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Research article

Gould-Hopper matrix-Bessel and Gould-Hopper matrix-Tricomi functions and related integral representations

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Abstract: The paper performs an investigation on the new class of functions, namely the Gould-Hopper-Bessel matrix functions and Gould-Hopper-Tricomi matrix functions via operational methods. The generating functions, operational representations and connection formulae are established. The generalized forms of the Gould-Hopper-Bessel matrix and Gould-Hopper-Tricomi matrix functions are introduced using integral transform. Several important properties related to these functions are also deduced.

Keywords: Gould-Hopper matrix polynomials; Bessel functions; Tricomi functions; Euler's integral; operational methods
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1. Introduction

The theory of special functions performs an essential role in the formalism of mathematical physics. The Bessel functions [16] are one of the most important special functions and have applications in number theory, lie theory and theoretical astronomy to some problems of engineering and physics. The Bessel functions $J_n(x)$ are specified by means of the following generating equation [16]:

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \quad t \neq 0;\ |x| < \infty \tag{1.1}$$

and have the following series form:

$$J_n(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k!\,\Gamma(1+n+k)}, \;\; |x| < \infty, \tag{1.2}$$

where n is a positive integer or zero.

The Tricomi functions are another important special function due to their intrinsic mathematical importance in numerous branches of applied mathematics and mathematical physics and possess the following series form:

$$C_n(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k\, x^k}{k!\,(n+k)!}. \tag{1.3}$$

The Tricomi function $C_n(x)$ are linked to the Bessel function $J_n(x)$ [16] as:

$$C_n(x) = x^{-n/2} J_n(2\sqrt{x}), \tag{1.4}$$

or

$$J_n(x) = \left(\frac{x}{2}\right)^n C_n\left(\frac{x^2}{4}\right) \tag{1.5}$$

and possess the following generating equation:

$$\exp\left(t - \frac{x}{t}\right) = \sum_{n=-\infty}^{\infty} C_n(x) t^n. \tag{1.6}$$

In last few decades, an increasing interest in the area of matrix special polynomials have been noticed and many fascinating results established for classical orthogonal polynomials have been extended to orthogonal matrix special polynomials [9, 10]. The introduction of the Gould-Hopper matrix polynomials (GHMaP) $\mathsf{g}_n^m(x, y; A, B)$ is very much captivating because of their intrinsic mathematical importance. The GHMaP $\mathsf{g}_n^m(x, y; A, B)$ possess the following generating function [2]:

$$\sum_{n=0}^{\infty} \mathsf{g}_n^m(x, y; A, B) \frac{t^n}{n!} = \exp(xt\sqrt{2A}) \exp(Byt^m), \tag{1.7}$$

where A, B are matrices in $\mathbb{C}^{N\times N}$, such that A is positive stable and m is a positive integer and specified by the following series expansion:

$$\mathsf{g}_n^m(x, y; A, B) = \sum_{k=0}^{[\frac{n}{m}]} \frac{n!\,(\sqrt{2A})^{n-mk} B^k}{(n-mk)!\,k!} x^{n-mk} y^k. \tag{1.8}$$

The special matrix polynomials are very essential as they turn out in matrix expansion problems, representation and prediction theory and in the matrix quadrature integration problems, see for instance [7, 8, 13]. The matrix polynomials have connection and applications in spectral analysis [12] and scattering theory [11].

Doing some calculations, we find that GHMaP $\mathsf{g}_n^m(x, y; A, B)$ are quasi-monomial [3, 4, 18] under the action of the following multiplicative and derivative operators:

$$\hat{M}_{\mathsf{g}} = x\sqrt{2A} + mBy(\sqrt{2A})^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}} \tag{1.9}$$

and

$$\hat{P}_{\mathsf{g}} = (\sqrt{2A})^{-1}\hat{D}_x, \tag{1.10}$$

respectively.

Many enthralling results for special polynomials must be determined in view of operational rules related to the appropriate multiplicative and derivative operators. The principle of quasi-monomiality are exploited to investigate the properties of new families of special polynomials [1]. A family of hybrid polynomials reveals a nature lying between two polynomials families which are constructed by means of appropriate operational rules.

Due to significance of the two variable forms of special matrix polynomials in various fields of mathematics and engineering, here in this article, we introduce certain mixed type special matrix functions. In Section 2, the Gould-Hopper matrix polynomials combined with the Bessel functions and Tricomi functions respectively, to construct the new class of hybrid matrix functions. The generating functions and operational representations for the Gould-Hopper-Bessel matrix functions and Gould-Hopper-Tricomi matrix functions are established. In Section 3, the generalized forms of the Gould-Hopper-Bessel matrix functions and Gould-Hopper-Tricomi matrix functions are derived via integral transforms and operational rule.

2. Gould-Hopper-Bessel matrix and Gould-Hopper-Tricomi matrix functions

First, the generating function of the Gould-Hopper-Bessel matrix functions (GHBMaF) ${}_{\mathsf{g}}J_n^m(x, y; A, B)$ is obtained by proving the following result:

Theorem 2.1. *For the Gould-Hopper-Bessel matrix functions ${}_{\mathsf{g}}J_n^m(x, y; A, B)$, the following generating function holds true:*

$$\exp\left(\frac{x\sqrt{2A}}{2}\left(t - \frac{1}{t}\right) + \frac{By}{2^m}\left(t - \frac{1}{t}\right)^m\right) = \sum_{n=-\infty}^{\infty} {}_{\mathsf{g}}J_n^m(x, y; A, B)\, t^n. \tag{2.1}$$

Proof. Replacing x by multiplicative operator $\hat{M}_{\mathsf{g}}$ of the Gould-Hopper matrix polynomials $\mathsf{g}_n^m(x, y; A, B)$ in Eq (1.1) and denoting the resultant Gould-Hopper-Bessel matrix functions in the r.h.s by ${}_{\mathsf{g}}J_n^m(x, y; A, B)$, it follows that

$$\exp\left(\frac{\hat{M}_{\mathsf{g}}}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} {}_{\mathsf{g}}J_n^m(x, y; A, B)\, t^n.$$

$$\exp\left(\left(x\sqrt{2A} + mBy\left(\sqrt{2A}\right)^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)\frac{1}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} {}_{\mathsf{g}}J_n^m(x, y; A, B)\, t^n. \tag{2.2}$$

Using Crofton-type identity [5, p. 12]:

$$f\Big(x+m\lambda\frac{d^{m-1}}{dx^{m-1}}\Big)\{1\}=\exp\Big(\lambda\frac{d^m}{dx^m}\Big)\{f(x)\},\tag{2.3}$$

in the l.h.s. of Eq (2.2), we get

$$\exp\left(By\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)\exp\left(\frac{x\sqrt{2A}}{2}\left(t-\frac{1}{t}\right)\right)=\sum_{n=-\infty}^{\infty}{}_{g}J_n^m(x,y;A,B)\,t^n,$$

which on simplification yields assertion (2.1).

Remark 2.1. *Taking $y=0$ in generating Eq* (2.1)*, we get*

$${}_{g}J_n^m(x,0;A,B)=J_n(x\sqrt{2A}),\tag{2.4}$$

which on taking $A=\frac{1}{2}I$ reduces to the Bessel function $J_n(x)$.

Remark 2.2. *On differentiating generating function* (2.1) *with respect to x, the following differential recurrence relation for Gould-Hopper-Bessel matrix functions ${}_{g}J_n^m(x,y;A,B)$ holds true:*

$$2\frac{\partial}{\partial x}\,{}_{g}J_n^m(x,y;A,B)=\sqrt{2A}\left({}_{g}J_{n-1}^m(x,y;A,B)-{}_{g}J_{n+1}^m(x,y;A,B)\right).\tag{2.5}$$

The operational representation for the GHBMaF ${}_{g}J_n^m(x,y;A,B)$ is obtained in the following theorem:

Theorem 2.2. *The following operational representation for the Gould-Hopper-Bessel matrix functions ${}_{g}J_n^m(x,y;A,B)$ holds true:*

$${}_{g}J_n^m(x,y;A,B)=\exp\left(yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)J_n\left(x\sqrt{2A}\right).\tag{2.6}$$

Proof. In view of generating relation (2.1), we can write

$$\frac{\partial^m}{\partial x^m}\left(\sqrt{2A}\right)^{-m}B\,{}_{g}J_n^m(x,y;A,B)=\frac{\partial}{\partial y}\,{}_{g}J_n^m(x,y;A,B).\tag{2.7}$$

Solving Eq (2.7) along with initial condition (2.4), assertion (2.6) is proved.

Lemma 2.3. *The following connection formulae for the Gould-Hopper-Bessel matrix functions ${}_{g}J_n^m(x,y;A,B)$ hold true:*

$${}_{g}J_n^m(x_1+x_2,y;A,B)=\sum_{l=-\infty}^{\infty}{}_{g}J_{n-l}^m(x_1,y;A,B)\;\;J_l(x_2\sqrt{2A})\tag{2.8}$$

$${}_{g}J_n^m(x_1-x_2,y;A,B)=\sum_{l=-\infty}^{\infty}(-1)^l{}_{g}J_{n-l}^m(x_1,y;A,B)\;\;J_l(x_2\sqrt{2A})\tag{2.9}$$

$${}_{g}J_n^m(x_1+x_2,y_1+y_2;A,B)=\sum_{l=-\infty}^{\infty}{}_{g}J_{n-l}^m(x_1,y_1;A,B)\,{}_{g}J_l^m(x_2,y_2;A,B).\tag{2.10}$$

Proof. Replacing x by $x_1 + x_2$ in generating Eq (2.1), it follows that

$$\begin{aligned}&\exp\left(\frac{x_1\sqrt{2A}}{2}\left(t-\frac{1}{t}\right)+\frac{By}{2^m}\left(t-\frac{1}{t}\right)^m\right)\exp\left(\frac{x_2\sqrt{2A}}{2}\left(t-\frac{1}{t}\right)\right)\\&=\sum_{n=-\infty}^{\infty}{}_{\mathsf{g}}J_n^m(x_1+x_2,y;A,B)\,t^n,\end{aligned}\tag{2.11}$$

which on making use of Eqs (1.1) and (2.1) and comparison of coefficients of same powers of t in both sides of the obtained relation gives assertion (2.8).

Next, replacing x by $x_1 - x_2$ in generating Eq (2.1) and proceeding same as above, assertion (2.9) follows. Again, replacing x by $x_1 + x_2$ and y by $y_1 + y_2$ in generating Eq (2.1) and proceeding same as above, assertion (2.10) is proved.

Remark 2.3. *Taking $t = \exp(i\phi)$ in generating Eq (2.1), the following Jacobi-Anger type expansion of the GHBMaF ${}_{\mathsf{g}}J_n^m(x,y;A,B)$ is obtained:*

$$\exp\left(x\sqrt{2A}(i\sin\phi)+By(i\sin\phi)^m\right)=\sum_{n=-\infty}^{\infty}{}_{\mathsf{g}}J_n^m(x_1+x_2,y;A,B)\exp(in\phi),\tag{2.12}$$

which can takes the form

$$\cos\alpha+i\sin\alpha=\sum_{n=-\infty}^{\infty}{}_{\mathsf{g}}J_n^m(x_1+x_2,y;A,B)\exp(in\phi),\tag{2.13}$$

where $\alpha = x\sqrt{2A}\sin\phi + Byi^{m-1}\sin^m\phi$. Also, we have

$$\cos\alpha=\sum_{n=-\infty}^{\infty}{}_{\mathsf{g}}J_n^m(x,y;A,B)(\cos n\phi),\tag{2.14}$$

$$\sin\alpha=\sum_{n=-\infty}^{\infty}{}_{\mathsf{g}}J_n^m(x,y;A,B)(\sin n\phi).\tag{2.15}$$

The Jacobi-Anger expansion is useful in physics (in conversion of plane waves and the cylindrical waves) and in signal processing (to describe frequency modulation signals).

Next, the generating function of the GHTMaF ${}_{\mathsf{g}}C_n^m(x,y;A,B)$ is obtained by proving the following result:

Theorem 2.4. *The Gould-Hopper-Tricomi matrix functions ${}_{\mathsf{g}}C_n^m(x,y;A,B)$ are specified by the following generating function:*

$$\exp\left(t-\frac{x\sqrt{2A}}{t}+\frac{By}{(-t)^m}\right)=\sum_{n=-\infty}^{\infty}{}_{\mathsf{g}}C_n^m(x,y;A,B)\,t^n.\tag{2.16}$$

Proof. Replacing x by multiplicative operator $\hat{M}_{\mathsf{g}}$ of the Gould-Hopper matrix polynomials $\mathsf{g}_n^m(x,y;A,B)$ in generating relation (1.6) and denoting the resultant GHTMaF in the r.h.s by ${}_{\mathsf{g}}C_n^m(x,y;A,B)$, it follows that

$$\exp\left(t-\frac{\hat{M}_{\mathsf{g}}}{t}\right)=\sum_{n=-\infty}^{\infty}{}_{\mathsf{g}}C_n^m(x,y;A,B)\,t^n.$$

$$\exp\left(t-\left(x\sqrt{2A}+mBy\left(\sqrt{2A}\right)^{-(m-1)}\frac{\partial^{m-1}}{\partial x^{m-1}}\right)\frac{1}{(-t)}\right)=\sum_{n=-\infty}^{\infty}{}_{\mathrm{g}}C_n^m(x,y;A,B)\,t^n, \tag{2.17}$$

which on using the Crofton-type identity proves assertion (2.16).

Remark 2.4. *Taking $y=0$ in generating Eq* (2.16)*, we get*

$$_{g}C_n^m(x,0;A,B)=C_n(x\sqrt{2A}), \tag{2.18}$$

which on taking $A=\frac{1}{2}I$ reduces to the Tricomi function $C_n(x)$.

The series representation for the GHTMaF $_{\mathrm{g}}C_n^m(x,y;A,B)$ is obtained by proving the following result:

Theorem 2.5. *For the Gould-Hopper-Tricomi matrix functions $_{g}C_n^m(x,y;A,B)$, the following series expansion holds true:*

$$_{g}C_n^m(x,y;A,B)=\sum_{k=0}^{\infty}\sum_{l=0}^{\left[\frac{k}{m}\right]}\frac{\left(x\sqrt{2A}\right)^{k-ml}(By)^l(-1)^k}{(k-ml)!\;l!\;(n+k)!}. \tag{2.19}$$

Proof. Using Eq (1.7) in generating relation (2.16), we find

$$\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\mathrm{g}_k^m(x,y;A,B)\frac{1}{(-t)^k\;k!}\right)=\sum_{n=-\infty}^{\infty}{}_{\mathrm{g}}C_n^m(x,y;A,B)t^n.$$

Comparing the coefficients of identical powers of t and on using relation (1.8), we get assertion (2.19).

The operational representations for the GHTMaF $_{\mathrm{g}}C_n^m(x,y;A,B)$ are obtained by proving the following results:

Theorem 2.6. *For the Gould-Hopper-Tricomi matrix functions $_{g}C_n^m(x,y;A,B)$, the following operational representation holds true:*

$$_{g}C_n^m(x,y;A,B)=\exp\left(yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)C_n\left(x\sqrt{2A}\right). \tag{2.20}$$

Proof. Differentiating generating relation (2.16) w.r.t. x and y respectively, we find

$$\frac{\partial^m}{\partial x^m}\left(\sqrt{2A}\right)^{-m}B\,{}_{\mathrm{g}}C_n^m(x,y;A,B)=\frac{\partial}{\partial y}\,{}_{\mathrm{g}}C_n^m(x,y;A,B). \tag{2.21}$$

Solving Eq (2.21) along with initial condition (2.18), assertion (2.20) follows.

An immediate consequence of Theorem 2.6 is given in the form of the following result:

Corollary 2.1. *The following operational representation between two forms of the Gould-Hopper-Tricomi matrix functions $_{g}C_n^m(x,y;A,B)$ holds true:*

$$_{g}C_n^m(x,y+z;A,B)=\exp\left(zB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right){}_{g}C_n^m(x,y;A,B). \tag{2.22}$$

Lemma 2.7. *The Gould-Hopper-Tricomi matrix functions* ${}_{g}C_n^m(x,y;A,B)$ *satisfies the following recurrence relation:*

$$\begin{aligned} {}_{g}C_{n+1}^m(x,y;A,B) =& \frac{1}{n+1}\Big({}_{g}C_n^m(x,y;A,B) + x\sqrt{2A}\,{}_{g}C_{n+2}^m(x,y;A,B) \\ & -mBy\,{}_{g}C_{n+m-1}^m(x,y;A,B)\Big). \end{aligned} \tag{2.23}$$

Proof. Differentiating generating relation (2.16) w.r.t. t and equating the coefficients of same powers of t in both sides, we get assertion (2.23).

In the next section, generalized form of the GHBMaF ${}_{g}J_n^m(x,y;A,B)$ and GHTMaF ${}_{g}C_n^m(x,y;A,B)$ are introduced by making use of integral transform and their properties are established.

3. Integral representations of Gould-Hopper-Bessel matrix functions and Gould-Hopper-Tricomi matrix functions

In recent years, the generalized and many-variable special functions have witnessed a significant evolution. These functions are proved to be very significant in purely mathematical and applied frameworks. The combined use of integral transforms and special polynomials imparts a powerful technique to deal with fractional derivatives, see for example [1, 6]. To detect the operational rule and generating relations for the generalized form of special polynomials, Dattoli and co-authors used the Euler's integral in [6]. The Euler's integral is given by [17, p.218]:

$$a^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}e^{-at}dt, \qquad \min\{Re(\nu), Re(a)\} \geq 0, \tag{3.1}$$

which consequently yields the following [6]:

$$\left(\alpha - \frac{\partial}{\partial x}\right)^{-\nu} f(x) = \frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}e^{-\alpha t}f(x+t)dt. \tag{3.2}$$

In order to introduce the generalized form of the GHBMaF ${}_{g}J_n^m(x,y;A,B)$, we give the following definition:

Definition 3.1. *The following operational rule for the generalized Gould-Hopper-Bessel matrix functions* ${}_{g}J_{n,\nu}^m(x,y;A,B;\alpha)$ *holds true:*

$$\left(\alpha - yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)^{-\nu} J_n(x\sqrt{2A}) = {}_{g}J_{n,\nu}^m(x,y;A,B;\alpha). \tag{3.3}$$

Theorem 3.1. *The following integral representation for the generalized Gould-Hopper-Bessel matrix functions* ${}_{g}J_{n,\nu}^m(x,y;A,B;\alpha)$ *holds true:*

$${}_{g}J_{n,\nu}^m(x,y;A,B;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\,{}_{g}J_n^m(x,yt;A,B)dt. \tag{3.4}$$

Proof. Replacing a by $\left(\alpha - yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)$ in relation (3.1) and then operating the resultant equation on $J_n(x\sqrt{2A})$, it follows that

$$\begin{aligned}\left(\alpha - yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)^{-\nu} J_n(x\sqrt{2A})\\ = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\exp\left(yBt\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)J_n(x\sqrt{2A})dt,\end{aligned} \tag{3.5}$$

which on using Eq (2.6) gives

$$\left(\alpha - yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)^{-\nu} J_n(x\sqrt{2A}) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}\ {}_{\mathsf{g}}J_n^m(x, yt; A, B)dt. \tag{3.6}$$

Indicating the transform on the r.h.s of Eq (3.6) by a new class of the generalized Gould-Hopper-Bessel matrix functions (gGHBMaF), denoted by ${}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha)$, we are led to assertion (3.4).

Next, the generating function of the gGHBMaF ${}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha)$ is obtained by proving the following theorem:

Theorem 3.2. *The following generating function for the generalized Gould-Hopper-Bessel matrix functions* ${}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha)$ *holds true:*

$$\frac{\exp\left(x\sqrt{2A}\left(u - \frac{1}{u}\right)\right)}{\left(\alpha - \frac{By}{2^m}\left(u - \frac{1}{u}\right)^m\right)^\nu} = \sum_{n=-\infty}^{\infty} {}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha)u^n. \tag{3.7}$$

Proof. Multiplying both sides of Eq (3.4) by u^n and summing over n, we find

$$\sum_{n=-\infty}^{\infty} {}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha)u^n = \sum_{n=-\infty}^{\infty}\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}{}_{\mathsf{g}}J_n^m(x, yt; A, B)u^n dt, \tag{3.8}$$

which on using Eq (2.1) in the r.h.s. gives

$$\sum_{n=-\infty}^{\infty} {}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha)u^n = \frac{\exp\left(\frac{x\sqrt{2A}}{2}\left(u - \frac{1}{u}\right)\right)}{\Gamma(\nu)}\int_0^\infty \exp\left(\frac{Byt}{2^m}\left(u - \frac{1}{u}\right)\right)e^{-\alpha t}t^{\nu-1}dt. \tag{3.9}$$

Using Eq (3.1) in the r.h.s. of above equation, we get assertion (3.7).

Differentiating generating function (3.7) w.r.t. α and x respectively, the following matrix recurrence relations for the generalized Gould-Hopper-Bessel matrix functions ${}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha)$ are obtained:

$$\frac{\partial}{\partial\alpha}\ {}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha) = -\nu\ {}_{\mathsf{g}}J_{n,\nu+1}^m(x, y; A, B; \alpha) \tag{3.10}$$

and

$$2\frac{\partial}{\partial x}\ {}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha) = \sqrt{2A}\left({}_{\mathsf{g}}J_{n-1,\nu}^m(x, y; A, B; \alpha) - {}_{\mathsf{g}}J_{n+1,\nu}^m(x, y; A, B; \alpha)\right). \tag{3.11}$$

Remark 3.1. *For* $\alpha = 1$, $\nu = 1$ *and* $y = D_y^{-1}$, *the generalized Gould-Hopper-Bessel matrix functions* ${}_{\mathsf{g}}J_{n,\nu}^m(x, y; A, B; \alpha)$ *reduce to the Gould-Hopper-Bessel matrix functions* ${}_{\mathsf{g}}J_n^m(x, y; A, B)$.

Next, the operational rule, generating function and recurrence relation for the gGHTMaF ${}_{\mathrm{g}}C^{m}_{n,\nu}(x, y; A, B; \alpha)$ are obtained. First, we give the following definition:

Definition 3.2. *The following operational rule for the generalized Gould-Hopper-Tricomi matrix functions* ${}_{g}C^{m}_{n,\nu}(x, y; A, B; \alpha)$ *holds:*

$$\left(\alpha - yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)^{-\nu} C_n(x\sqrt{2A}) = {}_{g}C^{m}_{n,\nu}(x, y; A, B; \alpha). \tag{3.12}$$

Theorem 3.3. *The following integral representation for the generalized Gould-Hopper-Tricomi matrix functions* ${}_{g}C^{m}_{n,\nu}(x, y; A, B; \alpha)$ *holds:*

$${}_{g}C^{m}_{n,\nu}(x, y; A, B; \alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t} t^{\nu-1}\, {}_{g}C^{m}_{n}(x, yt; A, B)dt. \tag{3.13}$$

Proof. Replacing a by $\left(\alpha - yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)$ in relation (3.1) and then operating the resultant equation on $C_n(x\sqrt{2A})$, it follows that

$$\begin{aligned}\left(\alpha - yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)^{-\nu} & C_n(x\sqrt{2A}) \\ &= \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t} t^{\nu-1} \exp\left(yBt\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right) C_n(x\sqrt{2A})dt,\end{aligned} \tag{3.14}$$

which on using Eq (2.20) gives

$$\left(\alpha - yB\left(\sqrt{2A}\right)^{-m}\frac{\partial^m}{\partial x^m}\right)^{-\nu} C_n(x\sqrt{2A}) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t} t^{\nu-1}\, {}_{\mathrm{g}}C^{m}_{n}(x, yt; A, B)dt. \tag{3.15}$$

Indicating the transform on the r.h.s of Eq (3.15) by a new class of the generalized Gould-Hopper-Tricomi matrix functions (gGHTMaF), denoted by ${}_{\mathrm{g}}C^{m}_{n,\nu}(x, y; A, B; \alpha)$, assertion (3.13) is proved.

Next, the generating function of the gGHTMaF ${}_{\mathrm{g}}C^{m}_{n,\nu}(x, y; A, B; \alpha)$ is obtained by proving the following theorem:

Theorem 3.4. *The following generating function for the generalized Gould-Hopper-Tricomi matrix functions* ${}_{g}C^{m}_{n,\nu}(x, y; A, B; \alpha)$ *holds:*

$$\frac{\exp\left(u - \frac{x\sqrt{2A}}{u}\right)}{\left(\alpha - \frac{By}{(-u)^m}\right)^{\nu}} = \sum_{n=-\infty}^{\infty} {}_{g}C^{m}_{n,\nu}(x, y; A, B; \alpha)u^n. \tag{3.16}$$

Proof. Multiplying both sides of Eq (3.13) by u^n and summing over n, we find

$$\sum_{n=-\infty}^{\infty} {}_{\mathrm{g}}C^{m}_{n,\nu}(x, y; A, B; \alpha)u^n = \sum_{n=-\infty}^{\infty}\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t} t^{\nu-1}{}_{\mathrm{g}}C^{m}_{n}(x, yt; A, B)u^n dt, \tag{3.17}$$

which on using Eq (2.16) in the r.h.s. gives

$$\sum_{n=-\infty}^{\infty} {}_{\mathrm{g}}C^{m}_{n,\nu}(x, y; A, B; \alpha)u^n = \frac{\exp\left(u - \frac{x\sqrt{2A}}{u}\right)}{\Gamma(\nu)}\int_0^\infty \exp\left(\frac{Byt}{(-u)^m}\right) e^{-\alpha t} t^{\nu-1}dt. \tag{3.18}$$

Using Eq (3.1) in the r.h.s. of Eq (3.18), we get assertion (3.16).

Corollary 3.1. *Differentiating generating function* (3.16) *w.r.t.* α*, the following differential recurrence relation for the generalized Gould-Hopper-Tricomi matrix functions* ${}_gC^m_{n,\nu}(x,y;A,B;\alpha)$ *holds true:*

$$\frac{\partial}{\partial\alpha}\,{}_gC^m_{n,\nu}(x,y;A,B;\alpha) = -\nu\,{}_gC^m_{n,\nu+1}(x,y;A,B;\alpha) \tag{3.19}$$

Remark 3.2. *For* $\alpha = 1$*,* $\nu = 1$ *and* $y = D_y^{-1}$*, the generalized Gould-Hopper-Tricomi matrix functions* ${}_gC^m_{n,\nu}(x,y;A,B;\alpha)$ *reduce to the Gould-Hopper-Tricomi matrix functions* ${}_gC^m_n(x,y;A,B)$*.*

4. Conclusion

The mixed families of special matrix functions are introduced as discrete convolution of the known special polynomials and these newly introduced polynomials possess the same properties as the parent polynomials or functions. Therefore, the Gould-Hopper-Bessel matrix and Gould-Hopper-Tricomi matrix functions have same properties as the Bessel and Tricomi functions.

It is known that the Bessel functions arise in astronomical and mechanical problems; the relevant theory is formulated in a coherent and organic body, displaying the wealth of properties and the connections with other special functions [19]. Several problems of chemistry, physics and mechanics are related to the second order matrix differential equations and special matrix polynomials and functions are basically the solutions of several matrix differential equations [13–15]. So, the mixed type special matrix functions introduced in this article will perform an indispensable role in the analysis of numerous problems of physics and engineering.

The significance and applications of mixed special polynomials in mathematical physics and engineering provides motivation to investigate mixed type special matrix functions associated with the Gould-Hopper matrix polynomials. The approach presented in this article is general and imparts a powerful technique for examining the properties of the hybrid special matrix functions and can be extended to establish the properties of other generalized families of special matrix functions.

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Conflict of interest

No conflict of interest was declared by the authors.

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