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Research article

Instability of standing waves for the inhomogeneous Gross-Pitaevskii equation

Yongbin Wang[1] **and Binhua Feng**[2,*]

[1] Department of Basic Teaching and Research, Qinghai University, Xining, 810016, China
[2] Department of Mathematics, Northwest Normal University, Lanzhou, 730070, China

* **Correspondence:** Email: binhuaf@163.com; Tel: +8613919836383; Fax: +8609317971143.

Abstract: In this paper, we consider the instability of standing waves for an inhomogeneous Gross-Pitaevskii equation

$$i\psi_t + \Delta\psi - a^2|x|^2\psi + |x|^{-b}|\psi|^p\psi = 0.$$

This equation arises in the description of nonlinear waves such as propagation of a laser beam in the optical fiber. We firstly proved that there exists $\omega_* > 0$ such that for all $\omega > \omega_*$, the standing wave $\psi(t, x) = e^{i\omega t}u_\omega(x)$ is unstable. Then, we deduce that if $\partial^2_\lambda S_\omega(u^\lambda_\omega)|_{\lambda=1} \leq 0$, the ground state standing wave $e^{i\omega t}u_\omega(x)$ is strongly unstable by blow-up, where $u^\lambda_\omega(x) = \lambda^{\frac{N}{2}}u_\omega(\lambda x)$ and S_ω is the action. This result is a complement to the partial result of Ardila and Dinh (Z. Angew. Math. Phys. 2020), where the strong instability of standing waves has been studied under a different assumption.

Keywords: inhomogeneous Gross-Pitaevskii equation; strong instability; ground state
Mathematics Subject Classification: 35Q55, 35A15

1. Introduction

In this paper, we consider the following inhomogeneous Gross-Pitaevskii equation

$$\begin{cases} i\psi_t + \Delta\psi - a^2|x|^2\psi + |x|^{-b}|\psi|^p\psi = 0, \ (t, x) \in [0, T^*) \times \mathbb{R}^N, \\ \psi(0, x) = \psi_0(x), \end{cases} \tag{1.1}$$

where $\psi : [0, T^*) \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $\psi_0 \in \Sigma$, $T^* \in (0, \infty]$, $b \in (0, \min\{2, N\})$, $p \in (0, \frac{4-2b}{N-2})$. Thus, the Cauchy problem (1.1) is local well-posedness in the energy space Σ, where Σ is defined by

$$\Sigma := \left\{ u \in L^2, \ \ \nabla u \in L^2 \ and \ \int_{\mathbb{R}^N} |x|^2|u(x)|^2 dx < \infty \right\}$$

with the norm

$$\|u\|_{\Sigma}^{2} = \|\nabla u\|_{L^2}^{2} + \|u\|_{L^2}^{2} + \int_{\mathbb{R}^N} |x|^2 |u(x)|^2 dx.$$

When $\omega \in (-N, \infty)$, we notice that

$$N \int_{\mathbb{R}^N} |u(x)|^2 dx = -\sum_{j=1}^{j=N} \int_{\mathbb{R}^N} x_j \partial_{x_j} |u(x)|^2 dx \le 2 \sum_{j=1}^{j=N} \|x_j u\|_{L^2} \|\partial_{x_j} u\|_{L^2},$$

there exist positive constants $C_1(\omega)$ and $C_2(\omega)$ such that

$$C_1(\omega)\|u\|_{\Sigma}^{2} \le \|\nabla u\|_{L^2}^{2} + \omega \|u\|_{L^2}^{2} + a^2 \int_{\mathbb{R}^N} |x|^2 |u(x)|^2 dx \le C_2(\omega) \|u\|_{\Sigma}^{2} \tag{1.2}$$

for all $u \in \Sigma$.

The inhomogeneous Gross-Pitaevskii equation (1.1) arises naturally in nonlinear optics for the propagation of laser beams. (1.1) also appears in Bose-Einstein condensation, where the harmonic potential $|x|^2$ may model a confining magnetic potential. The nonlinearity $|x|^{-b}|\psi|^p\psi$ describes the propagation of waves in the inhomogeneous medium, see [1–3] for the related physical backgrounds.

Recently, this type of equations has been studied extensively in [4–14]. Eq (1.1) enjoys a class of special solutions, which are called standing waves, namely solutions of the form $e^{i\omega t}u_\omega(x)$, where $\omega \in \mathbb{R}$ is a frequency and $u_\omega \in \Sigma$ is a nontrivial solution to the elliptic equation

$$-\Delta u_\omega + \omega u_\omega + a^2|x|^2 u_\omega - |x|^{-b}|u_\omega|^p u_\omega = 0. \tag{1.3}$$

Note that (1.3) can be written as $S'_\omega(u_\omega) = 0$, where

$$S_\omega(u) := \frac{1}{2}\|\nabla u\|_{L^2}^{2} + \frac{\omega}{2}\|u\|_{L^2}^{2} + \frac{a^2}{2}\int_{\mathbb{R}^N} |x|^2|u(x)|^2 dx - \frac{1}{p+2}\int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{p+2} dx, \tag{1.4}$$

is the action functional. We also define the following functionals

$$K_\omega(u) := \partial_\lambda S_\omega(\lambda u)|_{\lambda=1} = \|\nabla u\|_{L^2}^{2} + \omega\|u\|_{L^2}^{2} + a^2\int_{\mathbb{R}^N} |x|^2|u(x)|^2 dx - \int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{p+2} dx, \tag{1.5}$$

$$Q(u) := \partial_\lambda S_\omega(u^\lambda)|_{\lambda=1} = \|\nabla u\|_{L^2}^{2} - a^2\int_{\mathbb{R}^N} |x|^2|u(x)|^2 dx - \frac{\alpha}{p+2}\int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{p+2} dx, \tag{1.6}$$

where $\alpha = \frac{Np}{2} + b$ and $u^\lambda(x) := \lambda^{\frac{N}{2}} u(\lambda x)$. The ground state for (1.3) is defined by

$$\mathcal{G}_\omega = \{u_\omega \in \mathcal{A}_\omega : \ \ S_\omega(u_\omega) \le S_\omega(v_\omega) \ \ for \ all \ v_\omega \in \mathcal{A}_\omega\} \tag{1.7}$$

where

$$\mathcal{A}_\omega = \{v_\omega \in \Sigma \setminus \{0\} : \ S'_\omega(v_\omega) = 0\}$$

is the set of all nontrivial solutions for (1.3).

We firstly recall the definitions of stability and instability of standing waves, see [15].

Definition 1.1. *Let* $\psi(t, x) = e^{i\omega t}u_\omega(x)$ *be a standing wave solution of* (1.1).

1. *The solution* $\psi(t,x) = e^{i\omega t}u_\omega(x)$ *is said to be orbitally stable if, for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any initial data* ψ_0 *satisfying* $\|\psi_0 - u_\omega\|_\Sigma < \delta$*, then the corresponding solution* $\psi(t)$ *of* (1.1) *exists globally in time and satisfies*

$$\sup_{t\geq 0}\inf_{\theta\in\mathbb{R}} \|\psi(t) - e^{i\theta}u_\omega\|_\Sigma < \varepsilon.$$

2. *The solution* $\psi(t,x) = e^{i\omega t}u_\omega(x)$ *is said to be unstable if* $\psi(t,x) = e^{i\omega t}u_\omega(x)$ *is not stable.*
3. *The solution* $\psi(t,x) = e^{i\omega t}u_\omega(x)$ *is said to strongly unstable if for any* $\varepsilon > 0$*, there exists* $\psi_0 \in \Sigma$ *such that* $\|\psi_0 - u_\omega\|_\Sigma < \varepsilon$*, and the corresponding solution* $\psi(t)$ *of* (1.1) *blows up in finite time.*

Next, we recall some known instability results for nonlinear Schrödinger equations. The strong instability was first studied by Berestycki and Cazenave, see [15]. Later, Le Coz in [16] gave an alternative, simple proof of the classical result of Berestycki and Cazenave. The key point is to establish the finite time blow-up by using the variational characterization of ground states as minimizers of the action functional and the virial identity. More precisely, based on the variational characterization of ground states on the Pohozaev manifold $\mathcal{N} := \{v \in H^1,\ \ Q(v) = 0\}$ or the Nehari manifold, ones can obtain the key estimate $Q(\psi(t)) \leq 2(S_\omega(\psi_0) - S_\omega(u_\omega))$, where u_ω is the ground state solution. Then, it follows from the virial identity and the choice of initial data ψ_0 that

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|\psi(t,x)|^2dx = 8Q(\psi(t)) \leq 16(S_\omega(\psi_0) - S_\omega(u_\omega)) < 0.$$

This implies that the solution $\psi(t)$ blows up in a finite time. Thus, ones can prove the strong instability of ground state standing waves, see, e.g., [15–33].

For the nonlinear Schrödinger equation with a harmonic potential, i.e., $b = 0$ in (1.1), by constructing the cross-invariant manifolds of the evolution flow and defining the cross-invariant variational problems, Zhang in [32] studied the sharp threshold of global existence and blow-up, and proved the strong instability of standing waves. Recently, by using the idea of Zhang in [32], Ardila and Dinh in [34] studied the strong instability of standing waves of (1.1). More precisely, when $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$, defining the following variational problems

$$d(\omega) = inf\{S_\omega(v):\ v \in \Sigma\backslash\{0\},\ \ K_\omega(v) = 0\}. \tag{1.8}$$

and

$$m(\omega) := \inf\{S_\omega(u):\ \ u_\omega \in \Sigma\backslash\{0\},\ Q(u) = 0 \textit{ and } K_\omega(u) < 0\},$$

Ardila and Dinh in [34] obtained some sharp thresholds of global existence and blow-up. Under the assumption $d(\omega) \leq m(\omega)$, they proved that the standing wave $e^{i\omega t}u_\omega(x)$ is strongly unstable. However, this assumption is still vague. It is hard to determine for which ω, $d(\omega) \leq m(\omega)$ is true, see also Remark 5.1 in [32].

Motivated by this work, we will further study the strong instability of standing waves of (1.1) from a different perspective. Let $u \in \Sigma$, we define

$$\begin{aligned} f(\lambda) :=& S_\omega(u^\lambda) = \frac{\lambda^2}{2}\|\nabla u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 \\ &+ \frac{a^2\lambda^{-2}}{2}\int_{\mathbb{R}^N}|x|^2|u(x)|^2dx - \frac{\lambda^\alpha}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(x)|^{p+2}dx. \end{aligned} \tag{1.9}$$

It is obvious that $f(\lambda) \to +\infty$, as $\lambda \to 0^+$. Thus, there is no maximum point of $f(\lambda)$ on $(0, \infty)$. It is hard to establish the variational characterization of ground states in the Pohozaev manifold $\mathcal{N}$. On the other hand, we define

$$\begin{aligned} f_1(\lambda) := & S_\omega(\lambda u) = \frac{\lambda^2}{2}\|\nabla u\|_{L^2}^2 + \frac{\omega\lambda^2}{2}\|u\|_{L^2}^2 \\ & + \frac{a^2\lambda^2}{2}\int_{\mathbb{R}^N} |x|^2|u(x)|^2 dx - \frac{\lambda^{p+2}}{p+2}\int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{p+2} dx. \end{aligned}$$

It is obvious that equation $f_1'(\lambda) = 0$ has unique solution $\lambda_0 > 0$, and $f_1(\lambda)$ has the unique maximum point on $(0, \infty)$. Based on this fact, ones can easily obtain the variational characterization of ground states in the Nehari manifold by using the compact embedding $\Sigma \hookrightarrow L^q$ with $q \in [2, 2^*)$. But it is difficult to obtain the key estimate $Q(\psi(t)) \le 2(S_\omega(\psi_0) - S_\omega(u_\omega))$.

Since there is no any maximum of function $f(\lambda)$ on $(0, \infty)$, we assume that $\lambda = 1$ is the local maximum point of $f(\lambda)$, i.e., $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \le 0$. In this assumption, we will study the strong instability of standing waves of (1.1). Moreover, we obtain that there exists $\omega_* > 0$ such that for all $\omega > \omega_*$, the standing wave $\psi(t, x) = e^{i\omega t}u_\omega(x)$ is unstable.

Firstly, we establish the variational characterization of the ground states of (1.3). To this aim, we recall the existence of minimizing problem (1.8) established by Ardila and Dinh in [34].

Lemma 1.2. *Let $a > 0$, $N \ge 1$, $0 < b < \min\{2, N\}$, $0 < p < \frac{4-2b}{N-2}$, $\omega > -aN$. Then $d(\omega) > 0$ and $d(\omega)$ is attained by a function which is a solution to the elliptic equation* (1.3). *Moreover, every minimizer is the form $e^{i\theta}u$, where $u \in \Sigma$ is a real-valued, positive and spherically symmetric function.*

Since the embedding $\Sigma \hookrightarrow L^q$ with $q \in [2, 2^*)$ is compact, this result can be easily proved by (1.2). Based on this existence result, we can easily obtain the following variational characterization of the ground states to (1.3). So we omit the proof.

Theorem 1.3. *Let $N \ge 1$, $0 < b < \min\{2, N\}$, $\omega > -aN$, $0 < p < \frac{4-2b}{N-2}$. Then $u_\omega \in \mathcal{G}_\omega$ if and only if u_ω solves the minimizing problem* (1.8).

Based on this variational characterization of the ground states, we can obtain the key estimate $Q(\psi(t)) \le 2(S_\omega(\psi_0) - S_\omega(u_\omega))$ under the assumption $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \le 0$. Therefore, we can obtain the following instability and strong instability of standing waves of (1.1).

Theorem 1.4. *Let $N \ge 1$, $0 < b < \min\{2, N\}$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that u_ω is the ground state related to* (1.3). *Then, there exists $\omega_* > 0$ such that for all $\omega > \omega_*$, the standing wave $\psi(t, x) = e^{i\omega t}u_\omega(x)$ is unstable.*

Theorem 1.5. *Let $N \ge 1$, $0 < b < \min\{2, N\}$, $\omega > 0$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that u_ω is the ground state related to* (1.3) *and $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \le 0$. Then the standing wave $\psi(t, x) = e^{i\omega t}u_\omega(x)$ is strongly unstable by blow-up.*

Combining this theorem and Lemma 3.5, we can easily obtain the following corollary.

Corollary 1.6. *Let $N \ge 1$, $0 < b < \min\{2, N\}$, $\omega > 0$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that u_ω is the ground state related to* (1.3). *Then, there exists $\omega_* > 0$ such that for any $\omega > \omega_*$, the standing wave $\psi(t, x) = e^{i\omega t}u_\omega(x)$ is strongly unstable by blow-up.*

Remark. Ardila and Dinh in [34] proved the standing wave $e^{i\omega t}u_\omega(x)$ is strongly unstable under the assumption $d(\omega) \le m(\omega)$. However, it is hard to determine for which ω, $d(\omega) \le m(\omega)$ is true, see also Remark 5.1 in [32]. In this corollary, when ω is large, we prove that the standing wave $\psi(t,x) = e^{i\omega t}u_\omega(x)$ is strongly unstable by blow-up.

This paper is organized as follows: in Section 2, we will collect some preliminaries such as the local well-posedness of (1.1), the virial identity to (1.1), Pohozaev's identities related to (1.3). In section 3, we will prove the instability of the standing wave $e^{i\omega t}u_\omega(x)$. In section 4, we will prove the strong instability of the standing wave $e^{i\omega t}u_\omega(x)$.

2. Preliminaries

In this section, we recall some useful results. Firstly, we recall the following local well-posedness result for (1.1). By a similar argument as that in [15, Theorem 9.2.6], we can estabilish the following local well-posedness result for (1.1).

Lemma 2.1. *Let $N \ge 1$, $0 < b < \min\{2, N\}$, $0 < p < \frac{4-2b}{N-2}$, and $\psi_0 \in \Sigma$. Then, there exists $T = T(\|\psi_0\|_\Sigma)$ such that (1.1) admits a unique solution $\psi \in C([0,T),\Sigma)$. Assume that $[0,T^*)$ is the maximal time interval of solution $\psi(t)$. If $T^* < \infty$, then $\|\psi(t)\|_\Sigma \to \infty$ as $t \uparrow T^*$. Moreover, the solution $\psi(t)$ depends continuously on initial data ψ_0 and satisfies the following mass and energy conservation laws*

$$M(\psi(t)) = \int_{\mathbb{R}^N} |\psi(t,x)|^2 dx = M(\psi_0), \tag{2.1}$$

$$E(\psi(t)) = E(\psi_0), \tag{2.2}$$

for all $t \in [0,T^*)$, where

$$E(\psi(t)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\psi(t,x)|^2 dx + \frac{a^2}{2}\int_{\mathbb{R}^N} |x|^2|\psi(t,x)|^2 dx - \frac{1}{p+2}\int_{\mathbb{R}^N} |x|^{-b}|\psi(t,x)|^{p+2} dx. \tag{2.3}$$

In order to study the strong instability of standing waves, we need to prove the existence of blow-up solutions. In order to study the strong instability of standing waves, we need to prove the existence of blow-up solutions. Following the classical convexity method of Glassey, by some formal computations which are made rigorously in [15]), we can obtain the following lemma.

Lemma 2.2. *Let $N \ge 1$, $0 < b < \min\{2, N\}$, $0 < p < \frac{4-2b}{N-2}$. Assume that $\psi_0 \in \Sigma := \{v \in H^1$ and $|x|v \in L^2\}$ and $\psi \in C([0,T^*),\Sigma)$ is the corresponding solution of* (1.1). *Then, $\psi(t) \in \Sigma$ for all $t \in [0,T^*)$ and the function $J(t)$ belongs to $C^2[0,T^*)$, where $J(t) = \int_{\mathbb{R}^N} |x|^2|\psi(t,x)|^2 dx$. Furthermore, we have*

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^N} |x|^2|\psi(t,x)|^2 dx = 16E(\psi(t)) + \frac{4(4-Np-2b)}{p+2}\int_{\mathbb{R}^N} |\nabla\psi|^2 dx - 16a^2 J(t) = 8Q(\psi(t)), \tag{2.4}$$

for all $t \in [0,T^*)$, where $Q(u)$ is defined by (1.6).

Next, we recall the following Pohozaev's identities related to (1.3), see [34, Lemma 5.1].

Lemma 2.3. *[34, Lemma 5.1] If $u \in \Sigma$ and satisfies equation* (1.3)*, then the following properties hold:*

$$\|\nabla u\|_{L^2}^2 + \omega\|u\|_{L^2}^2 + \int_{\mathbb{R}^N} W(x)|u(x)|^2 dx - \int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{p+2} dx = 0, \tag{2.5}$$

and

$$\frac{(N-2)}{2}\|\nabla u\|_{L^2}^2+\frac{N\omega}{2}\|u\|_{L^2}^2+\frac{N+2}{2}\int_{\mathbb{R}^N}W(x)|u(x)|^2dx-\frac{N-b}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(x)|^{p+2}dx=0. \tag{2.6}$$

Lemma 2.4. *[35, Theorem 1.2] Let* $0<p<\frac{4-2b}{N-2}$ *and* $0<b<\min\{2,N\}$. *Then, for all* $u\in H^1$,

$$\int_{\mathbb{R}^N}|x|^{-b}|u(x)|^{p+2}dx\le C_{opt}\|\nabla u\|_{L^2}^{\frac{Np+b}{2}}\|u\|_{L^2}^{p+2-\frac{Np+2b}{2}}, \tag{2.7}$$

where the best constant C_{opt} *is given by*

$$C_{opt}=\left(\frac{Np+2b}{2(p+2)-(Np+2b)}\right)^{\frac{4-(Np+2b)}{4}}\frac{2(p+2)}{(Np+2b)\|Q\|_{L^2}^{p}},$$

where Q *is the ground state of the elliptic eqation*

$$-\Delta Q+Q-|x|^{-b}|Q|^pQ=0.$$

Moreover, the following Pohozaev's identities hold true:

$$\|\nabla Q\|_{L^2}^2=\frac{Np+2b}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|Q|^{p+2}dx=\frac{Np+2b}{2(p+2)-(Np+2b)}\|Q\|_{L^2}^2. \tag{2.8}$$

3. Non-linear instability

In this section, we will prove Theorem 1.4. Based on the variational characterization of the ground states, we can obtain the following result.

Lemma 3.1. *Let* $N\ge 1$, $0<b<\min\{2,N\}$, $\omega>-aN$, $\frac{4-2b}{N}<p<\frac{4-2b}{N-2}$, u_ω *be the ground state related to* (1.3). *Assume that* $v\in\Sigma$, *and* $\int_{\mathbb{R}^N}|x|^{-b}|v(x)|^{p+2}dx=\int_{\mathbb{R}^N}|x|^{-b}|u_\omega(x)|^{p+2}dx$, *then* $S_\omega(u_\omega)\le S_\omega(v)$.

Proof. Firstly, we notice that $S_\omega(v)$ can be written as

$$S_\omega(v)=\frac{1}{2}K_\omega(v)+\frac{p}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|v(x)|^{p+2}dx.$$

By the variational characterization of ground state u_ω, we have

$$d_1:=\inf\left\{\frac{p}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|v(x)|^{p+2}dx,\ v\in\Sigma\backslash\{0\},\ K_\omega(v)=0\right\}.$$

This implies that $d_1=S(u_\omega)=\frac{p}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|u_\omega(x)|^{p+2}dx$. We set

$$d_2:=\inf\left\{\frac{p}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|v(x)|^{p+2}dx,\ v\in\Sigma\backslash\{0\},\ K_\omega(v)\le 0\right\}.$$

Since it is clear that $d_2 \le d_1$, we show $d_1 \le d_2$. For any $v \in \Sigma\backslash\{0\}$ satisfying $K_\omega(v) < 0$, there exists $\mu_0 \in (0,1)$ such that $K_\omega(\mu_0 v) = 0$. Thus, we have

$$d_1 \le \frac{p}{2(p+2)} \int_{\mathbb{R}^N} |x|^{-b} |\mu_0 v(x)|^{p+2} dx < \frac{p}{2(p+2)} \int_{\mathbb{R}^N} |x|^{-b} |v(x)|^{p+2} dx.$$

Hence, we have $d_1 \le d_2$.

Finally, we define

$$d_3 := \inf\left\{ S(v),\ v \in \Sigma\backslash\{0\},\ \int_{\mathbb{R}^N} |x|^{-b}|v(x)|^{p+2}dx = \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2}dx \right\}.$$

Since $d_3 \le S(u_\omega)$, it suffices to prove $d_3 \ge S(u_\omega)$. By $d_1 = d_2$, for any $v \in \Sigma\backslash\{0\}$ satisfying $\int_{\mathbb{R}^N} |x|^{-b}|v(x)|^{p+2}dx = \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2}dx$, we have $K_\omega(v) \ge 0$. Thus, we have

$$S(v) \ge \frac{p}{2(p+2)} \int_{\mathbb{R}^N} |x|^{-b}|v(x)|^{p+2}dx = \frac{p}{2(p+2)} \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2}dx = S(u_\omega).$$

Therefore, we obtain $d_3 \ge S(u_\omega)$. This complete the proof. □

In order to study the instability, for any $u_\omega \in \Sigma$ and $\varepsilon > 0$, we define

$$U_\varepsilon(u_\omega) := \{v \in \Sigma : \inf_{\theta\in\mathbb{R}} \|v - e^{i\theta}u_\omega\|_\Sigma < \varepsilon\}.$$

Lemma 3.2. *Let $N \ge 1$, $0 < b < \min\{2, N\}$, $\omega > -aN$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that $u_\omega \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} < 0$. Then there exist $\varepsilon, \delta > 0$, and a mapping*

$$\lambda :\ U_\varepsilon(u_\omega) \to (1-\varepsilon, 1+\varepsilon)$$

such that $K_\omega(v^{\lambda(v)}) = 0$ for any $v \in U_\varepsilon(u_\omega)$.

Proof. Let

$$F(v,\lambda) = K_\omega(v^\lambda).$$

Since u_ω is a minimizer of $S_\omega(v)$ constrained on the manifold $\mathcal{N} := \{v \in \Sigma\backslash\{0\},\ K_\omega(v) = 0\}$, then

$$\langle S''_\omega(u_\omega)w, w\rangle \ge 0, \quad for\ \langle u_\omega, w\rangle = 0. \tag{3.1}$$

Next, since

$$\langle S'_\omega(u_\omega), \eta\rangle = 0, \quad for\ all\ \eta \in \Sigma,$$

then

$$\langle S''_\omega(u_\omega)\partial_\lambda u_\omega^\lambda|_{\lambda=1}, \partial_\lambda u_\omega^\lambda|_{\lambda=1}\rangle = \partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} < 0. \tag{3.2}$$

Combining (3.1) and (3.2), we have $\langle \partial_\lambda u_\omega^\lambda|_{\lambda=1}, u_\omega\rangle \ne 0$ and so

$$\partial_\lambda F(u_\omega, 1) = \partial_\lambda K_\omega(u_\omega^\lambda)|_{\lambda=1} = \langle K'_\omega(u_\omega), \partial_\lambda u_\omega^\lambda|_{\lambda=1}\rangle \ne 0.$$

Thanks to $\partial_\lambda F(u_\omega, 1) = K_\omega(u_\omega) = 0$, applying the implicit function theorem, there exist $\varepsilon, \delta > 0$, and a mapping

$$\lambda :\ U_\varepsilon(u_\omega) \to (1-\varepsilon, 1+\varepsilon)$$

such that $K_\omega(v^{\lambda(v)}) = 0$ for all $v \in U_\varepsilon(u_\omega)$. This completes the proof. □

Lemma 3.3. *Let $N \geq 1$, $0 < b < \min\{2, N\}$, $\omega > 0$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that $u_\omega \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} < 0$. Then there exist $\varepsilon_0, \delta_0 > 0$ such that, for any $v \in U_{\varepsilon_0}(u_\omega)$*

$$S_\omega(u_\omega) \leq S_\omega(v) + (\lambda(v) - 1)Q(v),$$

for some $\lambda(v) \in (1 - \delta_0, 1 + \delta_0)$.

Proof. Since $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} < 0$ and $\partial_\lambda^2 S_\omega(v^\lambda)$ is continuous in λ and v, we know that there exists $\varepsilon_0, \delta_0 > 0$ such that $\partial_\lambda^2 S_\omega(v^\lambda) < 0$ for any $\lambda \in (1-\delta_0, 1+\delta_0)$ and $v \in U_{\varepsilon_0}(u_\omega)$. Noticing that $\partial_\lambda S_\omega(v^\lambda)|_{\lambda=1} = Q(v)$, applying the Taylor expansion for the function $S_\omega(v^\lambda)$ at $\lambda = 1$, we have

$$S_\omega(v^\lambda) \leq S_\omega(v) + (\lambda - 1)Q(v), \quad \lambda \in (1 - \delta_0, 1 + \delta_0), \quad v \in U_{\varepsilon_0}(u_\omega). \tag{3.3}$$

By Lemma 3.2, we choose $\varepsilon_0 < \varepsilon$ and $\delta_0 < \delta$, then there exists $\lambda(v) \in (1 - \delta_0, 1 + \delta_0)$ such that $K_\omega(v^{\lambda(v)}) = 0$ for all $v \in U_{\varepsilon_0}(u_\omega)$. Therefore, we have $S_\omega(v^{\lambda(v)}) \geq S_\omega(u_\omega)$. This, together with (3.3) implies that

$$S_\omega(u_\omega) \leq S_\omega(v) + (\lambda(v) - 1)Q(v),$$

for some $\lambda(v) \in (1 - \delta_0, 1 + \delta_0)$. □

Let $u_\omega \in \mathcal{G}_\omega$, we define

$$C_\omega := \{v \in U_{\varepsilon_0}(u_\omega);\ S_\omega(v) < S_\omega(u_\omega),\ \ Q(v) < 0\},$$

and

$$T(\psi_0) = \sup\{T;\ \psi(t) \in U_{\varepsilon_0}(u_\omega),\ \ t \in [0, T)\},$$

where $\psi(t)$ is a solution of (1.1) with initial data ψ_0. Then, we have the following lemma.

Lemma 3.4. *Let $N \geq 1$, $0 < b < \min\{2, N\}$, $\omega > -aN$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that $u_\omega \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} < 0$. Then, for any $\psi_0 \in C_\omega$, there exists $\delta_2 = \delta_2(\psi_0) > 0$ such that $Q(\psi(t)) \leq -\delta_2$ for all $t \in [0, T(\psi_0))$.*

Proof. Let $\psi_0 \in C_\omega$, and $\delta_1 = S_\omega(u_\omega) - S_\omega(\psi_0) > 0$. We deduce from Lemma 3.3 that

$$S_\omega(u_\omega) \leq S_\omega(\psi(t)) + (\lambda(\psi(t)) - 1)Q(\psi(t)) = S_\omega(\psi_0) + (\lambda(\psi(t)) - 1)Q(\psi(t)),$$

which implies

$$0 < \delta_1 \leq (\lambda(\psi(t)) - 1)Q(\psi(t)), \quad 0 \leq t < T(\psi_0). \tag{3.4}$$

Thus, $Q(\psi(t)) \neq 0$. Due to $\psi_0 \in C_\omega$, then $Q(\psi_0) < 0$. It follows from the continuity of $Q(\psi(t))$ that

$$Q(\psi(t)) < 0, \quad for\ 0 \leq t < T(\psi_0).$$

Thus, $\lambda(\psi(t)) \in (1 - \delta_0, 1)$. Combining (3.4), we have

$$Q(\psi(t)) \leq \frac{\delta_1}{\lambda(\psi(t)) - 1} \leq -\frac{\delta_1}{\delta_0}, \quad for\ 0 \leq t < T(\psi_0).$$

Thus, $Q(\psi(t)) \leq \delta_2$ with $\delta_2 = -\frac{\delta_1}{\delta_0}$, for $0 \leq t < T(\psi_0)$. □

Lemma 3.5. *Let $N \geq 1$, $0 < b < \min\{2, N\}$, $\omega > -aN$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that $u_\omega \in \mathcal{G}_\omega$. Then, there exists $\omega_* > 0$ such that $\partial^2_\lambda S_\omega(u^\lambda_\omega)|_{\lambda=1} \leq 0$ for all $\omega > \omega_*$.*

Proof. Let $u_\omega \in \mathcal{G}_\omega$, $f(\lambda)$ be defined by (1.9), it follows from Lemma 2.3 that $Q(u_\omega) = 0$, i.e., $f'(1) = 0$. We consequently obtain

$$f''(1) = 4a^2 \int_{\mathbb{R}^N} |x|^2 |u_\omega(x)|^2 dx - \frac{\alpha(\alpha-2)}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u_\omega(x)|^{p+2} dx.$$

Thus, $\partial^2_\lambda S_\omega(u^\lambda_\omega)|_{\lambda=1} \leq 0$ if and only if

$$\frac{a^2 \int_{\mathbb{R}^N} |x|^2 |u_\omega(x)|^2 dx}{\int_{\mathbb{R}^N} |x|^{-b} |u_\omega(x)|^{p+2} dx} \leq \frac{\alpha(\alpha-2)}{4(p+2)}.$$

So it is sufficient to prove that

$$\lim_{\omega \to \infty} \frac{a^2 \int_{\mathbb{R}^N} |x|^2 |u_\omega(x)|^2 dx}{\int_{\mathbb{R}^N} |x|^{-b} |u_\omega(x)|^{p+2} dx} = 0.$$

Let $u_\omega(x) = \omega^{\frac{2-b}{2p}} \tilde{u}_\omega(\sqrt{\omega} x)$, then $\tilde{u}_\omega$ satisfies

$$-\Delta \tilde{u}_\omega + \tilde{u}_\omega + a^2 \omega^{-2} |x|^2 \tilde{u}_\omega - |x|^{-b} |\tilde{u}_\omega|^p \tilde{u}_\omega = 0.$$

Since

$$\frac{a^2 \int_{\mathbb{R}^N} |x|^2 |u_\omega(x)|^2 dx}{\int_{\mathbb{R}^N} |x|^{-b} |u_\omega(x)|^{p+2} dx} = \omega^{-2} \frac{a^2 \int_{\mathbb{R}^N} |x|^2 |\tilde{u}_\omega(x)|^2 dx}{\int_{\mathbb{R}^N} |x|^{-b} |\tilde{u}_\omega(x)|^{p+2} dx},$$

it is sufficient to prove that

$$\lim_{\omega \to \infty} \omega^{-2} \frac{a^2 \int_{\mathbb{R}^N} |x|^2 |\tilde{u}_\omega(x)|^2 dx}{\int_{\mathbb{R}^N} |x|^{-b} |\tilde{u}_\omega(x)|^{p+2} dx} = 0.$$

Let $V \in H^1 \setminus \{0\}$ be a ground state solution to the elliptic problem

$$-\Delta V + V - |x|^{-b} |V|^p V = 0,$$

then

$$S_0(V) = \inf\{S_0(v), \ \ v \in H^1 \setminus \{0\}, \ \ \tilde{K}_0(v) = 0\},$$

where

$$S_0(v) = \frac{1}{2} \|\nabla v\|^2_{L^2} + \frac{1}{2} \|v\|^2_{L^2} - \frac{1}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |v(x)|^{p+2} dx,$$

and

$$\tilde{K}_0(v) = \|\nabla v\|^2_{L^2} + \|v\|^2_{L^2} - \int_{\mathbb{R}^N} |x|^{-b} |v(x)|^{p+2} dx. \tag{3.5}$$

Then, by a similar argument as that in Lemma 3.1, we have

$$\int_{\mathbb{R}^N} |x|^{-b} |V(x)|^{p+2} dx = \inf\left\{ \int_{\mathbb{R}^N} |x|^{-b} |v(x)|^{p+2} dx, \ \ v \in H^1 \setminus \{0\}, \ \ \tilde{K}_0(v) \leq 0 \right\}, \tag{3.6}$$

and

$$\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx = \inf\left\{\int_{\mathbb{R}^N} |x|^{-b}|v(x)|^{p+2}dx,\ \ v \in \Sigma \setminus \{0\},\ \ \tilde{K}_\omega(v) \le 0\right\}, \tag{3.7}$$

where

$$\tilde{K}_\omega(v) = \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 + \omega^{-2}\|xv\|_{L^2}^2 - \int_{\mathbb{R}^N} |x|^{-b}|v(x)|^{p+2}dx.$$

In addition, we infer from $\tilde{K}_0(V) = 0$ that for $\lambda > 1$

$$\tilde{K}_\omega(\lambda V) = \lambda^2\left((1-\lambda^p)\int_{\mathbb{R}^N} |x|^{-b}|V(x)|^{p+2}dx + \omega^{-2}\|V\|_{L^2}^2\right).$$

Then, for any $\lambda > 1$, there exists $\omega(\lambda)$ such that $\tilde{K}_\omega(\lambda V) < 0$. This and (3.7) imply that

$$\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx \le \lambda^{p+2}\int_{\mathbb{R}^N} |x|^{-b}|V(x)|^{p+2}dx. \tag{3.8}$$

On the other hand, we deduce from $\tilde{K}_\omega(\tilde{u}_\omega) = 0$ that

$$\tilde{K}_0(\lambda\tilde{u}_\omega) = \lambda^2\left((1-\lambda^p)\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx - \omega^{-2}\|x\tilde{u}_\omega\|_{L^2}^2\right).$$

Then, for any $\lambda > 1$, $\tilde{K}_0(\lambda\tilde{u}_\omega) < 0$. We consequently deduce from (3.6) and (3.8) that for any $\omega > \omega(\lambda)$,

$$\lambda^{-(p+2)}\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx \le \int_{\mathbb{R}^N} |x|^{-b}|V(x)|^{p+2}dx \le \lambda^{p+2}\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx.$$

This implies

$$\lim_{\omega\to\infty}\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx = \int_{\mathbb{R}^N} |x|^{-b}|V(x)|^{p+2}dx. \tag{3.9}$$

Notice that

$$\tilde{K}_0(\lambda\tilde{u}_\omega) = \lambda^2\|\nabla\tilde{u}_\omega\|_{L^2}^2 + \lambda^2\|\tilde{u}_\omega\|_{L^2}^2 - \lambda^{p+2}\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx,$$

there exists $\lambda(\omega) > 0$ such that $\tilde{K}_0(\lambda(\omega)\tilde{u}_\omega) = 0$. This and (3.6) yield that

$$\int_{\mathbb{R}^N} |x|^{-b}|V(x)|^{p+2}dx \le \lambda(\omega)^{p+2}\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx.$$

This implies that $\liminf_{\omega\to\infty}\lambda(\omega) \ge 1$ and

$$\liminf_{\omega\to\infty}\tilde{K}_0(\tilde{u}_\omega) = \liminf_{\omega\to\infty}(\lambda(\omega)^p - 1)\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2}dx \ge 0. \tag{3.10}$$

On the other hand, we deduce from $\tilde{K}_\omega(\tilde{u}_\omega) = 0$ that $\tilde{K}_0(\tilde{u}_\omega) < 0$. This implies that

$$\limsup_{\omega\to\infty}\tilde{K}_0(\tilde{u}_\omega) \le 0.$$

Combining this and (3.10), it follows that

$$0 \le \liminf_{\omega\to\infty}\tilde{K}_0(\tilde{u}_\omega) \le \limsup_{\omega\to\infty}\tilde{K}_0(\tilde{u}_\omega) \le 0.$$

This implies that $\lim_{\omega\to\infty} \tilde{K}_0(\tilde{u}_\omega) = 0$. We consequently obtain that

$$\omega^{-2}\|x\tilde{u}_\omega\|_{L^2}^2 = -\tilde{K}_0(\tilde{u}_\omega) + \tilde{K}_\omega(\tilde{u}_\omega) = -\tilde{K}_0(\tilde{u}_\omega) \to 0,$$

as $\omega \to \infty$. Thus, we see from (3.9) that

$$\lim_{\omega\to\infty} \omega^{-2} \frac{a^2 \int_{\mathbb{R}^N} |x|^2|\tilde{u}_\omega(x)|^2 dx}{\int_{\mathbb{R}^N} |x|^{-b}|\tilde{u}_\omega(x)|^{p+2} dx} = 0.$$

This completes the proof. □

Proof of Theorem 1.4. Let $f(\lambda)$ be defined by (1.9), then

$$\begin{aligned} f''(\lambda) &= \|\nabla u_\omega\|_{L^2}^2 + 3\lambda^{-4}a^2 \int_{\mathbb{R}^N} |x|^2|u_\omega(x)|^2 dx - \frac{\alpha(\alpha-1)\lambda^{\alpha-2}}{p+2} \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2} dx \\ &\le f''(1) = \partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} < 0, \end{aligned}$$

for all $\lambda \ge 1$. This, together with Lemma 2.3 implies that

$$Q(u_\omega^\lambda) = \lambda f'(\lambda) < f'(1) = Q(u_\omega) = 0,$$

and

$$S_\omega(u_\omega^\lambda) < S_\omega(u_\omega),$$

for all $\lambda > 1$. On the other hand, it follows from Brezis-Lieb's lemma that $u_\omega^\lambda \to u_\omega$ as $\lambda \to 1$. Thus, for any $\varepsilon > 0$, there exists $\lambda_0 > 1$ such that $\|u_\omega^{\lambda_0} - u_\omega\|_\Sigma < \varepsilon$.

Next, let $\psi_0 = u_\omega^{\lambda_0}$, then $\psi_0 \in U_\varepsilon(u_\omega)$, $S_\omega(\psi_0) < S_\omega(u_\omega)$ and $Q(\psi_0) < 0$. Thus, $\psi_0 \in C_\omega$ and there exists $\delta_2 = \delta_2(\psi_0) > 0$ such that $Q(\psi(t)) \le -\delta_2$ for all $t \in [0, T(\psi_0))$. Then, we deduce from Lemma 2.2 that

$$J''(t) = 8Q(\psi(t)) \le -8\delta_2 < 0,$$

for all $t \in [0, T(\psi_0))$. If $e^{i\omega t}u_\omega$ is orbitally stable, then $T(\psi_0) = +\infty$ and $Q(\psi(t)) \le -\delta_2$ for all $t \in [0, \infty)$. This implies that $J(t)$ becomes negative for long time. This is an contradiction. Moreover, applying Lemma 3.5, there exists $\omega_* > 0$ such that $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} < 0$ for all $\omega > \omega_*$. Thus, when $\omega > \omega_*$, the standing wave $e^{i\omega t}u_\omega$ is unstable.

4. Strong instability

In this section, we will prove Theorem 1.5. To this end, we firstly establish the following key estimate.

Lemma 4.1. *Let $N \ge 1$, $0 < b < \min\{2, N\}$, $\omega > 0$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that $u_\omega \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \le 0$. Suppose further that $v \in \Sigma\backslash\{0\}$ such that*

$$\|v\|_{L^2} = \|u_\omega\|_{L^2}, \quad K_\omega(v) \le 0, \quad Q(v) \le 0.$$

Then it holds that

$$Q(v) \le 2(S_\omega(v) - S_\omega(u)). \tag{4.1}$$

Remark. It is easy to see from Lemma 4.1 that for $u_\omega \in \mathcal{G}_\omega$ satisfying $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \le 0$,

$$\{v \in \Sigma\backslash\{0\} : \ \|v\|_{L^2} = \|u_\omega\|_{L^2}, \ S_\omega(v) < S_\omega(u_\omega), \ K_\omega(v) < 0, \ Q(v) = 0\} = \emptyset, \tag{4.2}$$

Indeed, if there exists $v \in \Sigma\backslash\{0\}$ satisfying $\|v\|_{L^2} = \|u_\omega\|_{L^2}$, $S_\omega(v) < S_\omega(u_\omega)$, $K_\omega(v) < 0$ and $Q(v) = 0$, then by this lemma,

$$0 = Q(v) \le 2(S_\omega(v) - S_\omega(u_\omega)) < 0$$

which is a contradiction.

Proof. If $K_\omega(v) = 0$, we infer from Theorem 1.3 and $Q(v) < 0$ that

$$S_\omega(u_\omega) \le S_\omega(v) \le S_\omega(v) - \frac{1}{2}Q(v),$$

which is the desired estimate (4.1).

When $K_\omega(v) < 0$, we notice that

$$K_\omega(v^\lambda) := \lambda^2\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 + \lambda^{-2}a^2\int_{\mathbb{R}^N}|x|^2|v(x)|^2dx - \lambda^\alpha\int_{\mathbb{R}^N}|x|^{-b}|v(x)|^{p+2}dx.$$

Since $\lim_{\lambda\to 0} K_\omega(v^\lambda) = \omega\|v\|_{L^2}^2 > 0$ and $K_\omega(v) < 0$, there exists $\lambda_0 \in (0,1)$ such that $K_\omega(v^{\lambda_0}) = 0$. Applying Theorem 1.3, it follows that

$$\begin{aligned}
&\frac{p}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|u_\omega(x)|^{p+2}dx = S_\omega(u_\omega) \le S_\omega(v^{\lambda_0}) \\
&= \frac{p}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|v^{\lambda_0}(x)|^{p+2}dx = \frac{p\lambda_0^\alpha}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|v(x)|^{p+2}dx.
\end{aligned}$$

When $\int_{\mathbb{R}^N}|x|^2|v(x)|^2dx \ge \int_{\mathbb{R}^N}|x|^2|u_\omega(x)|^2dx$, it follows from $Q(u_\omega) = 0$ that

$$\begin{aligned}
S_\omega(u_\omega) &= S_\omega(u) - \frac{1}{2}Q(u_\omega) \\
&= \frac{\omega}{2}\|u_\omega\|_{L^2}^2 + a^2\int_{\mathbb{R}^N}|x|^2|u_\omega(x)|^2dx + \frac{\alpha-2}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|u_\omega(x)|^{p+2}dx \\
&\le \frac{\omega}{2}\|v\|_{L^2}^2 + a^2\int_{\mathbb{R}^N}|x|^2|v(x)|^2dx + \frac{\alpha-2}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|v(x)|^{p+2}dx \\
&= S_\omega(v) - \frac{1}{2}Q(v),
\end{aligned}$$

which is the desired estimate (4.1).

When $\int_{\mathbb{R}^N}|x|^2|v(x)|^2dx < \int_{\mathbb{R}^N}|x|^2|u_\omega(x)|^2dx$, we define

$$\begin{aligned}
f_1(\lambda) : &= S_\omega(v^\lambda) - \frac{\lambda^2}{2}Q(v) \\
&= \frac{\omega}{2}\|v\|_{L^2}^2 + \frac{a^2(\lambda^{-2}+\lambda^2)}{2}\int_{\mathbb{R}^N}|x|^2|v(x)|^2dx + \frac{\alpha\lambda^2 - 2\lambda^\alpha}{2(p+2)}\int_{\mathbb{R}^N}|x|^{-b}|v(x)|^{p+2}dx.
\end{aligned}$$

If $f_1(\lambda_0) \leq f_1(1)$, then we deduce from Theorem 1.3 and $Q(v) \leq 0$ that

$$S_\omega(u_\omega) \leq S_\omega(v^{\lambda_0}) \leq S_\omega(v^{\lambda_0}) - \frac{\lambda_0^2}{2}Q(v) \leq S_\omega(v) - \frac{1}{2}Q(v),$$

which is the desired estimate (4.1).

In what follows, we will prove $f_1(\lambda_0) \leq f_1(1)$, which is equivalent to

$$a^2 \int_{\mathbb{R}^N} |x|^2 |v(x)|^2 dx \leq \frac{\alpha - 2 - \alpha\lambda_0^2 + 2\lambda_0^\alpha}{(p+2)(\lambda_0^{-2} + \lambda_0^2 - 2)} \int_{\mathbb{R}^N} |x|^{-b} |v(x)|^{p+2} dx. \tag{4.3}$$

In views of (1.9), the condition $\partial_\lambda^2 S_\omega(u^\lambda)|_{\lambda=1} \leq 0$ is equivalent to

$$\|\nabla u_\omega\|_{L^2}^2 + 3a^2 \int_{\mathbb{R}^N} |x|^2 |u_\omega(x)|^2 dx - \frac{\alpha(\alpha-1)}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u_\omega(x)|^{p+2} dx \leq 0. \tag{4.4}$$

Combining (4.4) and $Q(u_\omega) = 0$, we can obtain that

$$\begin{aligned} 4a^2 \int_{\mathbb{R}^N} |x|^2 |v(x)|^2 dx < 4a^2 \int_{\mathbb{R}^N} |x|^2 |u_\omega(x)|^2 dx &\leq \frac{\alpha^2 - 2\alpha}{p+2} \int_{\mathbb{R}^N} |x|^{-b} |u_\omega(x)|^{p+2} dx \\ &\leq \frac{\alpha^2 - 2\alpha}{p+2} \lambda_0^\alpha \int_{\mathbb{R}^N} |x|^{-b} |v(x)|^{p+2} dx. \end{aligned}$$

This, together with (4.3), it suffices to show that

$$\frac{(\alpha^2 - 2\alpha)\lambda_0^\alpha}{4(p+2)} \leq \frac{\alpha - 2 - \alpha\lambda_0^2 + 2\lambda_0^\alpha}{(p+2)(\lambda_0^{-2} + \lambda_0^2 - 2)}. \tag{4.5}$$

Let $\alpha = 2\beta$, then (4.5) is equivalent to

$$\lambda_0^{2\beta} - \beta\lambda_0^2 + \beta - 1 \geq \frac{(\beta^2 - \beta)}{2}(\lambda_0 - \lambda_0^{-1})^2 \lambda_0^{2\beta+2}.$$

Let

$$h(\lambda) := \lambda^\beta - \beta\lambda + \beta - 1 - \frac{1}{2}(\beta^2 - \beta)(\lambda - 1)^2 \lambda^{\beta-1}, \ \ \lambda > 0.$$

From the Taylor expansion of λ^β at $\lambda = 1$, there exists $\xi \in (\lambda_0^2, 1)$ such that

$$h(\lambda_0^2) = \frac{(\beta^2 - \beta)}{2}(\lambda_0^2 - 1)^2 (\xi^{\beta-2} - \lambda_0^{2\beta-2}).$$

Due to $\beta > 1$ and $\xi \in (\lambda_0^2, 1)$, it follows that

$$\lambda_0^{2\beta-2} < \xi^{\beta-1} < \xi^{\beta-2}.$$

Thus, we have $h(\lambda_0) > 0$. This implies that (4.5) holds. This completes the proof. □

Based on this lemma, we define an invariant set $\mathcal{B}_\omega$ under the flow of (1.1).

$$\mathcal{B}_\omega := \{v \in \Sigma \backslash \{0\};\ S_\omega(v) < S_\omega(u_\omega),\ \|v\|_{L^2} = \|u_\omega\|_{L^2},\ \ K_\omega(v) < 0,\ \ Q(v) < 0\}.$$

Lemma 4.2. *Let $N \geq 1$, $0 < b < \min\{2, N\}$, $\omega > 0$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that $u_\omega \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \leq 0$. Then, the set $\mathcal{B}_\omega$ is invariant under the flow of* (1.1)*, that is, if $\psi_0 \in \mathcal{B}_\omega$, then the solution $\psi(t)$ to* (1.1) *with initial data ψ_0 belongs to $\mathcal{B}_\omega$.*

Proof. Let $\psi_0 \in \mathcal{B}_\omega$, it follows from Lemma 2.1 that there exists a unique solution $\psi \in C([0, T^*), \Sigma)$. By the conservations of mass and energy, we have $S_\omega(\psi(t)) = S_\omega(\psi_0) < S_\omega(u_\omega)$ and $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2} = \|u_\omega\|_{L^2}$ for any $t \in [0, T^*)$. In addition, by the continuity of the function $t \mapsto K_\omega(\psi(t))$ and Theorem 1.3, if there exists $t_0 \in [0, T^*)$ such that $K_\omega(\psi(t_0)) = 0$, then $S_\omega(u_\omega) \leq S_\omega(\psi(t_0))$, which contradicts with $S_\omega(u_\omega) > S_\omega(\psi(t))$ for all $t \in [0, T^*)$. Therefore, the solution $\psi(t)$ satisfies $K_\omega(\psi(t)) < 0$ for all $t \in [0, T^*)$.

Finally, we prove that if $Q(\psi_0) < 0$, then $Q(\psi(t)) < 0$ for all $t \in [0, T^*)$. Let us prove this by contradiction. If not, there exists $t_0 \in [0, T^*)$ such that $Q(\psi(t_0)) = 0$. Applying Lemma 4.1, we have

$$S_\omega(u_\omega) \leq S_\omega(\psi(t_0)) - \frac{1}{4}Q(\psi(t_0)) = S_\omega(\psi(t_0)), \tag{4.6}$$

which is a contradiction with $S_\omega(u_\omega) > S_\omega(\psi(t))$ for all $t \in [0, T^*)$. This ends the proof. □

Lemma 4.3. *Let $N \geq 1$, $0 < b < \min\{2, N\}$, $\omega > 0$, $\frac{4-2b}{N} < p < \frac{4-2b}{N-2}$. Assume that $u_\omega \in \mathcal{G}_\omega$ and $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \leq 0$. Then, $u_\omega^\lambda \in \mathcal{B}_\omega$ for any $\lambda > 1$.*

Proof. Firstly, it easily follows that

$$\|u_\omega^\lambda\|_{L^2} = \|u_\omega\|_{L^2}.$$

Next, we define

$$g(\lambda) := K_\omega(u_\omega^\lambda) = \lambda^2\|\nabla u_\omega\|_{L^2}^2 + \omega\|u_\omega\|_{L^2}^2 + \lambda^{-2}a^2 \int_{\mathbb{R}^N} |x|^2|u_\omega(x)|^2 dx - \lambda^\alpha \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2} dx.$$

Thus, it follows from the assumption

$$\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} = 2\|\nabla u\|_{L^2}^2 + 3a^2 \int_{\mathbb{R}^N} |x|^2|u_\omega(x)|^2 dx - \frac{\alpha(\alpha-1)}{p+2} \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2} dx \leq 0$$

that

$$g''(\lambda) = 2\|\nabla u_\omega\|_{L^2}^2 + 6\lambda^{-4}a^2 \int_{\mathbb{R}^N} |x|^2|u_\omega(x)|^2 dx - \alpha(\alpha-1)\lambda^{\alpha-2} \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2} dx < 0$$

for any $\lambda \geq 1$. This, together with Pohozaev identity related to (1.3), implies that

$$g'(\lambda) < g'(1) = 2\|\nabla u_\omega\|_{L^2}^2 - 2a^2 \int_{\mathbb{R}^N} |x|^2|u_\omega(x)|^2 dx - \alpha \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2} dx < 0.$$

We consequently obtain that $K_\omega(u_\omega^\lambda) < 0$ for any $\lambda \geq 1$.

Let $f(\lambda)$ be defined by (1.9), then

$$\begin{aligned} f''(\lambda) &= \|\nabla u_\omega\|_{L^2}^2 + 3\lambda^{-4}a^2 \int_{\mathbb{R}^N} |x|^2|u_\omega(x)|^2 dx - \frac{\alpha(\alpha-1)\lambda^{\alpha-2}}{p+2} \int_{\mathbb{R}^N} |x|^{-b}|u_\omega(x)|^{p+2} dx \\ &\leq f''(1) = \partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \leq 0, \end{aligned}$$

for all $\lambda \geq 1$. This, together with Pohozaev identity related to (1.3), implies that

$$\frac{Q(u_\omega^\lambda)}{\lambda} = f'(\lambda) < f'(1) = Q(u_\omega) = 0,$$

and

$$S_\omega(u_\omega^\lambda) < S_\omega(u_\omega),$$

for any $\lambda > 1$. Thus, $u^\lambda \in \mathcal{B}_\omega$ for any $\lambda > 1$. This finishes the proof. □

Proof of Theorem 1.5. Let $\omega > 0$ and u_ω be the ground state related to (1.3). By Lemma 4.3, we have $u_\omega^\lambda \in \mathcal{B}_\omega$. Let $\psi^\lambda \in C([0, T^*), \Sigma)$ be the solution of (1.1) with the initial data u_ω^λ, then $\psi^\lambda(t) \in \mathcal{B}_\omega$ for all $t \in [0, T^*)$. Thus, by a classical argument, it follows that $\psi^\lambda \in \Sigma$ and

$$\frac{d^2}{dt^2}\|x\psi^\lambda(t)\|_{L^2}^2 = 8Q(\psi^\lambda(t)) \leq 16(S(u_\omega^\lambda) - S(u_\omega)) < 0,$$

for all $t \in [0, T^*)$. This implies that the solution ψ^λ of (1.1) with the initial data u_ω^λ blows up in finite time. Hence, the result follows, since $u_\omega^\lambda \to u_\omega$ as $\lambda \downarrow 1$.

5. Conclusions

In this paper, we consider the instability of standing waves for an inhomogeneous Gross-Pitaevskii equation (1.1). We firstly proved that there exists $\omega_* > 0$ such that for all $\omega > \omega_*$, the ground state standing wave $\psi(t, x) = e^{i\omega t}u_\omega(x)$ is unstable. Then, we deduce that if $\partial_\lambda^2 S_\omega(u_\omega^\lambda)|_{\lambda=1} \leq 0$, the ground state standing wave $e^{i\omega t}u_\omega(x)$ is strongly unstable by blow-up. This result is a complement to the partial result of Ardila and Dinh in [34].

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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