^{\mu+1}}dt,\quad x<b. \tag{1.17}$$

(xiv) If we put $\phi(t)=t^{\mu}lnt$, then left and right sided-logarithmic fractional integrals defined in [21] are obtained and given as follows:

$$ {}_g^{\mu}\mathcal{L}_{a^+}f(x)=\int_a^x (g(x)-g(t))^{\mu-1}ln(g(x)-g(t))g'(t)dt, x>a, \tag{1.18}$$

$$ {}_g^{\mu}\mathcal{L}_{b_-}f(x)=\int_a^x (g(t)-g(x))^{\mu-1}ln(g(x)-g(t))g'(t)dt, x<b. \tag{1.19}$$

Fractional integrals and derivatives play very vital role in advancement of almost all subjects of sciences and engineering. Now a days fractional and conformable integral operators have been used frequently in the advancements of classical inequalities, see [7, 10, 18, 20–22, 28–33].
The aim of this paper is to derive general integral operator inequalities by using $(h-m)$ and (α,m)-convex functions which will hold for all types of fractional and conformable integral operators and functions which are explained in the above discussions.

In Section 2 we derive the bounds of integral operators given in (1.12) and (1.13) by using $(h-m)$-convexity of function f and $|f'|$. The bounds of various fractional integral operators and conformable integrals can be obtained by setting appropriate values of functions involved in (1.12) and (1.13). We also present results for (α,m) convex functions. The results of this paper hold for all kinds of functions explained in Remark 1 and Remark 2. In Section 3 we apply some of the results and get interesting consequences.

2. Results and discussions

First we give results for $(h-m)$-convex functions.

2.1. Inequalities for integral operators via $(h-m)$-convex function

Theorem 1. *Let $f:[a,b]\longrightarrow\mathbb{R}, 0\le a<b$, be a positive integrable $(h-m)$-convex function, $m\in(0,1]$ and $h\in L_{\infty}[0,1]$. Let $g:[a,b]\longrightarrow\mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function. Then for $x\in[a,b]$ the following inequalities for integral operators (1.12) and (1.13) hold:*

$$(F_{a^+}^{\phi,g}f)(x)\le\phi(g(x)-g(a))\left(mf\left(\frac{x}{m}\right)+f(a)\right)\|h\|_{\infty}, \tag{2.1}$$

$$(F_{b^-}^{\phi,g}f)(x)\le\phi(g(b)-g(x))\left(f(b)+mf\left(\frac{x}{m}\right)\right)\|h\|_{\infty}. \tag{2.2}$$

Hence

$$(F_{a^+}^{\phi,g}f)(x)+(F_{b^-}^{\phi,g}f)(x)\le\left(\phi(g(x)-g(a))\left(mf\left(\frac{x}{m}\right)+f(a)\right)+\phi(g(b)-g(x))\left(f(b)+mf\left(\frac{x}{m}\right)\right)\right)\|h\|_{\infty}. \tag{2.3}$$

Proof. Under given conditions, for $x \in (a, b]$ *and* $t \in [a, x)$ the following inequality holds true:

$$K_g(x, t; \phi)g'(t) \leq K_g(x, a; \phi)g'(t). \tag{2.4}$$

Since f is $(h - m)$-convex, we have

$$f(t) \leq h\left(\frac{x-t}{x-a}\right) f(a) + mh\left(\frac{t-a}{x-a}\right) f\left(\frac{x}{m}\right). \tag{2.5}$$

Inequalities (2.4) and (2.5) produce the following inequality:

$$\begin{aligned} &\int_a^x K_g(x, t; \phi)g'(t)f(t)dt \qquad (2.6)\\ &\leq f(a)K_g(x, a; \phi)\int_a^x h\left(\frac{x-t}{x-a}\right)g'(t)dt + mf\left(\frac{x}{m}\right)K_g(x, a; \phi)\int_a^x h\left(\frac{t-a}{x-a}\right)g'(t)dt. \end{aligned}$$

This further takes form as follows:

$$(F^{\phi,g}_{a^+} f)(x) \leq \phi(g(x) - g(a))\left(mf\left(\frac{x}{m}\right) + f(a)\right)\|h\|_\infty. \tag{2.7}$$

Now on the other hand for $t \in (x, b]$ and $x \in [a, b)$, the following inequality holds true:

$$K_g(t, x; \phi)g'(t) \leq K_g(b, x; \phi)g'(t). \tag{2.8}$$

Again by using $(h - m)$-convexity of f we have

$$f(t) \leq h\left(\frac{t-x}{b-x}\right) f(b) + mh\left(\frac{b-t}{b-x}\right) f\left(\frac{x}{m}\right). \tag{2.9}$$

From inequalities (2.8) and (2.9), the following integral operator inequality holds:

$$(F^{\phi,g}_{b^-} f)(x) \leq \phi(g(b) - g(x))\left(f(b) + mf\left(\frac{x}{m}\right)\right)\|h\|_\infty. \tag{2.10}$$

By adding (2.7) and (2.10), inequality (2.3) can be obtained.

Remark 4. *1. By setting $h(x) = x$ and $m = 1$ in (2.3), [34, Theorem 1] can be reproduced.*
2. By setting $h(x) = x$, $m = 1$ and $\phi(t) = \frac{t^x}{\Gamma(x)}, x > 0$ in (2.3), [33, Corollary 1] can be reproduced.
3. By setting $h(x) = x$, $m = 1$, $\phi(t) = \frac{t^x}{\Gamma(x)}, x > 0$ and $g(x) = x$ in (2.3), [28, Corollary 1] can be reproduced.

□

Theorem 2. *Let $f : [a, b] \longrightarrow \mathbb{R}, 0 \leq a < b$, be a differentiable function. Let $|f'|$ be $(h - m)$-convex, $m \in (0, 1]$, $h \in L_\infty[0, 1]$, $g : [a, b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function. Then for $x \in [a, b]$ the following inequality holds:*

$$\begin{aligned} \left|F^{\phi,g}_{a^+}(f * g)(x) + F^{\phi,g}_{b^-}(f * g)(x)\right| &\leq \left(\phi(g(x) - g(a))\left(m\left|f'\left(\frac{x}{m}\right)\right| + |f'(a)|\right)\right. \qquad (2.11)\\ &\left.+ \phi(g(b) - g(x))\left(|f'(b)| + m\left|f'\left(\frac{x}{m}\right)\right|\right)\right)\|h\|_\infty, \end{aligned}$$

where

$$F^{\phi,g}_{a^+}(f * g)(x) = \int_a^x K_g(x, t; \phi)f'(t)d(g(t)), F^{\phi,g}_{b^-}(f * g)(x) = \int_x^b K_g(t, x; \phi)f'(t)d(g(t)).$$

Proof. Let $x \in (a, b]$ and $t \in [a, x]$. Then $(h-m)$-convexity of $|f'|$ gives

$$|f'(t)| \leq h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.12}$$

From which we can write

$$f'(t) \leq h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.13}$$

Inequalities (2.4) and (2.13) produce the following integral inequality:

$$\int_a^x K_g(x,t;\phi)g'(t)f'(t)dt \leq |f'(a)|K_g(x,a;\phi)\int_a^x h\left(\frac{x-t}{x-a}\right)g'(t)dt$$
$$+ m\left|f'\left(\frac{x}{m}\right)\right|K_g(x,a;\phi)\int_a^x h\left(\frac{t-a}{x-a}\right)g'(t)dt.$$

This further takes form as follows:

$$F_{a^+}^{\phi,g}(f*g)(x) \leq \phi(g(x)-g(a))\left(m\left|f'\left(\frac{x}{m}\right)\right| + |f'(a)|\right)\|h\|_\infty. \tag{2.14}$$

From (2.12) we can also write

$$f'(t) \geq -\left(h\left(\frac{x-t}{x-a}\right)|f'(a)| + mh\left(\frac{t-a}{x-a}\right)\left|f'\left(\frac{x}{m}\right)\right|\right). \tag{2.15}$$

Adopting the same procedure as we did for (2.13), the following inequality holds:

$$F_{a^+}^{\phi,g}(f*g)(x) \geq -\phi(g(x)-g(a))\left(m\left|f'\left(\frac{x}{m}\right)\right| + |f'(a)|\right)\|h\|_\infty. \tag{2.16}$$

From (2.14) and (2.16), the following inequality is obtained:

$$\left|F_{a^+}^{\phi,g}(f*g)(x)\right| \leq \phi(g(x)-g(a))\left(m\left|f'\left(\frac{x}{m}\right)\right| + |f'(a)|\right)\|h\|_\infty. \tag{2.17}$$

Now using $(h-m)$-convexity of $|f'|$ on $(x, b]$ for $x \in (a, b)$ we have

$$|f'(t)| \leq h\left(\frac{t-x}{b-x}\right)|f'(b)| + mh\left(\frac{b-t}{b-x}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.18}$$

From which we can write

$$f'(t) \leq h\left(\frac{t-x}{b-x}\right)|f'(b)| + mh\left(\frac{b-t}{b-x}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.19}$$

Inequalities (2.8) and (2.19) produce the following integral inequality:

$$\int_x^b K_g(t,x;\phi)g'(t)f'(t)dt \leq |f'(b)|K_g(b,x;\phi)\int_a^x h\left(\frac{t-x}{b-x}\right)g'(t)dt$$

$$+\, m\left|f'\left(\frac{x}{m}\right)\right| K_g(b,x;\phi)\int_x^b h\left(\frac{b-t}{b-x}\right)g'(t)dt.$$

This further takes form as follows:

$$F_{b^-}^{\phi,g}(f*g)(x)\le \phi(g(b)-g(x))\left(m\left|f'\left(\frac{x}{m}\right)\right|+|f'(b)|\right)\|h\|_\infty. \tag{2.20}$$

From (2.18) we can also write

$$f'(t)\ge -\left(h\left(\frac{t-x}{b-x}\right)|f'(b)|+mh\left(\frac{b-t}{b-x}\right)\left|f'\left(\frac{x}{m}\right)\right|\right). \tag{2.21}$$

Adopting the same procedure as we did for (2.19), the following inequality holds:

$$F_{b^-}^{\phi,g}(f*g)(x)\ge -\phi(g(b)-g(x))\left(m\left|f'\left(\frac{x}{m}\right)\right|+|f'(b)|\right)\|h\|_\infty. \tag{2.22}$$

From (2.20) and (2.22), the following modulus inequality is obtained:

$$\left|F_{b^-}^{\phi,g}(f*g)(x)\right|\le \phi(g(b)-g(x))\left(m\left|f'\left(\frac{x}{m}\right)\right|+|f'(a)|\right)\|h\|_\infty. \tag{2.23}$$

By adding (2.17) and (2.23), inequality (2.11) can be obtained. □

Remark 5. *By setting $h(x)=x$ and $m=1$ in (2.11), [34, Theorem 1] can be reproduced.*

To prove the next result we need the following lemma.

Lemma 1. *[35] Let $f:[0,\infty)\to\mathbb{R}$ be a $(h-m)$-convex function, $m\in(0,1]$. If $0\le a<b$ and $f(x)=f\left(\frac{a+b-x}{m}\right)$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\le (m+1)h\left(\frac{1}{2}\right)f(x),\qquad x\in[a,b]. \tag{2.24}$$

Theorem 3. *Let $f:[a,b]\longrightarrow\mathbb{R}, 0\le a<b$, be a positive $(h-m)$-convex with $m\in(0,1]$, $f(x)=f\left(\frac{a+b-x}{m}\right)$, and $g:[a,b]\longrightarrow\mathbb{R}$ be a differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function. Then the following Hadamard inequality is valid:*

$$\frac{1}{(m+1)h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\left((F_{b^-}^{\phi,g}(1))(a)+(F_{a^+}^{\phi,g}(1))(b)\right)\le (F_{b^-}^{\phi,g}f)(a)+(F_{a^+}^{\phi,g}f)(b) \tag{2.25}$$

$$\le 2\phi(g(b)-g(a))\left(mf\left(\frac{b}{m}\right)+f(a)\right)\|h\|_\infty.$$

Proof. Under given conditions, the following inequality holds for $x\in(a,b]$:

$$K_g(x,a;\phi)g'(x)\le K_g(b,a;\phi)g'(x). \tag{2.26}$$

Using $(h - m)$-convexity of f for $x \in [a, b]$, we have

$$f(x) \leq mh\left(\frac{x-a}{b-a}\right)f\left(\frac{b}{m}\right) + h\left(\frac{b-x}{b-a}\right)f(a). \tag{2.27}$$

Inequalities (2.26) and (2.27) produce the following integral inequality:

$$\int_a^b K_g(x, a; \phi)g'(x)f(x)dx \leq mf\left(\frac{b}{m}\right)K_g(b, a; \phi)\int_a^b h\left(\frac{x-a}{b-a}\right)g'(x)dx$$
$$+ f(a)K_g(b, a; \phi)\int_a^b h\left(\frac{b-x}{b-a}\right)g'(x)dx.$$

This further takes form as follows:

$$(F_{b^-}^{\phi,g}f)(a) \leq \phi(g(b) - g(a))\left(mf\left(\frac{b}{m}\right) + f(a)\right)\|h\|_\infty. \tag{2.28}$$

On the other hand for $x \in [a, b)$, the following inequality holds true:

$$K_g(b, x; \phi)g'(x) \leq K_g(b, a; \phi)g'(x). \tag{2.29}$$

Adopting the same procedure as we did for (2.26) and (2.27), the following inequality can be obtained from (2.27) and (2.29):

$$(F_{b^-}^{\phi,g}f)(a) \leq \phi(g(b) - g(a))\left(mf\left(\frac{b}{m}\right) + f(a)\right)\|h\|_\infty. \tag{2.30}$$

By adding (2.28) and (2.30), the second inequality in (2.25) is obtained. Multiplying both sides of (2.24) by $K_g(x, a; \phi)g'(x)$, and integrating over $[a, b]$ we have

$$f\left(\frac{a+b}{2}\right)\int_a^b K_g(x, a; \phi)g'(x)dx \leq (m+1)h\left(\frac{1}{2}\right)\int_a^b K_g(x, a; \phi)g'(x)f(x)dx.$$

This further takes form as follows:

$$f\left(\frac{a+b}{2}\right)(F_{b^-}^{\phi,g}(1))(a) \leq (m+1)h\left(\frac{1}{2}\right)(F_{b^-}^{\phi,g}f)(a). \tag{2.31}$$

Similarly multiplying both sides of (2.24) by $K_g(b, x; \phi)g'(x)$, and integrating over $[a, b]$ we get

$$f\left(\frac{a+b}{2}\right)(F_{a^+}^{\phi,g}(1))(b) \leq (m+1)h\left(\frac{1}{2}\right)(F_{a^+}^{\phi,g}f)(b). \tag{2.32}$$

From (2.31) and (2.32), the first inequality in (2.25) is obtained.

□

Remark 6. *By setting $h(x) = x$ and $m = 1$ in (2.25), [34, Theorem 3] can be reproduced.*

2.2. *Inequalities for integral operators via* (α, m)*-convex function*

In this subsection we derive the bounds of integral operators defined in (1.12) and (1.13) by using (α, m)-convexity of functions f and $|f'|$.

Theorem 4. *Let* $f : [a, b] \longrightarrow \mathbb{R}, 0 \leq a < b$*, be a positive* (α, m)*-convex function with* $m \in (0, 1]$ *and* $g : [a, b] \longrightarrow \mathbb{R}$ *be a differentiable and strictly increasing function. Also let* $\frac{\phi}{x}$ *be an increasing function. Then for* $x \in [a, b]$ *the following inequality for integral operators (1.12) and (1.13) holds:*

$$\begin{aligned}&(F_{a^+}^{\phi,g} f)(x) + (F_{b^-}^{\phi,g} f)(x)\\ &\leq K_g(x, a; \phi)\Big(mf\Big(\frac{x}{m}\Big)g(x) - f(a)g(a) - \frac{\Gamma(\alpha + 1)}{(x - a)^\alpha}\Big(mf\Big(\frac{x}{m}\Big) - f(a)\Big)\, {}^{\alpha}I_{a^+} f(x)\Big)\\ &+ K_g(b, x; \phi)\Big(f(b)g(b) - mf\Big(\frac{x}{m}\Big)g(x) - \frac{\Gamma(\alpha + 1)}{(b - x)^\alpha}\Big(f(b) - mf\Big(\frac{x}{m}\Big)\Big)\, {}^{\alpha}I_{b^-} f(x)\Big).\end{aligned} \tag{2.33}$$

Proof. By using (α, m)-convexity of f we get

$$f(t) \leq \Big(\frac{x - t}{x - a}\Big)^\alpha f(a) + m\Big(1 - \Big(\frac{x - t}{x - a}\Big)^\alpha\Big)f\Big(\frac{x}{m}\Big). \tag{2.34}$$

Inequalities (2.4) and (2.34) produce the following integral inequality:

$$\int_a^x K_g(x, t; \phi)g'(t)f(t)dt \leq K_g(x, a; \phi)\Big[f(a)\int_a^x \Big(\frac{x - t}{x - a}\Big)^\alpha g'(t)dt + mf\Big(\frac{x}{m}\Big)\int_a^x \Big(1 - \Big(\frac{x - t}{x - a}\Big)^\alpha\Big)g'(t)dt\Big]. \tag{2.35}$$

This takes the form as follows:

$$(F_{a^+}^{\phi,g} f)(x) \leq \frac{K_g(x, a; \phi)}{(x - a)^\alpha}\Big[(x - a)^\alpha\Big(mf\Big(\frac{x}{m}\Big)g(x) - f(a)g(a)\Big) - \Gamma(\alpha + 1)\Big(mf\Big(\frac{x}{m}\Big) - f(a)\Big)\, {}^{\alpha}I_{a^+} f(x)\Big]. \tag{2.36}$$

Again by using (α, m)-convexity of f, we have

$$f(t) \leq \Big(\frac{t - x}{b - x}\Big)^\alpha f(b) + m\Big(1 - \Big(\frac{t - x}{b - x}\Big)^\alpha\Big)f\Big(\frac{x}{m}\Big). \tag{2.37}$$

From inequalities (2.8) and (2.37), the following integral inequality is obtained:

$$\int_x^b K_g(t, x; \phi)g'(t)f(t)dt \leq K_g(b, x; \phi)\Big[f(b)\int_x^b \Big(\frac{t - x}{b - x}\Big)^\alpha g'(t)dt + mf\Big(\frac{x}{m}\Big)\int_x^b \Big(1 - \Big(\frac{t - x}{b - x}\Big)^\alpha\Big)g'(t)dt\Big]. \tag{2.38}$$

This takes the form as follows:

$$(F_{b^-}^{\phi,g} f)(x) \leq \frac{K_g(b, x; \phi)}{(b - x)^\alpha}\Big[(b - x)^\alpha\Big(f(b)g(b) - mf\Big(\frac{x}{m}\Big)g(x)\Big) - \Gamma(\alpha + 1)\Big(f(b) - mf\Big(\frac{x}{m}\Big)\Big)\, {}^{\alpha}I_{b^-} f(x)\Big]. \tag{2.39}$$

By adding (2.36) and (2.39), (2.33) can be obtained. □

Remark 7. *1. By setting* $\phi(t) = \frac{t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ *in (2.33), [10, Corollary 1] can be reproduced.*
2. By setting $\phi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}, \alpha > 0$ *and* (α, m) *=(1,1) in (2.33), [33, Corollary 1] can be reproduced.*
3. By setting $\phi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}, \alpha > 0$*,* $g(x) = x$ *and* (α, m) *=(1,1) in (2.33), [28, Corollary 1] can be reproduced.*

Theorem 5. *Let* $f : [a, b] \longrightarrow \mathbb{R}, 0 \le a < b$*, be a differentiable function. Let* $|f'|$ *be* (α, m)*-convex,* $m \in (0, 1]$*,* $g : [a, b] \longrightarrow \mathbb{R}$ *be differentiable and strictly increasing function, also let* $\frac{\phi}{x}$ *be an increasing function. Then for* $x \in [a, b]$ *the following inequalities hold:*

$$|F^{\phi,g}_{a^+}(f * g)(x)| \le K_g(a, x; \phi)\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x) - |f'(a)|g(a) - \frac{\Gamma(\alpha + 1)}{(x - a)^{\alpha}}\left(m\left|f'\left(\frac{x}{m}\right)\right| - |f'(a)|\right)^{\alpha} I_{a^+}f(x)\right), \tag{2.40}$$

$$|F^{\phi,g}_{b^-}(f * g)(x)| \le K_g(a, x; \phi)\left(|f'(b)|g(b) - m\left|f'\left(\frac{x}{m}\right)\right|g(x) - \frac{\Gamma(\alpha + 1)}{(b - x)^{\alpha}}\left(|f'(b)| - m\left|f'\left(\frac{x}{m}\right)\right|\right)^{\alpha} I_{b^-}f(x)\right). \tag{2.41}$$

Proof. Since $|f'|$ is (α, m)-convex, we have

$$|f'(t)| \le \left(\frac{x - t}{x - a}\right)^{\alpha}|f'(a)| + m\left(1 - \left(\frac{x - t}{x - a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.42}$$

From which we can write

$$f'(t) \le \left(\frac{x - t}{x - a}\right)^{\alpha}|f'(a)| + m\left(1 - \left(\frac{x - t}{x - a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.43}$$

Inequalities (2.4) and (2.43) produce the following integral inequality:

$$\int_a^x K_g(x, t; \phi)g'(t)f'(t)dt \le K_g(x, a; \phi)\left[|f'(a)|\int_a^x \left(\frac{x - t}{x - a}\right)^{\alpha} g'(t)dt + m\left|f'\left(\frac{x}{m}\right)\right|\int_a^x \left(1 - \left(\frac{x - t}{x - a}\right)^{\alpha}\right)g'(t)dt\right]. \tag{2.44}$$

In compact form the following integral operator inequality holds:

$$F^{\phi,g}_{a^+}(f * g)(x) \le K_g(x, a; \phi)\left[\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x) - |f'(a)|g(a)\right) - \frac{\Gamma(\alpha + 1)}{(x - a)^{\alpha}}\left(m\left|f'\left(\frac{x}{m}\right)\right| - |f'(a)|\right)^{\alpha} I_{a^+}f(x)\right]. \tag{2.45}$$

From (2.42) we can write

$$f'(t) \ge -\left(\left(\frac{x - t}{x - a}\right)^{\alpha}|f'(a)| + m\left(1 - \left(\frac{x - t}{x - a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|\right). \tag{2.46}$$

Adopting the same procedure as we did for (2.43), the following inequality holds:

$$F^{\phi,g}_{a^+}(f * g)(x) \ge -K_g(x, a; \phi)\left[\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x) - |f'(a)|g(a)\right) - \frac{\Gamma(\alpha + 1)}{(x - a)^{\alpha}}\left(m\left|f'\left(\frac{x}{m}\right)\right| - |f'(a)|\right)^{\alpha} I_{a^+}f(x)\right]. \tag{2.47}$$

From (2.45) and (2.47), (2.40) can be obtained.
By using (α, m)-convexity of $|f'|$, we have

$$|f'(t)| \leq \left(\frac{t-x}{b-x}\right)^{\alpha} |f'(b)| + m\left(1 - \left(\frac{t-x}{b-x}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.48}$$

From which we can write

$$f'(t) \leq \left(\frac{t-x}{b-x}\right)^{\alpha} |f'(b)| + m\left(1 - \left(\frac{t-x}{b-x}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|. \tag{2.49}$$

Inequalities (2.49) and (2.8) produce the following integral inequality:

$$\int_x^b K_g(t,x;\phi)g'(t)f(t)dt \leq K_g(b,x;\phi)\left[|f'(b)| \int_a^x \left(\frac{x-t}{x-a}\right)^{\alpha} g'(t)dt + m\left|f'\left(\frac{x}{m}\right)\right| \int_a^x \left(1 - \left(\frac{x-t}{x-a}\right)^{\alpha}\right) g'(t)dt\right]. \tag{2.50}$$

In compact form the following integral operator inequality holds:

$$F_{b^-}^{\phi,g}(f*g)(x) \leq K_g(b,x;\phi)\left[\left(|f'(b)|g(b) - m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right) - \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\left(|f'(b)| - m\left|f'\left(\frac{x}{m}\right)\right|\right)^{\alpha} I_{b^-}f(x)\right]. \tag{2.51}$$

From (2.48) we can write

$$f'(t) \geq -\left(\left(\frac{t-x}{b-x}\right)^{\alpha} |f'(b)| + m\left(1 - \left(\frac{t-x}{b-x}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|\right). \tag{2.52}$$

Adopting the same procedure as we did for (2.49), the following inequality holds:

$$F_{b^-}^{\phi,g}(f*g)(x) \geq -K_g(b,x;\phi)\left[\left(|f'(b)|g(b) - m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right) - \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\left(|f'(b)| - m\left|f'\left(\frac{x}{m}\right)\right|\right)^{\alpha} I_{b^-}f(x)\right]. \tag{2.53}$$

From (2.51) and (2.53), (2.41) can be obtained.

□

Lemma 2. *[10] Let $f : [0,\infty] \longrightarrow \mathbb{R}$, be an (α, m)-convex function. If $f(x) = f(\frac{a+b-x}{m})$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\alpha}}(1 + m(2^{\alpha}-1))f(x) \quad x \in [a,b]. \tag{2.54}$$

Theorem 6. *Let $f : [a,b] \longrightarrow \mathbb{R}, 0 \leq a < b$ be a positive (α, m)-convex, $m \in (0,1]$, $f(x) = f(\frac{a+b-x}{m})$ and $g : [a,b] \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also let $\frac{\phi}{x}$ be an increasing function. Then for $(\alpha, m) \in [0,1]^2$, the following Hadamard type inequality holds:*

$$\begin{aligned} &\frac{2^{\alpha}}{1+m(2^{\alpha}-1)} f\left(\frac{a+b}{2}\right)\left[(F_{b^-}^{\phi,g}(1))(a) + (F_{a^+}^{\phi,g}(1))(b)\right] \\ &\leq (F_{b^-}^{\phi,g}f)(a) + (F_{a^+}^{\phi,g}f)(b) \\ &\leq 2K_g(b,a;\phi)\left[\left(f(b)g(b) - mf\left(\frac{a}{m}\right)g(a)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(f(b) - mf\left(\frac{a}{m}\right)\right)^{\alpha} I_{b^-}g(b)\right]. \end{aligned} \tag{2.55}$$

Proof. Under given conditions, the following inequality holds for $x \in (a, b]$:

$$K_g(x, a; \phi)g'(x) \leq K_g(b, a; \phi)g'(x). \tag{2.56}$$

By using (α, m)-convexity of f, we have

$$f(x) \leq \left(\frac{x-a}{b-a}\right)^{\alpha} f(b) + m\left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right) f\left(\frac{x}{m}\right). \tag{2.57}$$

Inequalities (2.56) and (2.57) produce the following integral inequality:

$$\int_a^b K_g(x, a; \phi)g'(x)f(x)dx \leq K_g(b, a; \phi)\left[f(b) \int_a^b \left(\frac{x-a}{b-a}\right)^{\alpha} g'(x)dx + mf\left(\frac{a}{m}\right) \int_a^b \left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right) g'(x)dx\right]. \tag{2.58}$$

In compact form the following integral operator inequality holds:

$$(F_{b^-}^{\phi,g} f)(a) \leq \frac{K_g(b, a; \phi)}{(b-a)^{\alpha}}\left[\left(f(b)g(b) - mf\left(\frac{a}{m}\right)g(a)\right)(b-a)^{\alpha} - \Gamma(\alpha+1)\left(f(b) - mf\left(\frac{a}{m}\right)\right)^{\alpha} I_{b^-} g(b)\right]. \tag{2.59}$$

On the other hand, under given conditions, the following inequality holds for $x \in [a, b)$:

$$K_g(b, x; \phi)g'(x) \leq K_g(b, a; \phi)g'(x). \tag{2.60}$$

Inequalities (2.57) and (2.60) produce the following integral inequality:

$$\int_a^b K_g(b, x; \phi)g'(x)f(x)dx \leq K_g(b, a; \phi)\left[f(b) \int_a^b \left(\frac{x-a}{b-a}\right)^{\alpha} g'(x)dx + mf\left(\frac{a}{m}\right) \int_a^b \left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right) g'(x)dx\right].$$

In compact form the following integral operator inequality holds:

$$(F_{a^+}^{\phi,g} f)(b) \leq 2\frac{K_g(b, a; \phi)}{(b-a)^{\alpha}}\left[\left(f(b)g(b) - mf\left(\frac{a}{m}\right)g(a)\right)(b-a)^{\alpha} - \Gamma(\alpha+1)\left(f(b) - mf\left(\frac{a}{m}\right)\right)^{\alpha} I_{b^-} g(b)\right]. \tag{2.61}$$

From (2.59) and (2.61), the second inequality of (2.55) can be obtained.

Now using Lemma 2 and multiplying (2.54) with $K_g(x, a; \phi)g'(x)$, then integrating over $[a, b]$ we have

$$\int_a^b K_g(x, a; \phi)f\left(\frac{a+b}{2}\right)g'(x)dx \leq \frac{1}{2^{\alpha}}(1 + m(2^{\alpha} - 1)) \int_a^b K_g(x, a; \phi)g'(x)f(x)dx.$$

From which we get

$$f\left(\frac{a+b}{2}\right)(F_{b^-}^{\phi,g}(1))(a) \leq \frac{1}{2^{\alpha}}(1 + m(2^{\alpha} - 1))(F_{b^-}^{\phi,g} f)(a). \tag{2.62}$$

Again by using Lemma 2 and multiplying (2.54) with $K_g(b, x; \phi)g'(x)$, then integrating over $[a, b]$ we get

$$\int_a^b K_g(b, x; \phi)f\left(\frac{a+b}{2}\right)g'(x)dx \leq \frac{1}{2^{\alpha}}(1 + m(2^{\alpha} - 1)) \int_a^b K_g(b, x; \phi)g'(x)f(x)dx.$$

From which we have

$$f\left(\frac{a+b}{2}\right)(F_{a^+}^{\phi,g}(1))(b) \leq \frac{1}{2^{\alpha}}(1 + m(2^{\alpha} - 1))(F_{a^+}^{\phi,g} f)(b). \tag{2.63}$$

From (2.62) and (2.63), the first inequality of (2.55) can be obtained. □

3. Applications

In this section we will apply Theorem 4 for some particular functions and get upper bounds of several fractional and conformable integral operators. By applying Theorems 4, 5 & 6 we give bounds for m-convex functions. We also apply Theorem 1 to give the boundedness and continuity of all kinds of these operators. Further by applying other theorems reader can obtain specific results for these operators.

3.1. Some special cases

Corollary 1. *If we take $\alpha = 1$ in Theorem 4, then following inequality for m-convex functions holds:*

$$(F_{a^+}^{\phi,g} f)(x) + (F_{b^-}^{\phi,g} f)(x) \leq \phi(g(x) - g(a))\left(mf\left(\frac{x}{m}\right) + f(a)\right) + \phi(g(b) - g(x))\left(mf\left(\frac{x}{m}\right) + f(b)\right).$$

Corollary 2. *If we take $\alpha = 1$ in Theorem 5, then following inequality for m-convex functions holds:*

$$\left|(F_{a^+}^{\phi,g} f * g)(x) + (F_{b^-}^{\phi,g} f * g)(x)\right| \leq \phi(g(x) - g(a))\left(m\left|f'\left(\frac{x}{m}\right)\right| + |f'(a)|\right)$$
$$+ \phi(g(b) - g(x))\left(m\left|f'\left(\frac{x}{m}\right)\right| + |f'(b)|\right).$$

Corollary 3. *If we take $\alpha = 1$ in Theorem 6, then following inequality for m-convex functions holds:*

$$\frac{2}{1+m} f\left(\frac{a+b}{2}\right)\left[(F_{b^-}^{\phi,g}(1))(a) + (F_{a^+}^{\phi,g}(1))(b)\right] \leq (F_{b^-}^{\phi,g} f)(a) + (F_{a^+}^{\phi,g} f)(b)$$
$$\leq 2\phi(g(b) - g(a))\left(f(b) + mf\left(\frac{a}{m}\right)\right).$$

3.2. Consequences of Theorem 4

Proposition 1. *Let $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$. Then (1.12) and (1.13) produce the fractional integral operators (1.8) and (1.9) as follows:*

$$\left(F_{a^+}^{\frac{t^\alpha}{\Gamma(\alpha)},g} f\right)(x) := {}_g^\alpha I_{a^+} f(x), \quad \left(F_{b^-}^{\frac{t^\alpha}{\Gamma(\alpha)},g} f\right)(x) := {}_g^\alpha I_{b^-} f(x).$$

Further they satisfy the following bound for $\alpha \geq 1$:

$$({}_g^\alpha I_{a^+} f)(x) + ({}_g^\alpha I_{b^-} f)(x)$$
$$\leq \frac{(g(x) - g(a))^{\alpha-1}}{\Gamma(\alpha)}\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(mf\left(\frac{x}{m}\right) - f(a)\right){}^\alpha I_{a^+} f(x)\right)$$
$$+ \frac{(g(b) - g(x))^{\alpha-1}}{\Gamma(\alpha)}\left[f(b)g(b) - mf\left(\frac{x}{m}\right)g(x) - \frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(f(b) - mf\left(\frac{x}{m}\right)\right){}^\alpha I_{b^-} f(x)\right].$$

Proposition 2. *Let $g(x) = I(x) = x$. Then (1.12) and (1.13) produce integral operators defined in [36] as follows:*

$$(F_{a^+}^{\phi,I} f)(x) := ({}_{a^+}I_\phi f)(x) = \int_a^x \frac{\phi(x-t)}{(x-t)} f(t)dt, \tag{3.1}$$

$$(F_{b^-}^{\phi,I}f)(x) := ({}_{b^-}I_\phi f)(x) = \int_x^b \frac{\phi(t-x)}{(t-x)} f(t)dt. \tag{3.2}$$

Further they satisfy the following bound:

$$\begin{aligned}
&({}_{a^+}I_\phi f)(x) + ({}_{b^-}I_\phi f)(x)\\
&\le \frac{\phi(x-a))}{(x-a)^{\alpha+1}}\Big[(x-a)^\alpha\Big(mf\Big(\frac{x}{m}\Big)g(x) - f(a)g(a)\Big) - \Gamma(\alpha+1)\Big(mf\Big(\frac{x}{m}\Big) - f(a)\Big)\,{}^\alpha I_{a^+}f(x)\Big]\\
&+ \frac{\phi(b-x))}{(b-x)^{\alpha+1}}\Big[(b-x)^\alpha\Big(f(b)g(b) - mf\Big(\frac{x}{m}\Big)g(x)\Big) - \Gamma(\alpha+1)\Big(f(b) - mf\Big(\frac{x}{m}\Big)\Big)\,{}^\alpha I_{b^-}f(x)\Big].
\end{aligned}$$

Corollary 4. *If we take $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$. Then (1.12) and (1.13) produce the fractional integral operators (1.10) and (1.11) as follows:*

$$\left(F_{a^+}^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},g}f\right)(x) :=_g^\alpha I_{a^+}^k f(x), \quad \left(F_{b^-}^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},g}f\right)(x) :=_g^\alpha I_{b^-}^k f(x).$$

Further, the following bound holds for $\alpha \ge k$:

$$\begin{aligned}
&({}_g^\alpha I_{a^+}^k f)(x) + ({}_g^\alpha I_{b^-}^k f)(x)\\
&\le \frac{(g(x)-g(a))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\Big[\Big(mf\Big(\frac{x}{m}\Big)g(x) - f(a)g(a)\Big) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\Big(mf\Big(\frac{x}{m}\Big) - f(a)\Big)\,{}^\alpha I_{a^+}f(x)\Big]\\
&+ \frac{(g(b)-g(x))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\Big[\Big(f(b)g(b) - mf\Big(\frac{x}{m}\Big)g(x)\Big) - \frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\Big(f(b) - mf\Big(\frac{x}{m}\Big)\Big)\,{}^\alpha I_{b^-}f(x)\Big].
\end{aligned}$$

Corollary 5. *If we take $\phi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\alpha > 0$ and $g(x) = I(x) = x$. Then (1.12) and (1.13) produce left and right Riemann-Liouville fractional integral operators (1.4) and (1.5) as follows:*

$$\left(F_{a^+}^{\frac{t^\alpha}{\Gamma(\alpha)},I}f\right)(x) := {}^\alpha I_{a^+}f(x), \quad \left(F_{b^-}^{\frac{t^\alpha}{\Gamma(\alpha)},I}f\right)(x) := {}^\alpha I_{b^-}f(x).$$

Further, the following bound holds for $\alpha \ge 1$:

$$\begin{aligned}
&({}^\alpha I_{a^+}f)(x) + ({}^\alpha I_{b^-}f)(x)\\
&\le \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\Big[\Big(mf\Big(\frac{x}{m}\Big)g(x) - f(a)g(a)\Big) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\Big(mf\Big(\frac{x}{m}\Big) - f(a)\Big)\,{}^\alpha I_{a^+}f(x)\Big]\\
&+ \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}\Big[\Big(f(b)g(b) - mf\Big(\frac{x}{m}\Big)g(x)\Big) - \frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\Big(f(b) - mf\Big(\frac{x}{m}\Big)\Big)\,{}^\alpha I_{b^-}f(x)\Big].
\end{aligned}$$

Corollary 6. *If we take $\phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $g(x) = I(x) = x$. Then (1.12) and (1.13) produce the fractional integral operators (1.6) and (1.7) as follows:*

$$\left(F_{a^+}^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},I}f\right)(x) := {}^\alpha I_{a^+}^k f(x), \quad \left(F_{b^-}^{\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},I}f\right)(x) := {}^\alpha I_{b^-}^k f(x).$$

Further, the following bound holds for $\alpha \geq k$:

$$
\begin{aligned}
&({}^{\alpha}I^{k}_{a^+}f)(x) + ({}^{\alpha}I^{k}_{b^-}f)(x) \\
&\leq \frac{(x-a)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\Big[\Big(mf\Big(\frac{x}{m}\Big)g(x) - f(a)g(a)\Big) - \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\Big(mf\Big(\frac{x}{m}\Big) - f(a)\Big)\,{}^{\alpha}I_{a^+}f(x)\Big] \\
&+ \frac{(b-x)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\Big[\Big(f(b)g(b) - mf\Big(\frac{x}{m}\Big)g(x)\Big) - \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\Big(f(b) - mf\Big(\frac{x}{m}\Big)\Big)\,{}^{\alpha}I_{b^-}f(x)\Big].
\end{aligned}
$$

Corollary 7. *If we take $\phi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, $\alpha > 0$ and $g(x) = \frac{x^{\rho}}{\rho}$, $\rho > 0$. Then (1.12) and (1.13) produce the fractional integral operators defined in [20], as follows:*

$$
\Big(F^{\frac{t^{\alpha}}{\Gamma(\alpha)},g}_{a^+}f\Big)(x) = ({}^{\rho}I^{\alpha}_{a^+}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{\rho} - t^{\rho})^{\alpha-1}t^{\rho-1}f(t)dt, \tag{3.3}
$$

$$
\Big(F^{\frac{t^{\alpha}}{\Gamma(\alpha)},g}_{b^-}f\Big)(x) = ({}^{\rho}I^{\alpha}_{b^-}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^{\rho} - x^{\rho})^{\alpha-1}t^{\rho-1}f(t)dt. \tag{3.4}
$$

Further, they satisfy the following bound:

$$
\begin{aligned}
&({}^{\rho}I^{\alpha}_{a^+}f)(x) + ({}^{\rho}I^{\alpha}_{b^-}f)(x) \\
&\leq \frac{(x^{\rho}-a^{\rho})^{\alpha-1}}{\Gamma(\alpha)\rho^{\alpha-1}}\Big[\Big(mf\Big(\frac{x}{m}\Big)g(x) - f(a)g(a)\Big) - \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\Big(mf\Big(\frac{x}{m}\Big) - f(a)\Big)\,{}^{\alpha}I_{a^+}f(x)\Big] \\
&+ \frac{(b^{\rho}-x^{\rho})^{\alpha-1}}{\Gamma(\alpha)\rho^{\alpha-1}}\Big[\Big(f(b)g(b) - mf\Big(\frac{x}{m}\Big)g(x)\Big) - \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\Big(f(b) - mf\Big(\frac{x}{m}\Big)\Big)\,{}^{\alpha}I_{b^-}f(x)\Big].
\end{aligned}
$$

Corollary 8. *If we take $\phi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, $\alpha > 0$ and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0$. Then (1.12) and (1.13) produce the fractional integral operators define as follows:*

$$
\Big(F^{\frac{t^{\alpha}}{\Gamma(\alpha)},g}_{a^+}f\Big)(x) = ({}^{s}I^{\alpha}_{a^+}f)(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_a^x (x^{s+1} - t^{s+1})^{\alpha-1}t^{s}f(t)dt, \tag{3.5}
$$

$$
\Big(F^{\frac{t^{\alpha}}{\Gamma(\alpha)},g}_{b^-}f\Big)(x) = ({}^{s}I^{\alpha}_{b^-}f)(x) = \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)}\int_x^b (t^{s+1} - x^{s+1})^{\alpha-1}t^{s}f(t)dt. \tag{3.6}
$$

Further, they satisfy the following bound:

$$
\begin{aligned}
&({}^{s}I^{\alpha}_{a^+}f)(x) + ({}^{s}I^{\alpha}_{b^-}f)(x) \\
&\leq \frac{(x^{s+1}-a^{s+1})^{\alpha-1}}{\Gamma(\alpha)(s+1)^{\alpha-1}}\Big[\Big(mf\Big(\frac{x}{m}\Big)g(x) - f(a)g(a)\Big) - \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\Big(mf\Big(\frac{x}{m}\Big) - f(a)\Big)\,{}^{\alpha}I_{a^+}f(x)\Big] \\
&+ \frac{(b^{s+1}-x^{s+1})^{\alpha-1}}{\Gamma(\alpha)(s+1)^{\alpha-1}}\Big[\Big(f(b)g(b) - mf\Big(\frac{x}{m}\Big)g(x)\Big) - \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\Big(f(b) - mf\Big(\frac{x}{m}\Big)\Big)\,{}^{\alpha}I_{b^-}f(x)\Big].
\end{aligned}
$$

3.3. Boundedness and continuity

We apply Theorem 1 and prove the boundedness and continuity of integral operators.

Theorem 7. *Under the assumptions of Theorem 1, the following inequality holds for m-convex functions:*

$$(F_{a^+}^{\phi,g}f)(x) + (F_{b^-}^{\phi,g}f)(x) \leq \left(\phi(g(x)-g(a))\left(mf\left(\frac{x}{m}\right)+f(a)\right)+\phi(g(b)-g(x))\left(f(b)+mf\left(\frac{x}{m}\right)\right)\right). \quad (3.7)$$

Proof. If we put $h(x) = x$ in (2.6), we have

$$\int_a^x K_g(x,t;\phi)g'(t)f(t)dt \leq f(a)K_g(x,a;\phi)\int_a^x\left(\frac{x-t}{x-a}\right)g'(t)dt + mf\left(\frac{x}{m}\right)K_g(x,a;\phi)\int_a^x\left(\frac{t-a}{x-a}\right)g'(t)dt. \quad (3.8)$$

This further takes form as follows:

$$(F_{a^+}^{\phi,g}f)(x) \leq \phi(g(x)-g(a))\left(mf\left(\frac{x}{m}\right)+f(a)\right). \quad (3.9)$$

Similarly, from (2.10) we have

$$(F_{b^-}^{\phi,g}f)(x) \leq \phi(g(b)-g(x))\left(mf\left(\frac{x}{m}\right)+f(b)\right). \quad (3.10)$$

From (3.9) and (3.10), (3.7) can be obtained. □

Corollary 9. *If $m = 1$, then the following inequality holds for convex functions:*

$$(F_{a^+}^{\phi,g}f)(x) + (F_{b^-}^{\phi,g}f)(x) \leq \left(\phi(g(x)-g(a))\left(f(x)+f(a)\right)+\phi(g(b)-g(x))\left(f(b)+f(x)\right)\right). \quad (3.11)$$

Theorem 8. *With assumptions of Theorem 7, if $f \in L_\infty[a,b]$, then integral operators (1.12) and (1.13) for m-convex functions are bounded and continuous.*

Proof. From (3.9), we have

$$\left|(F_{a^+}^{\phi,g}f)(x)\right| \leq \phi(g(x)-g(a))(m+1)\|f\|_\infty,$$

which further gives $\left|(F_{a^+}^{\phi,g}f)(x)\right| \leq K\|f\|_\infty$, where $K = (m+1)\phi(g(b)-g(a))$. Similarly, from (3.10) we get $\left|(F_{b^-}^{\phi,g}f)(x)\right| \leq K\|f\|_\infty$. Hence the integral operators (1.12) and (1.13) are bounded, also they are linear. Therefore continuity is followed. □

Corollary 10. *If $m = 1$ and $f \in L_\infty[a,b]$, then integral operators (1.12) and (1.13) for convex functions are bounded and continuous:*

$$\left|(F_{a^+}^{\phi,g}f)(x)\right| \leq K\|f\|_\infty,$$

where $K = 2\phi(g(b)-g(a))$. Similarly we have $\left|(F_{b^-}^{\phi,g}f)(x)\right| \leq K\|f\|_\infty$.

4. Conclusions

An integral operator has been studied for $(h-m)$-convex and (α, m)-convex functions. The bounds of this operator are obtained in different forms. The Hadamard inequality is presented for $(h-m)$-convex as well as (α, m)-convex functions. Further the results are deducible for various kinds of known functions given in Remarks 1 & 2. Some of the results are applied to obtain bounds of different kinds of conformable and fractional integral operators. The boundedness and continuity of integral operators (1.12) and (1.13) are given for m-convex and convex functions. Reader can deduce results for integral operators comprised in Remarks 3 for all functions given in Remarks 1 & 2.

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Conflict of interests

It is declared that authors have no competing interests.

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