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Research article

Existence of least energy nodal solution for Kirchhoff-type system with Hartree-type nonlinearity

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Abstract: This paper deals with following Kirchhoff-type system with critical growth

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u + V(x)u + \phi|u|^{p-2}u = |u|^4u + \mu f(u), & x\in\mathbb{R}^3,\\ (-\Delta)^{\alpha/2}\phi = l|u|^p, & x\in\mathbb{R}^3,\end{cases}$$

where $a,\mu > 0$, $b,l \geq 0$, $\alpha \in (0,3)$, $p \in [2,3)$ and $\phi|u|^{p-2}u$ is a Hartree-type nonlinearity. By the minimization argument on the nodal Nehari manifold and the quantitative deformation lemma, we prove that the above system has a least energy nodal solution. Our result improve and generalize some interesting results which were obtained in subcritical case.

Keywords: nonlocal term; variation methods; nodal solutions
Mathematics Subject Classification: 35J60, 35J20

1. Introduction and main result

In this article, we are interested in the least energy nodal solution for the following Kirchhoff-type system

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u + V(x)u + \phi|u|^{p-2}u = |u|^4u + \mu f(u), & x\in\mathbb{R}^3,\\ (-\Delta)^{\alpha/2}\phi = l|u|^p, & x\in\mathbb{R}^3,\end{cases} \tag{1.1}$$

where $a,\mu > 0$, $b,l \geq 0$, $\alpha \in (0,3)$, $p \in [2,3)$ and $\phi|u|^{p-2}u$ is a Hartree-type nonlinearity (in fact, $\phi = I * |u|^p$, where I is the Riesz potential defined by (1.10)), $(-\Delta)^{\alpha/2}$ is the fractional Laplacian. The potential function $V \in C(\mathbb{R}^3,\mathbb{R}_+)$ and function $f \in C^1(\mathbb{R},\mathbb{R})$ satisfy the following hypotheses:

(V) for every $M > 0$, the set $V_M := \{x \in \mathbb{R}^3 : V(x) \leq M\}$ has a finite Lebesgue measure, i.e.$m(V_M) < \infty$;

(f_1) $\lim_{|t|\to 0} \frac{f(t)}{|t|^{2p-1}} = 0$;

(f_2) there exist $q \in (2p, 6)$ and $C > 0$ such that $|f'(t)| \leq C(1 + |t|^{q-2})$ for all $t \in \mathbb{R}$;

(f_3) $\frac{f(t)}{t^{2p-1}}$ is strictly increasing for $t > 0$ and is strictly decreasing for $t < 0$.

In the past decades, many mathematicians pay their much attention to nonlocal problems. The appearance of nonlocal terms in the equations not only marks its importance in many physical applications but also causes some difficulties and challenges from a mathematical point of view. Certainly, this fact makes the study of nonlocal problems particularly interesting. The following Schrödinger-Poisson system is a typical nonlocal problem

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

Recently, many authors have been devoted to the study for system (1.2) or similar problems. Especially on nodal solutions to problems like (1.2), and indeed some interesting results were obtained, see for examples, [1–14] and the references therein. In fact, there are very few results about nodal solutions to Schrödinger-Poisson system with critical growth. In [14], Zhong and Tang [14] considered the existence of ground state nodal solution for following system with critical growth

$$\begin{cases} -\Delta u + u + k(x)\phi u = |u|^4 u + \lambda f(x)u, & x \in \mathbb{R}^3, \\ -\Delta \phi = k(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $k, f \geq 0$, $0 < \lambda < \lambda_1$(where λ_1 is the first eigenvalue of the problem $-\Delta u + u = \lambda f(x)u$ in $H^1(\mathbb{R}^3)$). However, if $k(x) \equiv 1$, their methods seems not valid because their results depends on the case $k \in L^p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for some $p \in [2, \infty)$.

In [11], Wang, Zhang ang Guan considered the existence of least energy nodal solution for following system with critical growth

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^4 u + \lambda f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

Via the constraint variational method and quantitative deformation lemma, they obtained the existence and asymptotic behavior of least energy nodal solution for system (1.4).

As another typical nonlocal problem, the following Kirchhoff-type equation

$$-(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = f(u), x \in \mathbb{R}^3, \tag{1.5}$$

has also aroused many mathematicians's wide concern. Especially, There are many papers about nodal solutions to problems like (1.5) [15–32]. However, to the best of our knowledge, the most results seem to be obtained in the subcritical case. It is noticed that the second author [27] considered the least energy nodal solution for following Kirchhoff equation with critical growth

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^2 dx)\Delta u = |u|^4 u + \lambda f(x, u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$, $\lambda, a, b > 0$. By using the constraint variational method and quantitative deformation lemma, the author studied the existence and energy characteristics of least energy nodal solution to Eq 1.6.

In [32], Zhao and Liu studied the following Kirchhoff equation with critical growth

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = u^5 + \mu|u|^{q-2}u, & x \in \mathbb{R}^3, \\ u(x) \to 0, \text{ as } |x| \to \infty, \end{cases} \tag{1.7}$$

where $a, b, \mu > 0, 5 < q < 6$ are constants and V is a radial function and is bounded from below by a positive constant. By using the truncation method, they proved that, for any given positive integer k, the problem has a radial solution with k nodal domains exactly.

When $\alpha = 2$, system (1.1) is related to following system

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u + \phi u = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.8}$$

where a, b are positive constants. Since there are both nonlocal operator and nonlocal nonlinear term, the study of system (1.7) become more complicated. In recent years, there are some scholars began to show interest to problem like (1.7), see [33–44] and references therein. However, to our best knowledge, few papers considered nodal solutions to problem like (1.7). Via gluing the function methods, Deng and Yang [34] studied the nodal solutions for system (1.8) with $f(u) = |u|^{p-2}u, p \in (4, 6)$. In [39], Wang, Li and Hao studied the existence and the asymptotic behavior of least energy nodal solution for system (1.8) by using the constraint variation methods.

Recently, in order to research uniformly Kirchhoff-type equation and Schrödinger-Poisson system, Li, Gao and Zhu [45] considered the following Kirchhoff-type system

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + \lambda V(x)u + \phi|u|^{p-2}u = f(u), & x \in \mathbb{R}^3, \\ (-\Delta)^{\alpha/2}\phi = l|u|^p, \quad x \in \mathbb{R}^3, \end{cases} \tag{1.9}$$

where $a > 0$, $b, l \geq 0$, $\alpha \in (0, 3)$ and $p \in [2, 3 + \alpha)$. More precisely, they studied the existence and asymptotic behavior of least energy nodal solution for system (1.9).

Inspired by the works mentioned above, especially by [11, 27, 45], in this paper, we investigate the existence of the least energy nodal solution to Kirchhoff-type system (1.1).

Before presenting our main results, we denote $L^r(\mathbb{R}^3)$ the Lebesgue space with the norm $|u|_r := (\int_{\mathbb{R}^3} |u|^r dx)^{\frac{1}{r}}$, $1 \leq r < \infty$. Let $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ be a Hilbert space with the inner product and corresponding norm

$$(u, v)_1 = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx, ||u||_1 = (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^{\frac{1}{2}}.$$

Denote $E := H^1_V(\mathbb{R}^3)$ is given Hilbert space

$$H^1_V(\mathbb{R}^3) = \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} Vu^2 dx < \infty\},$$

by equipped with the inner product and norm

$$(u,v)=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+Vuv)dx,\|u\|=(\int_{\mathbb{R}^3}(|\nabla u|^2+Vu^2)dx)^{\frac{1}{2}}.$$

Under the condition (V), according to Remark 3.5 of [46], the embedding $E\hookrightarrow L^r(\mathbb{R}^3)$ is continuous for each $r\in[2,6]$, and is compact for each $r\in[2,6)$.

It follows from the second equation $(-\Delta)^{\alpha/2}\phi=l|u|^p$ of system (1.1) that the unique solution is $\phi=I*|u|^p$, where I is the Riesz potential defined by

$$I(x)=\frac{\Gamma((3-\alpha)/2)}{\Gamma(\alpha/2)\pi^{3/2}2^\alpha}\frac{1}{|x|^{3-\alpha}},x\in\mathbb{R}^3\backslash\{0\} \tag{1.10}$$

and $*$ is the convolution of two functions in $\mathbb{R}^3$. Hence, system (1.1) can also be rewritten as a Hartree-type equation

$$-(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u+V(x)u+l(I*|u|^p)|u|^{p-2}u=f(u),\quad \text{in } \mathbb{R}^3. \tag{1.11}$$

So, the energy functional associated with system (1.1) is defined by

$$J_\mu(u)=\frac{1}{2}\|u\|^2+\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla u|^2dx)^2+\frac{l}{2p}\int_{\mathbb{R}^3}(I*|u|^p)|u|^pdx-\mu\int_{\mathbb{R}^3}F(u)dx-\frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx,$$

for any $u\in E$.

Moreover, under our conditions, $J_\mu(u)$ belongs to C^1, and the Fréchet derivative of $J_\mu(u)$ is

$$\begin{aligned}\langle J'_\mu(u),v\rangle=&a\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+Vuv)dx+b(\int_{\mathbb{R}^3}|\nabla u|^2dx)(\int_{\mathbb{R}^3}\nabla u\cdot\nabla vdx)\\&+l\int_{\mathbb{R}^3}(I*|u|^p)|u|^{p-2}uvdx-\mu\int_{\mathbb{R}^3}f(u)vdx-\int_{\mathbb{R}^3}|u|^4uvdx\end{aligned}$$

for any $u,v\in E$.

The weak solution of system (1.1) is the critical point of the functional $J_\mu(u)$. Furthermore, if $u\in E$ is a weak solution of system (1.1) with $u^\pm\neq 0$, then we say that u is a nodal solution of system (1.1), where $u^+=\max\{u(x),0\}, u^-=\min\{u(x),0\}$. In this paper, we borrow some ideas from [11,24,26,27,45,47] and seek a minimizer of the energy functional J_μ over the constraint $\mathcal{M}_\mu=\{u\in E, u^\pm\neq 0 \text{ and } \langle J'_\mu(u),u^\pm\rangle=0\}$, and then prove that the minimizer is a nodal solution of system (1.1).

The main results can be stated as follows.

Theorem 1.1. *Suppose that* (V) *and* $(f_1)-(f_3)$ *are satisfied. Then, there exists* $\mu^\star>0$ *such that for all* $\mu\geq\mu^\star$*, the system* (1.1) *has a least energy nodal solution* u_μ.

Theorem 1.2. *Suppose that* (V) *and* $(f_1)-(f_3)$ *are satisfied. Then, there exists* $\mu^{\star\star}>0$ *such that for all* $\mu\geq\mu^{\star\star}$*, then the* $c^*>0$ *is achieved and*

$$J_\mu(u_\mu)>2c^*,$$

where $c^*=\inf_{u\in\mathcal{N}_\mu}J_\mu(u)$, $\mathcal{N}_\mu=\{u\in E\backslash\{0\}|\langle J'_\mu(u),u\rangle=0\}$*, and* u_μ *is the least energy nodal solution obtained in Theorem 1.1. In particular,* $c^*>0$ *is achieved either by a positive or a negative function.*

2. Technical lemmas

Lemma 2.1. *([45]) Under the condition (V), if $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in E, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(I*|u_n|^p)|v_n|^p dx = \int_{\mathbb{R}^3}(I*|u|^p)|v|^p dx.$$

In particular,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(I*|u_n|^p)|u_n|^p dx = \int_{\mathbb{R}^3}(I*|u|^p)|u|^p dx,$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(I*|u_n|^p)|u_n^{\pm}|^p dx = \int_{\mathbb{R}^3}(I*|u|^p)|u^{\pm}|^p dx.$$

Now, fixed $u \in E$ with $u^{\pm} \neq 0$, we define function $G_u : [0,\infty)\times[0,\infty) \to \mathbb{R}$ and mapping $H_u : [0,\infty)\times[0,\infty) \to \mathbb{R}^2$ by

$$G_u(s,t) = J_\mu(su^+ + tu^-),$$
$$H_u(s,t) = (\langle J'_\mu(su^+ + tu^-), su^+\rangle, \langle J'_\mu(su^+ + tu^-), tu^-\rangle).$$

Inspired by [1,11,24,27] and similar to that of in [1,11,24,27], we have following Lemmas 2.2–2.3. For reader convenient, we give the details of proof.

Lemma 2.2. *Assume that $(f_1)-(f_3)$ hold, if $u \in E$ with $u^{\pm} \neq 0$, then G_u has the following properties:*
(i) The pair (s,t) is a critical point of G_u with $s,t > 0$ if and only if $su^+ + tu^- \in \mathcal{M}_\mu$;
(ii) The function G_u has a unique critical point (s_u,t_u) on $(0,\infty)\times(0,\infty)$, which is also the unique maximum point of G_u on $[0,\infty)\times[0,\infty)$; Furthermore, if $\langle J'_\mu(u), u^{\pm}\rangle \leq 0$, then $0 < s_u, t_u \leq 1$.

Proof. *(i)* It follows from definition of G_u that

$$\nabla G_u(s,t) = (\frac{1}{s}\langle J'_\mu(su^+ + tu^-), su^+\rangle, \frac{1}{t}\langle J'_\mu(su^+ + tu^-), tu^-\rangle),$$

which implies that *(i)* holds.

In the following, we prove *(ii)*. We shall proceed through several steps to complete the proof.

Step 1. We prove the existence of s_u and t_u.

From (f_1) and (f_2), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^{q-1}, \text{for all } t \in \mathbb{R}. \tag{2.1}$$

So, together with Sobolev embedding theorem, one gets that

$$\begin{aligned}\langle J'_\mu(su^+ + tu^-), su^+\rangle &= s^2\|u^+\|^2 + bs^4(\int_{\mathbb{R}^3}|\nabla u^+|^2dx)^2 + bs^2t^2\int_{\mathbb{R}^3}|\nabla u^-|^2dx\int_{\mathbb{R}^3}|\nabla u^+|^2dx\\ &+ ls^{2p}\int_{\mathbb{R}^3}(I*|u^+|^p)|u^+|^pdx + ls^pt^p\int_{\mathbb{R}^3}(I*|u^-|^p)|u^+|^pdx\\ &-\mu\int_{\mathbb{R}^3}f(su^+)su^+dx - s^6\int_{\mathbb{R}^3}|u^+|^6dx\\ &\geq s^2\|u^+\|^2 - s^6\int_{\mathbb{R}^3}|u^+|^6dx - \mu\varepsilon s^2\int_{\mathbb{R}^3}|u^+|^2dx - \mu C_\varepsilon s^q\int_{\mathbb{R}^3}|u^+|^qdx\end{aligned}$$

$$\geq s^2\|u^+\|^2 - C_1 s^6\|u^+\|^6 - \mu\varepsilon C_2 s^2\|u^+\|^2 - \mu C_\varepsilon C_3 s^q\|u^+\|^q$$
$$\geq (1-\mu\varepsilon C_4)s^2\|u^+\|^2 - C_4 s^6\|u^+\|^6 - \mu C_4 s^q\|u^+\|^q.$$

Choosing $\varepsilon > 0$ such that $(1-\mu\varepsilon C_4) > 0$, it follows from $2 < q < 6$ that

$$\langle J'_\mu(su^+ + tu^-), su^+\rangle > 0 \text{ for } s \text{ small enough and all } t \geq 0. \tag{2.2}$$

By similarly arguments, we have that

$$\langle J'_\mu(su^+ + tu^-), tu^-\rangle > 0 \text{ for } t \text{ small enough and all } s \geq 0. \tag{2.3}$$

So, from (2.2) and (2.3), there exists $\gamma_1 > 0$ such that

$$\langle J'_\mu(\gamma_1 u^+ + tu^-), \gamma_1 u^+\rangle > 0, \langle J'_\mu(su^+ + \gamma_1 u^-), \gamma_1 u^-\rangle > 0 \tag{2.4}$$

for all $s, t \geq 0$.

Thanks to (f_1) and (f_3), we conclude that

$$f(t)t > 0, t \neq 0;\ F(t) \geq 0, t \in \mathbb{R} \tag{2.5}$$

for a.e. $x \in \mathbb{R}^3$.

Let $s = \gamma'_2 > \gamma_1$ and γ'_2 large enough, by (2.5), we have that

$$\begin{aligned}\langle J'_\mu(\gamma'_2 u^+ + tu^-), \gamma'_2 u^+\rangle &\leq (\gamma'_2)^2\|u^+\|^2 + b(\gamma'_2)^4\Big(\int_{\mathbb{R}^3}|\nabla u^+|^2dx\Big)^2 + b(\gamma'_2)^4\int_{\mathbb{R}^3}|\nabla u^-|^2dx\int_{\mathbb{R}}|\nabla u+|^2dx\\ &+ l(\gamma'_2)^{2p}\int_{\mathbb{R}^3}(I*|u^+|^p)|u^+|^pdx + l(\gamma'_2)^{2p}\int_{\mathbb{R}^3}(I*|u^-|^p)|u^+|^pdx\\ &- (\gamma'_2)^6\int_{\mathbb{R}^3}|u^+|^6dx\\ &\leq 0,\end{aligned} \tag{2.6}$$

for any $t \in [\gamma_1, \gamma'_2]$.

Similarly, let $t = \gamma'_2 > \gamma_1$ and γ'_2 large enough, we conclude that

$$\langle J'_\mu(su^+ + \gamma'_2 u^-), \gamma'_2 u^-\rangle \leq 0, \tag{2.7}$$

for any $s \in [\gamma_1, \gamma'_2]$.

Combining (2.6) and (2.7), choose $\gamma_2 > \gamma'_2$ large enough, we have that

$$\langle J'_\mu(\gamma_2 u^+ + tu^-), \gamma_2 u^+\rangle < 0, \langle J'_\mu(su^+ + \gamma_2 u^-), \gamma_2 u^-\rangle < 0 \tag{2.8}$$

for all $s, t \in [\gamma_1, \gamma_2]$.

Thanks to (2.4) and (2.8), it follows from Miranda's Theorem [48] that there is $(s_u, t_u) \in (0,\infty) \times (0,\infty)$ such that $H_u(s_u, t_u) = (0,0)$, and then $s_u u^+ + t_u u^- \in \mathcal{M}_\mu$.

Step 2. We prove the uniqueness of (s_u, t_u).

By standard arguments, we only prove the uniqueness in case of $u \in \mathcal{M}_\mu$ here.

For any $u \in \mathcal{M}_\mu$, we have that

$$\|u^\pm\|^2 + b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^\pm|^2dx + l\int_{\mathbb{R}^3}(I*|u|^p)|u^\pm|^pdx = \mu\int_{\mathbb{R}^3}f(u^\pm)u^\pm dx + \int_{\mathbb{R}^3}|u^\pm|^6dx. \quad (2.9)$$

Suppose (s_0, t_0) be an other pair of numbers such that $s_0u^+ + t_0u^- \in \mathcal{M}_\mu$ with $0 < s_0 \le t_0$. So, one has that

$$\begin{aligned}&\frac{\|u^+\|^2}{s_0^{2p-2}} + \frac{b}{s_0^{2p-4}}\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^+|^2dx + l\int_{\mathbb{R}^3}(I*|u|^p)|u^+|^pdx \le \frac{1}{s_0^{2p-6}}\int_{\mathbb{R}^3}|u^+|^6dx\\&+\mu\int_{\mathbb{R}^3}[\frac{f(s_0u^+)}{(s_0u^+)^{2p-1}}](u^+)^{2p}dx. \end{aligned} \quad (2.10)$$

and

$$\begin{aligned}&\frac{\|u^-\|^2}{t_0^{2p-2}} + \frac{b}{t_0^{2p-4}}\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx + l\int_{\mathbb{R}^3}(I*|u|^p)|u^-|^pdx \ge \frac{1}{t_0^{2p-6}}\int_{\mathbb{R}^3}|u^-|^6dx\\&+\mu\int_{\mathbb{R}^3}[\frac{f(t_0u^-)}{(t_0u^-)^{2p-1}}](u^-)^{2p}dx. \end{aligned} \quad (2.11)$$

It follows from (2.9) and (2.11) that

$$\begin{aligned}&(\frac{1}{t_0^{2p-2}} - 1)\|u^-\|^2 + b(\frac{1}{t_0^{2p-4}} - 1)\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx \ge (\frac{1}{t_0^{2p-6}} - 1)\int_{\mathbb{R}^3}|u^-|^6dx\\&+\mu\int_{\mathbb{R}^3}[\frac{f(t_0u^-)}{(t_0u^-)^{2p-1}} - \frac{f(u^-)}{(u^-)^{2p-1}}](u^-)^{2p}dx. \end{aligned} \quad (2.12)$$

Thanks to (f_3), we obtain that $t_0 \le 1$.

Similarly, by (2.9), (2.10) and (f_3), we conclude that $s_0 \ge 1$.

Consequently, $s_0 = t_0 = 1$.

Step 3. If $\langle J'_\mu(u), u^\pm\rangle \le 0$, then $0 < s_u, t_u \le 1$.

Suppose $s_u \ge t_u > 0$. Combining $s_uu^+ + t_uu^- \in \mathcal{M}_\mu$ and $\langle J'_\mu(u), u^\pm\rangle \le 0$, one has

$$\begin{aligned}&(\frac{1}{s_u^{2p-2}} - 1)\|u^+\|^2 + b(\frac{1}{s_u^{2p-4}} - 1)\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^+|^2dx \ge (\frac{1}{s_u^{2p-6}} - 1)\int_{\mathbb{R}^3}|u^+|^6dx\\&+\mu\int_{\mathbb{R}^3}[\frac{f(s_uu^+)}{(s_uu^+)^{2p-1}} - \frac{f(u^+)}{(u^+)^{2p-1}}](u^+)^{2p}dx. \end{aligned} \quad (2.13)$$

So, according to condition (f_3), we get $s_u \le 1$. Thus, we have that $0 < s_u, t_u \le 1$.

Step 4. (s_u, t_u) is the unique maximum point of G_u on $[0, \infty) \times [0, \infty)$.

Obviously, it follows from (2.5) that

$$\lim_{|(s,t)|\to\infty} G_u(s, t) = -\infty.$$

Hence, (s_u, t_u) is the unique critical point of G_u in $[0, \infty) \times [0, \infty)$.

At same time, let $t_0 \ge 0$ be fixed, we infer that

$$(G_u(s, t_0))'_s > 0, \text{ if } s \text{ is small enough.}$$

That is, $G_u(s,t)$ is an increasing function with respect to s if s is small enough.

Similarly, we conclude that $G_u(s,t)$ is an increasing function with respect to t if t is small enough.

Therefore, we conclude that maximum point of G_u cannot be achieved on the boundary of $[0,\infty)\times[0,\infty)$. That is, (s_u,t_u) is the unique maximum point of G_u on $[0,\infty)\times[0,\infty)$. □

Lemma 2.3. *There exist* $\rho>0$ *such that* $\|u^{\pm}\|\geq\rho$ *for all* $u\in\mathcal{M}_\mu$.

Proof. For any $u\in\mathcal{M}_\mu$, we have

$$\|u^{\pm}\|^2+b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^{\pm}|^2dx+l\int_{\mathbb{R}^3}(I*|u|^p)|u^{\pm}|^pdx=\mu\int_{\mathbb{R}^3}f(u^{\pm})u^{\pm}dx+\int_{\mathbb{R}^3}|u^{\pm}|^6dx.$$

Thanks to (2.1), one has

$$\begin{aligned}\|u^{\pm}\|^2&\leq\mu\int_{\mathbb{R}^3}f(u^{\pm})u^{\pm}dx+\int_{\mathbb{R}^3}|u^{\pm}|^6dx.\\&\leq\mu\varepsilon C_1\|u^{\pm}\|^2+\mu C_2\|u^{\pm}\|^q+C_3\|u^{\pm}\|^6.\end{aligned}$$

So, $(1-\mu\varepsilon C_1)\|u^{\pm}\|^2\leq\mu C_2\|u^{\pm}\|^q+C_3\|u^{\pm}\|^6$. Choosing ε small enough such that $(1-\lambda\varepsilon C_1)>0$, we get the conclusion. □

Lemma 2.4. *Let* $c_\mu=\inf_{u\in\mathcal{M}_\mu}J_\mu(u)$, *then we have that* $\lim_{\mu\to\infty}c_\mu=0$.

Proof. For any $u\in\mathcal{M}_\mu$, $\langle J'_\mu(u),u\rangle=0$. Thanks to (f_3), it is easy to obtain that

$$\overline{F}(t):f(t)t-2pF(t)\geq0, \tag{2.14}$$

and is increasing when $t>0$ and decreasing when $t<0$. Then, one gets

$$\begin{aligned}J_\mu(u)&=J_\mu(u)-\frac{1}{2p}\langle J'_\mu(u),u\rangle\\&=(\frac12-\frac{1}{2p})\|u\|^2+(\frac b4-\frac{b}{2p})(\int_{\mathbb{R}^3}|\nabla u|^2dx)^2\\&+\frac{\mu}{2p}\int_{\mathbb{R}^3}[f(u)t-2pF(u)]dx+(\frac{1}{2p}-\frac16)\int_{\mathbb{R}^3}|u|^6dx\\&\geq(\frac12-\frac{1}{2p})\|u\|^2.\end{aligned}$$

So, by Lemma 2.3 we have that $J_\mu(u)>0$, for all $u\in\mathcal{M}_\mu$. Hence, $c_\mu=\inf_{u\in M_\mu}J_\mu(u)$ is well-defined.

Let $u\in E$ with $u^{\pm}\neq0$ be fixed. According to Lemma 2.2, for each $\mu>0$, there exist $s_\mu,t_\mu>0$ such that $s_\mu u^++t_\mu u^-\in\mathcal{M}_\mu$.

By using Lemma 2.2 again and Hardy-Littlewood-Sobolev inequality (Page 106 of [49]), we have that

$$\begin{aligned}0\leq c_\mu&=\inf_{u\in\mathcal{M}_\mu}J_\mu(u)\leq J_\mu(s_\mu u^++t_\mu u^-)\\&\leq\frac12\|s_\mu u^++t_\mu u^-\|^2+\frac b4(\int_{\mathbb{R}^3}|\nabla(s_\mu u^++t_\mu u^-)|^2dx)^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{l}{2p}\int_{\mathbb{R}^3}(I*|s_\mu u^+ + t_\mu u^-|^p)|s_\mu u^+ + t_\mu u^-|^p dx\\
&=\frac{s_\mu^2}{2}\|u^+\|^2+\frac{t_\mu^2}{2}\|u^+\|^2+\frac{bs_\mu^4}{4}(\int_{\mathbb{R}^3}|\nabla u^+|^2dx)^2+\frac{bs_\mu^2t_\mu^2}{2}\int_{\mathbb{R}^3}|\nabla u^+|^2dx\int_{\mathbb{R}^3}|\nabla u^-|^2dx\\
&+\frac{bt_\mu^4}{4}(\int_{\mathbb{R}^3}|\nabla u^-|^2dx)^2+\frac{ls_\mu^{2p}}{2p}\int_{\mathbb{R}^3}(I*|u^+|^p)|u^+|^pdx\\
&+\frac{ls_\mu^pt_\mu^p}{p}\int_{\mathbb{R}^3}(I*|u^+|^p)|u^-|^pdx+\frac{lt_\mu^{2p}}{2p}\int_{\mathbb{R}^3}(I*|u^-|^p)|u^-|^pdx\\
&\le\frac{s_\mu^2}{2}\|u^+\|^2+\frac{t_\mu^2}{2}\|u^+\|^2+\frac{bs_\mu^4}{4}(\int_{\mathbb{R}^3}|\nabla u^+|^2dx)^2+\frac{bt_\mu^4}{4}(\int_{\mathbb{R}^3}|\nabla u^-|^2dx)^2\\
&+\frac{l}{2p}C_1s_\mu^{2p}\|u^+\|^{2p}+\frac{l}{2p}C_2t_\mu^{2p}\|u^-\|^{2p}
\end{aligned}
$$

To our end, we just prove that $s_\mu \to 0$ and $t_\mu \to 0$, as $\mu \to \infty$.
Let

$$B_u = \{(s_\mu, t_\mu) \in [0,\infty)\times[0,\infty) : H_u(s_\mu, t_\mu) = (0,0), \mu > 0\}.$$

By (2.5), we get

$$
\begin{aligned}
s_\mu^6\int_{\mathbb{R}^3}|u^+|^2dx+t_\mu^6\int_{\mathbb{R}^3}|u^-|^2dx &\le s_\mu^6\int_{\mathbb{R}^3}|u^+|^2dx+t_\mu^6\int_{\mathbb{R}^3}|u^-|^2dx\\
&+\mu\int_{\mathbb{R}^3}f(s_\mu u^+)s_\mu u^+dx+\mu\int_{\mathbb{R}^3}f(t_\mu u^-)t_\mu u^-dx\\
&=\|s_\mu u^+ + t_\mu u^-\|^2+b(\int_{\mathbb{R}^3}|\nabla(s_\mu u^+ + t_\mu u^-)|^2dx)^2\\
&+l\int_{\mathbb{R}^3}(I*|s_\mu u^+ + t_\mu u^-|^p)|s_\mu u^+ + t_\mu u^-|^pdx\\
&\le 2s_\mu^2\|u^+\|^2+2t_\mu^2\|u^+\|^2+4bs_\mu^4(\int_{\mathbb{R}^3}|\nabla u^+|^2dx)^2\\
&+4bt_\mu^4(\int_{\mathbb{R}^3}|\nabla u^-|^2dx)^2+lC_1s_\mu^{2p}\|u^+\|^{2p}+lC_2t_\mu^{2p}\|u^-\|^{2p},
\end{aligned}
$$

which implies that B_u is bounded.

Let $\{\mu_n\} \subset (0,\infty)$ be such that $\mu_n \to \infty$ as $n \to \infty$. Then, there exist s_0 and t_0 such that

$$(s_{\mu_n}, t_{\mu_n}) \to (s_0, t_0),$$

as $n \to \infty$ (in subsequence sense).

We claim $s_0 = t_0 = 0$.

Suppose, by contradiction, that $s_0 > 0$ or $t_0 > 0$. Thanks to $s_{\mu_n}u^+ + t_{\mu_n}u^- \in \mathcal{M}_{\mu_n}$, for any $n \in \mathbb{N}$, we have

$$\|s_{\mu_n}u^+ + t_{\mu_n}u^-\|^2 + b(\int_{\mathbb{R}^3} |\nabla(s_{\mu_n}u^+ + t_{\mu_n}u^-)|^2 dx)^2 + l\int_{\mathbb{R}^3}(I * |s_{\mu_n}u^+ + t_{\mu_n}u^-|^p)|s_{\mu_n}u^+ + t_{\mu_n}u^-|^p dx$$
$$= \mu_n \int_{\mathbb{R}^3} f(s_{\mu_n}u^+ + t_{\mu_n}u^-)(s_{\mu_n}u^+ + t_{\mu_n}u^-)dx + \int_{\mathbb{R}^3} |s_{\mu_n}u^+ + t_{\mu_n}u^-|^6 dx \tag{2.15}$$

According to $s_{\mu_n}u^+ \to s_0u^+$ and $t_{\mu_n}u^- \to t_0u^-$ in E, (2.3) and (2.5), we conclude that

$$\int_{\mathbb{R}^3} f(s_{\mu_n}u^+ + t_{\mu_n}u^-)(s_{\mu_n}u^+ + t_{\mu_n}u^-)dx$$
$$\to \int_{\mathbb{R}^3} f(s_0u^+ + t_0u^-)(s_0u^+ + t_0u^-)dx > 0,$$

as $n \to \infty$.

It follows from $\mu_n \to \infty$ as $n \to \infty$ and $\{s_{\mu_n}u^+ + t_{\mu_n}u^-\}$ is bounded in E that we have a contradiction with the equality (2.15). Hence, $s_0 = t_0 = 0$.

That is, $\lim_{\mu\to\infty} c_\mu = 0$. □

Lemma 2.5. *There exist $\mu^\star > 0$ such that for all $\mu \geq \mu^\star$, the infimum c_μ is achieved.*

Proof. According to definition of c_μ, there is a sequence $\{u_n\} \subset \mathcal{M}_\mu$ such that

$$\lim_{n\to\infty} J_\mu(u_n) = c_\mu.$$

Obviously, $\{u_n\}$ is a bounded in E. Then, up to a subsequence, still denoted by $\{u_n\}$, there exist $u \in E$ such that $u_n \rightharpoonup u$.

Since the embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ is compact, for all $r \in [2, 6)$, we have

$$u_n \to u \ \text{ in } L^r(\mathbb{R}^3),$$
$$u_n(x) \to u(x) \ \text{ a.e. } x \in \mathbb{R}^3.$$

So,

$$u_n^\pm \rightharpoonup u^\pm \ \text{ in } E,$$
$$u_n^\pm \to u^\pm \ \text{ in } L^r(\mathbb{R}^3),$$
$$u_n^\pm(x) \to u^\pm(x) \ \text{ a.e. } x \in \mathbb{R}^3.$$

Denote $\beta := \frac{1}{3}S^{\frac{3}{2}}$, where

$$S := \inf_{u\in E\setminus\{0\}} \frac{\|u\|^2}{(\int_{\mathbb{R}^3} |u|^6 dx)^{\frac{1}{3}}}.$$

According to Lemma 2.4, there is $\mu^\star > 0$ such that $c_\mu < \beta$ for all $\mu \geq \mu^\star$.

Fix $\mu \geq \mu^\star$, it follows from Lemma 2.2 that

$$J_\mu(su_n^+ + tu_n^-) \leq J_\mu(u_n)$$

for all $s, t \geq 0$.

By the weak lower semi-continuity of norm, Brezis-Lieb Lemma and Lemma 2.1, we have that

$$
\begin{aligned}
\liminf_{n\to\infty} J_\mu(su_n^+ + tu_n^-) \geq\ & \frac{s^2}{2}\lim_{n\to\infty}(\|u_n^+ - u^+\|^2 + \|u^+\|^2) + \frac{t^2}{2}\lim_{n\to\infty}(\|u_n^- - u^-\|^2 + \|u^-\|^2)\\
&+ \frac{bs^4}{4}(\lim_{n\to\infty}(\|u_n^+ - u^+\|_1^2 + \|u^+\|_1^2))^2 + \frac{bt^4}{4}(\lim_{n\to\infty}(\|u_n^- - u^-\|_1^2 + \|u^-\|_1^2))^2\\
&+ \frac{bs^2t^2}{2}\liminf_{n\to\infty}(\|u_n^+\|_1^2\|u_n^-\|_1^2) + \lim_{n\to\infty}\int_{\mathbb{R}^3}(I * |su_n^+ + tu_n^-|^p)|su_n^+ + tu_n^-|^p dx\\
&- \mu\int_{\mathbb{R}^3} F(su^+)dx - \mu\int_{\mathbb{R}^3} F(tu^-)dx - \frac{s^6}{6}\lim_{n\to\infty}(|u_n^+ - u^+|_6^6 - |u^+|_6^6)\\
&- \frac{t^6}{6}\lim_{n\to\infty}(|u_n^- - u^-|_6^6 - |u^-|_6^6)\\
&\geq J_\mu(su^+ + tu^-) + \frac{s^2}{2}A_1 + \frac{bs^4}{2}A_3^2\|u^+\|_1^2 + \frac{bs^4}{4}A_3^4 - \frac{s^6}{6}B_1\\
&+ \frac{t^2}{2}A_2 + \frac{bt^4}{2}A_4^2\|u^-\|_1^2 + \frac{bt^4}{4}A_4^4 - \frac{t^6}{6}B_2,
\end{aligned}
$$

where

$$
A_1 = \lim_{n\to\infty}\|u_n^+ - u^+\|^2, A_2 = \lim_{n\to\infty}\|u_n^- - u^-\|^2, A_3 = \lim_{n\to\infty}\|u_n^+ - u^+\|_1^2,
$$
$$
A_4 = \lim_{n\to\infty}\|u_n^- - u^-\|_1^2, B_1 = \lim_{n\to\infty}|u_n^+ - u^+|_6^6, B_2 = \lim_{n\to\infty}|u_n^- - u^-|_6^6.
$$

From above fact, one has that

$$
\begin{aligned}
c_\mu \geq\ & J_\mu(su^+ + tu^-) + \frac{s^2}{2}A_1 + \frac{bs^4}{2}A_3^2\|u^+\|_1^2 + \frac{bs^4}{4}A_3^4 - \frac{s^6}{6}B_1 + \frac{t^2}{2}A_2\\
&+ \frac{bt^4}{2}A_4^2\|u^-\|_1^2 + \frac{bt^4}{4}A_4^4 - \frac{t^6}{6}B_2,
\end{aligned}
\tag{2.16}
$$

for all $s \geq 0$ and all $t \geq 0$.

Firstly, we prove that $\mathbf{u^\pm \neq 0}$.

Since the situation $u^- \neq 0$ is analogous, we just prove $u^+ \neq 0$. By contradiction, we suppose $u^+ = 0$.

Case 1 : $B_1 = 0$.

If $A_1 = 0$, that is, $u_n^+ \to u^+$ in E. In view of Lemma 2.3, we obtain $\|u^+\| > 0$, which contradicts our supposition. If $A_1 > 0$, let $t = 0$ in (2.16), one hase

$$
\frac{s^2}{2}A_1 \leq \frac{s^2}{2}A_1 + \frac{bs^4}{2}A_3^2\|u^+\|_1^2 + \frac{bs^4}{4}A_3^4 \leq c_\mu
$$

for all $s \geq 0$. Thanks to $c_\mu < \beta$, we have a contradiction.

Case 2 : $B_1 > 0$.

According to definition of S, we have that

$$
\beta = \frac{1}{3}S^{\frac{3}{2}} \leq \frac{1}{3}(\frac{A_1}{(B_1)^{\frac{1}{3}}})^{\frac{3}{2}} = \max_{s\geq 0}\{\frac{s^2}{2}A_1 - \frac{s^6}{6}B_1\} \leq \max_{s\geq 0}\{\frac{as^2}{2}A_1 + \frac{bs^4}{2}A_3^2\|u^+\|_1^2 + \frac{bs^4}{4}A_3^4 - \frac{s^6}{6}B_1\}.
$$

According to (2.16), we have a contradiction.

From above discussions, we have that $u^{\pm} \neq 0$.

Secondly, we prove that $\mathbf{B_1 = B_2 = 0}$.

Since the situation $B_2 = 0$ is analogous, we only prove $B_1 = 0$. By contradiction, we suppose that $B_1 > 0$.

Case 1 : $B_2 > 0$.

Let $\widetilde{s}$ and $\widetilde{t}$ satisfy

$$\frac{\widetilde{\alpha}^2}{2}A_1 + \frac{b\widetilde{s}^4}{2}A_3^2\|u^+\|_1^2 + \frac{b\widetilde{s}^4}{4}A_3^4 - \frac{\widetilde{s}^6}{6}B_1 = \max_{s\geq 0}\{\frac{s^2}{2}A_1 + \frac{bs^4}{2}A_3^2\|u^+\|_1^2 + \frac{bs^4}{4}A_3^4 - \frac{s^6}{6}B_1\},$$

$$\frac{\widetilde{t}^2}{2}A_2 + \frac{b\widetilde{t}^4}{2}A_4^2\|u^-\|_1^2 + \frac{b\widetilde{t}^4}{4}A_4^4 - \frac{\widetilde{t}^6}{6}B_2 = \max_{t\geq 0}\{\frac{t^2}{2}A_2 + \frac{bt^4}{2}A_4^2\|u^-\|_1^2 + \frac{bt^4}{4}A_4^4 - \frac{t^6}{6}B_2\}.$$

Since G_u is continuous, there exists $(s_u, t_u) \in [0, \widetilde{s}] \times [0, \widetilde{t}]$ such that

$$G_u(s_u, t_u) = \max_{(s,t)\in[0,\widetilde{s}]\times[0,\widetilde{t}]} G_u(s, t).$$

In the following, we prove that $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \widetilde{t})$.

Note that, if t is small enough, we have that

$$G_u(s, 0) = J_\mu(su^+) < J_\mu(su^+) + J_\mu(tu^-) \leq J_\mu(su^+ + tu^-) = G_u(s, t),$$

for all $s \in [0, \widetilde{s}]$. That is, there exists $t_0 \in [0, \widetilde{t}]$ such that $G_u(s, 0) \leq G_u(s, t_0)$ for all $s \in [0, \widetilde{s}]$.

Hence, we conclude that any point of $(s, 0)$ with $0 \leq s \leq \widetilde{s}$ is not the maximizer of G_u, and then $(s_u, t_u) \notin [0, \widetilde{s}] \times \{0\}$. Similarly, we have that $(s_u, t_u) \notin \{0\} \times [0, \widetilde{s}]$.

On the other hand, it is easy to see that

$$\frac{s^2}{2}A_1 + \frac{bs^4}{2}A_3^2\|u^+\|_1^2 + \frac{bs^4}{4}A_3^4 - \frac{s^6}{6}B_1 > 0, s \in (0, \widetilde{s}], \tag{2.17}$$

$$\frac{t^2}{2}A_2 + \frac{bt^4}{2}A_4^2\|u^-\|_1^2 + \frac{bt^4}{4}A_4^4 - \frac{t^6}{6}B_2 > 0, t \in (0, \widetilde{t}]. \tag{2.18}$$

Then,

$$\beta \leq \frac{\widetilde{s}^2}{2}A_1 + \frac{b\widetilde{s}^4}{2}A_3^2\|u^+\|_1^2 + \frac{b\widetilde{s}^4}{4}A_3^4 - \frac{\widetilde{s}^6}{6}B_1 + \frac{t^2}{2}A_2 + \frac{bt^4}{2}A_4^2\|u^-\|_1^2 + \frac{bt^4}{4}A_4^4 - \frac{t^6}{6}B_2,$$

$$\beta \leq \frac{\widetilde{t}^2}{2}A_2 + \frac{b\widetilde{t}^4}{2}A_4^2\|u^-\|_1^2 + \frac{b\widetilde{t}^4}{4}A_4^4 - \frac{\widetilde{t}^6}{6}B_2 + \frac{s^2}{2}A_1 + \frac{bs^4}{2}A_3^2\|u^+\|_1^2 + \frac{bs^4}{4}A_3^4 - \frac{s^6}{6}B_1,$$

for all $s \in [0, \widetilde{s}]$ and all $t \in [0, \widetilde{t}]$.

Therefore, according to (2.16), we have that

$$G_u(s, \widetilde{t}) < 0, G_u(\widetilde{s}, t) < 0$$

for all $s \in [0, \widetilde{s}]$ and all $t \in [0, \widetilde{t}]$. So, $(s_u, t_u) \notin \{\widetilde{s}\} \times [0, \widetilde{t}]$ and $(s_u, t_u) \notin \times[0, \widetilde{s}] \times \{\widetilde{t}\}$.

At last, we get that $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \widetilde{t})$. Hence, it follows that (s_u, t_u) is a critical point of G_u. By Lemma 2.1, we get $s_u u^+ + t_u u^- \in \mathcal{M}_\mu$.

Combining (2.16), (2.17) with (2.18), we infer that

$$c_\mu \geq J_\mu(s_u u^+ + t_u u^-) + \frac{s_u^2}{2}A_1 + \frac{b s_u^4}{2}A_3^2\|u^+\|_1^2 + \frac{b s_u^4}{4}A_3^4 - \frac{s_u^6}{6}B_1 + \frac{t_u^2}{2}A_2 + \frac{b t_u^4}{2}A_4^2\|u^-\|_1^2 + \frac{b t_u^4}{4}A_4^4 - \frac{t_u^6}{6}B_2 > J_\mu(s_u u^+ + t_u u^-) \geq c_\mu,$$

which is a contradiction.

Case 2 : $B_2 = 0$.

In this case, we can maximize in $[0, \widetilde{s}] \times [0, \infty)$. Indeed, it is possible to show that there exist $t_0 \in [0, \infty)$ such that $J_\mu(s_u u^+ + t_u u^-) \leq 0$, for all $(s, t) \in [0, \widetilde{s}] \times [t_0, \infty)$. Hence, there are $(s_u, t_u) \in [0, \widetilde{s}] \times [0, \infty)$ satisfy

$$G_u(s_u, t_u) = \max_{s \in [0,\widetilde{s}] \times [0,\infty)} G_u(s, t).$$

Following, we prove that $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \infty)$.

It is noticed that $G_u(s, 0) < G_u(s, t)$ for $s \in [0, \widetilde{s}]$ and t small enough, so we have $(s_u, t_u) \notin [0, \widetilde{s}] \times \{0\}$.

Meantime, $G_u(0, t) < G_u(s, t)$ for $t \in [0, \infty)$ and s small enough, then we have $(s_u, t_u) \notin \{0\} \times [0, \infty)$.

On the other hand, it is obvious that

$$\beta \leq \frac{\widetilde{s}^2}{2}A_1 + \frac{b\widetilde{s}^4}{2}A_3^2\|u^+\|_1^2 + \frac{b\widetilde{s}^4}{4}A_3^4 - \frac{s^6}{6}B_1 + \frac{t^2}{2}A_2 + \frac{bt^4}{2}A_4^2\|u^-\|_1^2 + \frac{bt^4}{4}A_4^4,$$

for all $t \in [0, \infty)$.

Hence, we have that $G_u(\widetilde{s}, t) \leq 0$ for all $t \in [0, \infty)$. Thus, $(s_u, t_u) \notin \{\widetilde{s}\} \times [0, \infty)$. And so $(s_u, t_u) \in (0, \widetilde{s}) \times (0, \infty)$. That is, (s_u, t_u) is an inner maximizer of G_u in $[0, \widetilde{s}) \times [0, \infty)$. So, $s_u u^+ + t_u u^- \in \mathcal{M}_\mu$.

Therefore, according to (2.17), we get

$$c_\mu \geq J_\mu(s_u u^+ + t_u u^-) + \frac{s_u^2}{2}A_1 + \frac{b s_u^4}{2}A_3^2\|u^+\|_1^2 + \frac{b s_u^4}{4}A_3^4 - \frac{s_u^6}{6}B_1 + \frac{t_u^2}{2}A_2 + \frac{b t_u^4}{2}A_4^2\|u^-\|_1^2 + \frac{b t_u^4}{4}A_4^4 > J_\mu(s_u u^+ + t_u u^-) \geq c_\mu.$$

That is, we have a contradiction.

Therefore, from above discussion, we have that $B_1 = B_2 = 0$.

Lastly, we prove that c_μ is achieved.

For $u^\pm \neq 0$, according to Lemma 2.2, there exist $s_u, t_u > 0$ such that $\bar{u} : s_u u^+ + t_u u^- \in \mathcal{M}_\mu$. Furthermore, it is easy to see that

$$\langle J'_\mu(u), u^\pm \rangle \leq 0.$$

So,we have that $0 < s_u, t_u \leq 1$.

Since $u_n \in \mathcal{M}_\mu$, using Lemma 2.2 again, we get

$$J_\mu(s_u u_n^+ + t_u u_n^-) \leq J_\mu(u_n^+ + u_n^-) = J_\mu(u_n).$$

Thanks to (2.14), $B_1 = B_2 = 0$ and the norm in E is weak lower semicontinuous, we conclude that

$$\begin{aligned} c_\mu &\le J_\mu(\overline{u})) - \frac{1}{2p}\langle J'_\mu(\overline{u}), \overline{u}\rangle \\ &= (\frac{1}{2} - \frac{1}{2p})\|\overline{u}\|^2 + (\frac{b}{4} - \frac{b}{2p})(\int_{\mathbb{R}^3} |\nabla \overline{u}|^2 dx)^2 \\ &+ \frac{\mu}{2p}\int_{\mathbb{R}^3} [f(\overline{u})t - 2pF(\overline{u})]dx + (\frac{1}{2p} - \frac{1}{6})\int_{\mathbb{R}^3} |\overline{u}|^6 dx \\ &= (\frac{1}{2} - \frac{1}{2p})(\|s_u u^+\|^2 + \|t_u u^-\|^2) + (\frac{1}{2p} - \frac{1}{6})(|s_u u^+|_6^6 + |t_u u^-|_6^6) \\ &+ \frac{\mu}{2p}\int_{\mathbb{R}^3} [f(s_u u^+)s_u u^+ - 2pF(s_u u+)]dx + \frac{\mu}{2p}\int_{\mathbb{R}^3} [f(t_u u^-)s_u u^+ - 2pF(t_u u-)]dx \\ &+ (\frac{b}{4} - \frac{b}{2p})s_u^4(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx)^2 + (\frac{b}{4} - \frac{b}{2p})2s_u^2 t_u^2 \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \int_{\mathbb{R}^3} |\nabla u^-|^2 dx \\ &+ (\frac{b}{4} - \frac{b}{2p})t_u^4(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx)^2 \\ &\le (\frac{1}{2} - \frac{1}{2p})\|u\|^2 + (\frac{b}{4} - \frac{b}{2p})(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 \\ &+ \frac{\mu}{2p}\int_{\mathbb{R}^3} [f(u)u - 2pF(u)]dx + (\frac{1}{2p} - \frac{1}{6})\int_{\mathbb{R}^3} |u|^6 dx \\ &\le \liminf_{n\to\infty}[J_\mu(u_n) - \frac{1}{2p}\langle J'_\mu(u_n), u_n\rangle] = c_\mu. \end{aligned}$$

Therefore, $s_u = t_u = 1$, and c_μ is achieved by $u_\mu := u^+ + u^- \in \mathcal{M}_\mu$. □

3. The proof of main results

Proof. (**Proof of Theorem 1.1**) From Lemma 2.5, we just prove that the minimizer u_μ for c_μ is indeed a sign-changing solution of problem (1.1). That is, we need prove $J'_\mu(u_\mu) = 0$. Suppose, by contradiction, that $J'_\mu(u_\mu) \neq 0$. Then there exist $\delta > 0$ and $\theta > 0$ such that

$$\|J'_\mu(v)\| \ge \theta, \text{ for all } \|v - u_\mu\| \le 3\delta.$$

Choose $\sigma \in (0, \min\{1/2, \frac{\delta}{\sqrt{2}\|u_\mu\|}\})$. Let $D := (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$ and $g(s,t) = su_\mu^+ + tu_\mu^-$, $(s,t) \in D$.

Thanks to Lemma 2.3, for $(s,t) \in (\mathbb{R}_+ \times \mathbb{R}_+)\backslash(1,1)$, we have

$$J_\mu(su_\mu^+ + tu_\mu^-) < J_\mu(u_\mu^+ + u_\mu^-) = c_\mu.$$

So, it follows that

$$\overline{c}_\lambda^\mu := \max_{\partial D} I \circ g < c_\mu. \tag{3.1}$$

Let $\varepsilon := min\{(c_\mu - \overline{c}_\lambda^\mu)/2, \theta\delta/8\}$ and $S_\delta := B(u_\mu, \delta)$, according to Lemma 2.3 in [50], there exists a deformation $\eta \in C([0,1] \times E, E)$ satisfy

(a) $\eta(1, v) = v$ if $v \notin J_\mu^{-1}([c_\mu - 2\varepsilon, c_\mu + 2\varepsilon] \cap S_{2\delta})$;

(b) $\eta(1, J_\mu^{c_\mu+\varepsilon} \cap S_\delta) \subset J_\mu^{c_\mu-\varepsilon}$, where $J_\mu^c = \{u \in E : J_\mu(u) \leq c\}$;

(c) $J_\mu(\eta(1, v)) \leq J_\mu(v)$ for all $v \in E$.

From (b) and Lemma 2.2, it is easy to see that

$$\max_{(s,t)\in\bar{D}} J_\mu(\eta(1, g(s,t))) < c_\mu. \tag{3.2}$$

Next, we prove that $\eta(1, g(D)) \cap \mathcal{M}_\mu \neq \emptyset$, which contradicts the definition of c_μ.

Let $h(s,t) := \eta(1, g(s,t))$ and

$$\begin{aligned}\Psi_0(s,t) &:= (\langle (J_\mu)'(g(s,t)), u_\mu^+\rangle, \langle (J_\mu)'(g(s,t)), u_\mu^-\rangle)\\ &= (\langle (J_\mu)'(su_\mu^+ + tu_\mu^-), su_\mu^+\rangle, \langle (J_\mu)'(su_\mu^+ + tu_\mu^-), tu_\mu^-\rangle)\\ &:= (\varphi_u^1(s,t), \varphi_u^2(s,t))\end{aligned}$$

and

$$\Psi_1(s,t) := (\langle (J_\mu)'(h(s,t)), (h(s,t))^+\rangle, \langle (J_\mu)'(h(s,t)), (h(s,t))^-\rangle).$$

Let

$$M = \begin{bmatrix} \frac{\varphi_u^1(s,t)}{\partial s}|_{(1,1)} & \frac{\varphi_u^2(s,t)}{\partial s}|_{(1,1)} \\ \frac{\varphi_u^1(s,t)}{\partial t}|_{(1,1)} & \frac{\varphi_u^2(s,t)}{\partial t}|_{(1,1)} \end{bmatrix}.$$

From condition (f_3), for $t \neq 0$, we have

$$f'(t)t^2 - (2p-1)f(t)t > 0.$$

Therefore, by direct calculation, we can conclude that $\det M > 0$.

Since $\Psi_0(s,t)$ is a C^1 function and $(1,1)$ is the unique isolated zero point of Ψ_0, by using the degree theory, we deduce that $\deg(\Psi_0, D, 0) = 1$.

So, combining (3.1) with (a), we obtain $g(s,t) = h(s,t)$ on ∂D. Consequently, we obtain $\deg(\Psi_1, D, 0) = 1$. Therefore, $\Psi_1(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$. By similar discussion as in [45], we can prove that $h(s,t)^\pm \neq 0$. So, we obtain that $\eta(1, g(s_0, t_0)) = h(s_0, t_0) \in \mathcal{M}_\mu$, which is contradicted to (3.2). □

Proof. (**Proof of Theorem 1.2**) Similar as the proof of Lemma 2.5, there exists $\mu_1^\star > 0$ such that for all $\mu \geq \mu_1^\star$, there exists $v_\mu \in \mathcal{N}_\mu$ such that $J_\mu(v_\mu) = c^* > 0$. By standard arguments, the critical points of the functional J_μ on $\mathcal{N}_\mu$ are critical points of J_μ in E and we obtain $J_\mu'(v_\mu) = 0$. That is, v_μ is a least energy solution of system (1.1). According to Theorem 1.1, we know that the system (1.1) has a least energy nodal solution u_μ. Let $\mu^{\star\star} = \max\{\mu^\star, \mu_1^\star\}$. As the proof of Lemma 2.2, there exist $s_{u^+}, t_{u^-} \in (0,1)$ such that $s_{u^+}u^+ \in \mathcal{N}_\mu$, $t_{u^-}u^- \in \mathcal{N}_\mu$. Therefore, in view of Lemma 2.2 again, we infer that

$2c^* \leq J_\mu(s_{u^+}u^+) + J_\mu(t_{u^-}u^-) \leq J_\mu(s_{u^+}u^+ + t_{u^-}u^-) < J_\mu(u^+ + u^-) = c_\mu$. □

4. Conclusions

In this paper, by the minimization argument on the nodal Nehari manifold and the quantitative deformation lemma, we discussed the existence of least energy nodal solution for a class of Kirchhoff-type system with Hartree-type nonlinearity and critical growth. Our results improve and generalize some interesting results which were obtained in subcritical case.

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Conflict of interest

We declare that we have no conflict of interest.

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