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Research article

Homomorphism-derivation functional inequalities in C^*-algebras

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Abstract: In this paper, we introduce and solve the following additive-additive (s,t)-functional inequality

$$\begin{aligned} &\|g(x+y)-g(x)-g(y)\| + \left\|2h\left(\frac{x+y}{2}\right)-h(x)-h(y)\right\| \\ &\le \left\|s\left(2g\left(\frac{x+y}{2}\right)-g(x)-g(y)\right)\right\| + \|t(h(x+y)-h(x)-h(y))\|, \end{aligned} \tag{0.1}$$

where s and t are fixed nonzero complex numbers with $|s|<1$ and $|t|<1$. Furthermore, we investigate homomorphisms and derivations in complex Banach algebras and unital C^*-algebras, associated to the additive-additive (s,t)-functional inequality (0.1) under some extra condition.

Moreover, we introduce and solve the following additive-additive (s,t)-functional inequality

$$\begin{aligned} &\|g(x+y+z)-g(x)-g(y)-g(z)\| + \left\|3h\left(\frac{x+y+z}{3}\right)+h(x-2y+z)+h(x+y-2z)-3h(x)\right\| \\ &\le \left\|s\left(3g\left(\frac{x+y+z}{3}\right)-g(x)-g(y)-g(z)\right)\right\| \\ &+ \|t\left(h(x+y+z)+h(x-2y+z)+h(x+y-2z)-3h(x)\right)\|, \end{aligned} \tag{0.2}$$

where s and t are fixed nonzero complex numbers with $|s|<1$ and $|t|<1$. Furthermore, we investigate C^*-ternary derivations and C^*-ternary homomorphisms in C^*-ternary algebras, associated to the additive-additive (s,t)-functional inequality (0.2) under some extra condition.

Keywords: additive-additive (s,t)-functional inequality; derivation and homomorphism in C^*-ternary algebra; derivation and homomorphism in unital C^*-algebra

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1. Introduction and preliminaries

Let A be a complex Banach algebra. A $\mathbb{C}$-linear mapping $g : A \to A$ is a derivation if $g : A \to A$ satisfies

$$g(xy) = g(x)y + xg(y),$$

for all $x, y \in A$, and a $\mathbb{C}$-linear mapping $h : A \to A$ is a homomorphism if $h : A \to A$ satisfies

$$h(xy) = h(x)h(y),$$

for all $x, y \in A$,

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [18]).

Let A be a C^*-ternary algebra. A $\mathbb{C}$-linear mapping $g : A \to A$ is a C^*-ternary derivation if $g : A \to A$ satisfies

$$g([x, y, z]) = [g(x), y, z] + [x, g(y), z] + [x, y, g(z)],$$

for all $x, y, z \in A$, and a $\mathbb{C}$-linear mapping $h : A \to A$ is a C^*-ternary homomorphism if $h : A \to A$ satisfies

$$h([x, y, z]) = [h(x), h(y), h(z)],$$

for all $x, y, z \in A$.

In 1940, Ulam [16] raised a question the stability problem for functional equations, and Hyers [8], Aoki [1], Rassias [14] and Găvruta [7] have given positive answers for additive additive functional equations in Banach spaces. Park [11, 12] introduced new additive functional inequalities and proved the Hyers-Ulam stability of the additive functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations in various spaces have been extensively investigated by a number of authors (see [2–6, 13, 15, 17]).

This paper is organized as follows: In Sections 2 and 3, we solve the additive-additive (s, t)-functional inequality (0.1). Furthermore, we investigate homomorphisms and derivations on compex Banach algebras and unital C^*-algebras associated to the additive-additive (s, t)-functional inequality (0.1) and the following functional inequality

$$\|g(xy) - g(x)y - xg(y)\| + \|h(xy) - h(x)h(y)\| \le \varphi(x, y). \tag{1.1}$$

In Section 4, we solve the additive-additive (s, t)-functional inequality (0.2). Furthermore, we investigate C^*-ternary homomorphisms and C^*-ternary derivations on C^*-ternary algebras associated to the additive-additive (s, t)-functional inequality (0.2) and the following functional inequality

$$\begin{aligned} &\|g([x, y, z]) - [g(x), y, z] - [x, g(y), z] - [x, y, g(z)]\| \\ &+\|h([x, y, z]) - [h(x), h(y), h(z)]\| \le \varphi(x, y, z). \end{aligned} \tag{1.2}$$

Throughout this paper, assume that A is a complex Banach algebra and that B is a C^*-ternary algebra and that s and t are fixed nonzero complex numbers with $|s| < 1$ and $|t| < 1$.

2. Additive-additive (s,t)-functional inequality (0.1) in Banach algebras

In this section, we solve and investigate the additive-additive (s,t)-functional inequality (0.1) in complex Banach algebras.

Lemma 2.1. *If mappings $g,h : A \to A$ satisfy $g(0) = h(0) = 0$ and*

$$\|g(x+y) - g(x) - g(y)\| + \left\|2h\left(\frac{x+y}{2}\right) - h(x) - h(y)\right\| \le \left\|s\left(2g\left(\frac{x+y}{2}\right) - g(x) - g(y)\right)\right\| + \|t(h(x+y) - h(x) - h(y))\|, \tag{2.1}$$

for all $x,y \in A$, then the mappings $g,h : A \to A$ are additive.

Proof. Letting $x = y$ in (2.1), we get

$$\|g(2x) - 2g(x)\| \le \|t(h(2x) - 2h(x))\|,$$

for all $x \in A$.

Letting $y = 0$ in (2.1), we get

$$\left\|2h\left(\frac{x}{2}\right) - h(x)\right\| \le \left\|s\left(2g\left(\frac{x}{2}\right) - g(x)\right)\right\|$$

and so

$$\|2h(x) - h(2x)\| \le \|s(2g(x) - g(2x))\|,$$

for all $x \in A$. Thus

$$\|g(2x) - 2g(x)\| \le \|st(2g(x) - g(2x))\|,$$

$$\|2h(x) - h(2x)\| \le \|st(h(2x) - 2h(x))\|,$$

for all $x \in A$. So $h(2x) = 2h(x)$ and $g(2x) = 2g(x)$ for all $x \in A$, since $|st| < 1$. It follows from (2.1) that

$$\|g(x+y) - g(x) - g(y)\| + \|h(x+y) - h(x) - h(y)\| \le \|s(g(x+y) - g(x) - g(y))\| + \|t(h(x+y) - h(x) - h(y))\|,$$

for all $x,y \in A$. Thus $g(x+y) - g(x) - g(y) = 0$ and $h(x+y) - h(x) - h(y) = 0$ for all $x \in A$, since $|s| < 1$ and $|t| < 1$. So the mappings $g,h : A \to A$ are additive. □

Lemma 2.2. *[10, Theorem 2.1] Let $f : A \to A$ be a mapping such that*

$$f(\lambda(a+b)) = \lambda f(a) + \lambda f(b),$$

for all $\lambda \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and all $a,b \in A$. Then the mapping $f : A \to A$ is $\mathbb{C}$-linear.

Now, we investigate homomorphisms and derivations in complex Banach algebras associated to the additive-additive (s,t)-functional inequality (2.1).

Theorem 2.3. *Let $\varphi : A^2 \to [0,\infty)$ be a function such that*

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{2.2}$$

for all $x, y \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and

$$\begin{aligned} &\|g(\lambda(x+y)) - \lambda g(x) - \lambda g(y)\| + \left\|2h\left(\lambda\frac{x+y}{2}\right) - \lambda h(x) - \lambda h(y)\right\| \\ &\le \left\|s\left(2g\left(\lambda\frac{x+y}{2}\right) - \lambda g(x) - \lambda g(y)\right)\right\| + \|t(h(\lambda(x+y)) - \lambda h(x) - \lambda h(y))\|, \end{aligned} \tag{2.3}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. If $g, h : A \to A$ satisfy

$$\|g(xy) - g(x)y - xg(y)\| + \|h(xy) - h(x)h(y)\| \le \varphi(x, y), \tag{2.4}$$

for all $x, y \in A$, then the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism.

Proof. Let $\lambda = 1$ in (2.3). By Lemma 2.1, the mappings $g, h : A \to A$ are additive.

It follows from (2.3) that

$$\begin{aligned} &\|g(\lambda(x+y)) - \lambda g(x) - \lambda g(y)\| + \|h(\lambda(x+y)) - \lambda h(x) - \lambda h(y)\| \\ &\le \|s(g(\lambda(x+y)) - \lambda g(x) - \lambda g(y))\| + \|t(h(\lambda(x+y)) - \lambda h(x) - \lambda h(y))\|, \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Since $|s| < 1$ and $|t| < 1$,

$$\begin{aligned} g(\lambda(x+y)) - \lambda g(x) - \lambda g(y) &= 0, \\ h(\lambda(x+y)) - \lambda h(x) - \lambda h(y) &= 0, \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Thus by Lemma 2.2, the mappings $g, h : A \to A$ are $\mathbb{C}$-linear.

It follows from (2.4) and the additivity of g, h that

$$\begin{aligned} &\|g(xy) - g(x)y - xg(y)\| + \|h(xy) - h(x)h(y)\| \\ &= 4^n \left\|g\left(\frac{xy}{4^n}\right) - g\left(\frac{x}{2^n}\right)\frac{y}{2^n} - \frac{x}{2^n}g\left(\frac{y}{2^n}\right)\right\| + 4^n \left\|h\left(\frac{xy}{4^n}\right) - h\left(\frac{x}{2^n}\right)h\left(\frac{y}{2^n}\right)\right\| \\ &\le 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \end{aligned}$$

which tends to zero as $n \to \infty$, by (2.2). So

$$\begin{aligned} g(xy) - g(x)y - xg(y) &= 0, \\ h(xy) - h(x)h(y) &= 0, \end{aligned}$$

for all $x, y \in A$. Hence the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism. □

Theorem 2.4. *Let $\varphi : A^2 \to [0,\infty)$ be a function and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$,* (2.3)*,* (2.4) *and*

$$\sum_{j=1}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty, \tag{2.5}$$

for all $x, y \in A$. Then the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.3, one can show that the mappings $g, h : A \to A$ are $\mathbb{C}$-linear.

It follows from (2.4) and the additivity of g, h that

$$\begin{aligned}
&\|g(xy) - g(x)y - xg(y)\| + \|h(xy) - h(x)h(y)\| \\
&= \frac{1}{4^n}\|g(4^n xy) - g(2^n x)(2^n y) - (2^n x)g(2^n y)\| + \frac{1}{4^n}\|h(4^n xy) - h(2^n x)h(2^n y)\| \\
&\le \frac{1}{4^n}\varphi(2^n x, 2^n y),
\end{aligned}$$

which tends to zero as $n \to \infty$, by (2.5). So

$$\begin{aligned}
g(xy) - g(x)y - xg(y) &= 0, \\
h(xy) - h(x)h(y) &= 0,
\end{aligned}$$

for all $x, y \in A$. Hence the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism. □

Next, we investigate homomorphisms and derivations in complex Banach algebras associated to the additive-additive (s,t)-functional inequality (2.1) by using a similar method to the fixed point alternative.

Theorem 2.5. *Let $\varphi : A^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y), \tag{2.6}$$

for all $x, y \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3) *and* (2.4)*. Then the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism.*

Proof. It follows from (2.6) that

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \le \sum_{j=1}^{\infty} 4^j \frac{L^j}{4^j}\varphi(x,y) = \frac{L}{1-L}\varphi(x,y) < \infty,$$

for all $x, y \in A$. By Theorem 2.3, the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism. □

Theorem 2.6. *Let $\varphi : A^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right), \tag{2.7}$$

for all $x, y \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3) and (2.4). Then the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism.

Proof. It follows from (2.7) that

$$\sum_{j=1}^{\infty} \frac{1}{4^j} \varphi\left(2^j x, 2^j y\right) \leq \sum_{j=1}^{\infty} \frac{1}{4^j} (4L)^j \varphi(x, y) = \frac{L}{1 - L} \varphi(x, y) < \infty,$$

for all $x, y \in A$. By Theorem 2.4, the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism. □

Remark 2.7. In (2.4), the pair (g, h) of a derivation g and a homomorphism h can be replaced by the pair of a derivation and a derivation or the pair of a homomorphism and a homomorphism.

3. Additive-additive (s, t)-functional inequality (0.1) in C^*-algebras

In this section, we study homorphisms and derivations in unital C^*-algebras associated to the additive-additive (s, t)-functional inequality (2.1) by using a similar method to the fixed point alternative. Throughout this scetion, assume that A is a unital C^*-algebra with unitary group $U(A)$.

Theorem 3.1. *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (2.2). Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and (2.3). If $g, h : A \to A$ satisfy*

$$\|g(uv) - g(u)v - ug(v)\| + \|h(uv) - h(u)h(v)\| \leq \varphi(u, v), \tag{3.1}$$

for all $u, v \in U(A)$, then the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism.

Proof. Let $\lambda = 1$ in (2.3). By Lemma 2.1, the mappings $g, h : A \to A$ are additive.

It follows from (2.3) that

$$\begin{aligned} &\|g(\lambda(x + y)) - \lambda g(x) - \lambda g(y)\| + \|h(\lambda(x + y)) - \lambda h(x) - \lambda h(y)\| \\ &\leq \|s(g(\lambda(x + y)) - \lambda g(x) - \lambda g(y))\| + \|t(h(\lambda(x + y)) - \lambda h(x) - \lambda h(y))\|, \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Since $|s| < 1$ and $|t| < 1$,

$$\begin{aligned} g(\lambda(x + y)) - \lambda g(x) - \lambda g(y) &= 0, \\ h(\lambda(x + y)) - \lambda h(x) - \lambda h(y) &= 0, \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Thus by Lemma 2.2, the mappings $g, h : A \to A$ are $\mathbb{C}$-linear.

Since D is $\mathbb{C}$-linear in the first variable and each $y \in A$ is a finite linear combination of unitary elements (see [9]), i.e., $y = \sum_{i=1}^{k} \lambda_i v_i$ ($\lambda_i \in \mathbb{C}$, $v_i \in U(A)$).

$$\begin{aligned} g(uy) &= g(u\sum_{i=1}^{k}\lambda_i v_i) = \sum_{i=1}^{k}\lambda_i g(uv_i) = \sum_{i=1}^{k}\lambda_i (g(u)v_i + ug(v_i)) \\ &= (\sum_{i=1}^{k}\lambda_i) g(u)v_i + (\sum_{i=1}^{k}\lambda_i u) g(v_i) = g(u)y + ug(y), \end{aligned}$$

for all $x, y \in A$.

Similarly, let $x = \sum_{j=1}^{m}\lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$). Then

$$\begin{aligned} g(xy) &= g(\sum_{j=1}^{m}\lambda_j u_j y) = \sum_{j=1}^{m}\lambda_j g(u_j y) = \sum_{j=1}^{m}\lambda_j (g(u_j)y + u_j g(y)) \\ &= (\sum_{j=1}^{m}\lambda_j) g(u_j)y + (\sum_{j=1}^{m}\lambda_j u_j) g(y) = g(x)y + xg(y), \end{aligned}$$

for all $x, y \in A$. So

$$g(xy) - g(x)y - xg(y) = 0,$$

for all $x, y \in A$. Hence the $\mathbb{C}$-linear mapping $g : A \to A$ is a derivation.

Since D is $\mathbb{C}$-linear in the first variable and each $y \in A$ is a finite linear combination of unitary elements (see [9]), $y = \sum_{i=1}^{k}\lambda_i v_i$ ($\lambda_i \in \mathbb{C}$, $v_i \in U(A)$),

$$h(uy) = h(u\lambda\sum_{i=1}^{k} v_i) = \sum_{i=1}^{k}\lambda_i h(uv_i) = \sum_{i=1}^{k}\lambda_i (h(u)h(v_i)) = h(u)h(y),$$

for all $x, y \in A$.

Similarly, let $x = \sum_{j=1}^{m}\lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$). Then

$$h(xy) = h(\lambda\sum_{i=1}^{m} u_j y) = \sum_{j=1}^{m}\lambda_j h(u_j y) = \sum_{j=1}^{m}\lambda_j (h(u_j)h(y)) = h(x)h(y),$$

for all $x, y \in A$. So

$$h(xy) - h(x)h(y) = 0$$

for all $x, y \in A$. Hence the $\mathbb{C}$-linear mapping $g : A \to A$ is a homomorphism. □

Theorem 3.2. *Let $\varphi : A^2 \to [0, \infty)$ be a function and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3), (3.1) and (2.5). Then the mapping $g : A \to A$ is a derivation and the mapping $h : A \to A$ is a homomorphism.*

Proof. The proof is similar to the proofs of Theorems 2.4 and 3.1. □

Remark 3.3. Using the fixed point method given in Theorems 2.5 and 2.6, one can obtain the same results as in Theorems 3.1 and 3.2.

4. Additive-additive (s,t)-functional inequality (0.2) in C^*-ternary algebras

In this section, we solve and investigate the additive-additive (s,t)-functional inequality (0.2) in C^*-ternary algebras.

Lemma 4.1. *If mappings $g,h : B \to B$ satisfy $g(0) = h(0) = 0$ and*

$$\begin{aligned} &\|g(x+y+z) - g(x) - g(y) - g(z)\| \\ &+\left\|3h\left(\frac{x+y+z}{3}\right) + h(x-2y+z) + h(x+y-2z) - 3h(x)\right\| \\ &\le \left\|s\left(3g\left(\frac{x+y+z}{3}\right) - g(x) - g(y) - g(z)\right)\right\| \\ &+ \|t(h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x))\|, \end{aligned} \tag{4.1}$$

for all $x,y,z \in B$, then the mappings $g,h : B \to B$ are additive.

Proof. Letting $x = y = z$ in (4.1), we get

$$\|g(3x) - 3g(x)\| \le \|t(h(3x) - 3h(x))\|,$$

for all $x \in B$.

Letting $y = z = 0$ in (4.1), we get

$$\left\|3h\left(\frac{x}{3}\right) - h(x)\right\| \le \left\|s\left(3g\left(\frac{x}{3}\right) - g(x)\right)\right\|$$

and so

$$\|3h(x) - h(3x)\| \le \|s(3g(x) - g(3x))\|,$$

for all $x \in B$. Thus

$$\|g(3x) - 3g(x)\| \le \|st(3g(x) - g(3x))\|,$$

$$\|3h(x) - h(3x)\| \le \|st(h(3x) - 3h(x))\|,$$

for all $x \in B$. So $h(3x) = 3h(x)$ and $g(3x) = 3g(x)$ for all $x \in B$, since $|st| < 1$. It follows from (4.1) that

$$\begin{aligned} &\|g(x+y+z) - g(x) - g(y) - g(z)\| \\ &+\|h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x)\| \\ &\le \|s(g(x+y+z) - g(x) - g(y) - g(z))\| \\ &+ \|t(h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x))\|, \end{aligned}$$

for all $x,y,z \in B$. Thus

$$g(x+y+z) - g(x) - g(y) - g(z) = 0,$$

$$h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x) = 0,$$

for all $x,y,z \in A$, since $|s| < 1$ and $|t| < 1$. So the mappings $g,h : B \to B$ are additive. □

Now, we investigate C^*-ternary homomorphisms and C^*-ternary derivations in C^*-ternary algebras associated to the additive-additive (s,t)-functional inequality (4.1).

Theorem 4.2. *Let $\varphi : B^3 \to [0,\infty)$ be a function such that*

$$\sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) < \infty, \tag{4.2}$$

for all $x, y, z \in B$. Let $g, h : B \to B$ be mappings satisfying $g(0) = h(0) = 0$ and

$$\begin{aligned} &\|g(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z))\| \\ &+\left\|3h\left(\lambda\frac{x+y+z}{3}\right) + \lambda(h(x-2y+z)+h(x+y-2z)-3h(x))\right\| \\ &\le \left\|s\left(3g\left(\lambda\frac{x+y+z}{3}\right) - \lambda(g(x)+g(y)+g(z))\right)\right\| \\ &+\|t(h(\lambda(x+y+z)) + \lambda(h(x-2y+z)+h(x+y-2z)-3h(x)))\|, \end{aligned} \tag{4.3}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in B$. If $g, h : B \to B$ satisfy

$$\begin{aligned} &\|g([x,y,z]) - [g(x),y,z] - [x,g(y),z] - [x,y,g(z)]\| \\ &+\|h([x,y,z]) - [h(x),h(y),h(z)]\| \le \varphi(x,y,z), \end{aligned} \tag{4.4}$$

for all $x, y, z \in B$, then the mapping $g : B \to B$ is a C^*-ternary derivation and the mapping $h : B \to B$ is a C^*-ternary homomorphism.

Proof. Let $\lambda = 1$ in (4.3). By Lemma 4.1, the mappings $g, h : B \to B$ are additive.
It follows from (4.3) that

$$\begin{aligned} &\|g(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z))\| \\ &+\|h(\lambda(x+y+z)) + \lambda(h(x-2y+z)+h(x+y-2z)-3h(x))\| \\ &\le \|s(g(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z)))\| \\ &+\|t(h(\lambda(x+y+z)) + \lambda(h(x-2y+z)+h(x+y-2z)-3h(x)))\|, \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in B$. Since $|s| < 1$ and $|t| < 1$,

$$\begin{aligned} g(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z)) &= 0, \\ h(\lambda(x+y+z)) + \lambda(h(x-2y+z)+h(x+y-2z)-3h(x)) &= 0, \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in B$. Thus by Lemma 2.2, the mappings $g, h : B \to B$ are $\mathbb{C}$-linear.
It follows from (4.4) and the additivity of g, h that

$$\begin{aligned} &\|g([x,y,z]) - [g(x),y,z] - [x,g(y),z] - [x,y,g(z)]\| \\ &+\|h([x,y,z]) - [h(x),h(y),h(z)]\| \\ &= 8^n \left\|g\left(\frac{[x,y,z]}{8^n}\right) - \left[g\left(\frac{x}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, g\left(\frac{y}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, g\left(\frac{z}{2^n}\right)\right]\right\| \end{aligned}$$

$$+8^n \left\| h\left(\frac{[x,y,z]}{8^n}\right) - \left[h\left(\frac{x}{2^n}\right), h\left(\frac{y}{2^n}\right), h\left(\frac{z}{2^n}\right)\right]\right\|$$
$$\leq 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right),$$

which tends to zero as $n \to \infty$, by (4.2). So

$$\begin{aligned} g([x,y,z]) - [g(x),y,z] - [x,g(y),z] - [x,y,g(z)] &= 0, \\ h([x,y,z]) - [h(x),h(y),h(z)] &= 0, \end{aligned}$$

for all $x,y,z \in B$. Hence the mapping $g : B \to B$ is a C^*-ternary derivation and the mapping $h : B \to B$ is a C^*-ternary homomorphism. □

Theorem 4.3. *Let $\varphi : B^3 \to [0,\infty)$ be a function and $g,h : B \to B$ be mappings satisfying $g(0) = h(0) = 0$, (4.3), (4.4) and*

$$\sum_{j=1}^{\infty} \frac{1}{8^j} \varphi(2^j x, 2^j y, 2^j z) < \infty, \tag{4.5}$$

for all $x,y,z \in B$. Then the mapping $g : B \to B$ is a C^*-ternary derivation and the mapping $h : B \to B$ is a C^*-ternary homomorphism.

Proof. By the same reasoning as in the proof of Theorem 4.2, one can show that the mappings $g,h : B \to B$ are $\mathbb{C}$-linear.

It follows from (4.4) and the additivity of g,h that

$$\|g([x,y,z]) - [g(x),y,z] - [x,g(y),z] - [x,y,g(z)]\|$$
$$+\|h([x,y,z]) - [h(x),h(y),h(z)]\|$$
$$= \frac{1}{8^n} \|g(8^n[x,y,z]) - [g(2^n x), 2^n y, 2^n z] - [2^n x, g(2^n y), 2^n z] - [2^n x, 2^n y, g(2^n z)]\|$$
$$+\frac{1}{8^n} \|h(8^n[x,y,z]) - [h(2^n x), h(2^n y), h(2^n z)]\|$$
$$\leq \frac{1}{8^n} \varphi(2^n x, 2^n y, 2^n z),$$

which tends to zero as $n \to \infty$, by (4.5). So

$$\begin{aligned} g([x,y,z]) - [g(x),y,z] - [x,g(y),z] - [x,y,g(z)] &= 0, \\ h([x,y,z]) - [h(x),h(y),h(z)] &= 0, \end{aligned}$$

for all $x,y,z \in B$. Hence the mapping $g : B \to B$ is a C^*-ternary derivation and the mapping $h : B \to B$ is a C^*-ternary homomorphism. □

Next, we investigate C^*-ternary homomorphisms and C^*-ternary derivations in C^*-ternary algebras associated to the additive-additive (s,t)-functional inequality (4.1) by using a similar method to the fixed point alternative.

Theorem 4.4. *Let* $\varphi : B^3 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{L}{8}\varphi(x, y, z), \tag{4.6}$$

for all $x, y, z \in B$. *Let* $g, h : B \to B$ *be mappings satisfying* $g(0) = h(0) = 0$, (4.3) *and* (4.4). *Then the mapping* $g : B \to B$ *is a* C^**-ternary derivation and the mapping* $h : B \to B$ *is a* C^**-ternary homomorphism.*

Proof. It follows from (4.6) that

$$\sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}\right) \le \sum_{j=1}^{\infty} 8^j \frac{L^j}{8^j}\varphi(x, y, z) = \frac{L}{1-L}\varphi(x, y, z) < \infty,$$

for all $x, y, z \in B$. By Theorem 4.2, the mapping $g : B \to B$ is a C^*-ternary derivation and the mapping $h : B \to B$ is a C^*-ternary homomorphism. □

Theorem 4.5. *Let* $\varphi : B^3 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y, z) \le 8L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), \tag{4.7}$$

for all $x, y, z \in B$. *Let* $g, h : B \to B$ *be mappings satisfying* $g(0) = h(0) = 0$, (4.3) *and* (4.4). *Then the mapping* $g : B \to B$ *is a* C^**-ternary derivation and the mapping* $h : B \to B$ *is a* C^**-ternary homomorphism.*

Proof. It follows from (4.7) that

$$\sum_{j=1}^{\infty} \frac{1}{8^j}\varphi\left(2^j x, 2^j y, 2^j z\right) \le \sum_{j=1}^{\infty} \frac{1}{8^j}(8L)^j\varphi(x, y, z) = \frac{L}{1-L}\varphi(x, y, z) < \infty,$$

for all $x, y, z \in B$. By Theorem 4.3, the mapping $g : B \to B$ is a C^*-ternary derivation and the mapping $h : B \to B$ is a C^*-ternary homomorphism. □

Remark 4.6. In (4.4), the pair (g, h) of a C^*-ternary derivation g and a C^*-ternary homomorphism h can be replaced by the pair of a C^*-ternary derivation and a C^*-ternary derivation or the pair of a C^*-ternary homomorphism and a C^*-ternary homomorphism.

5. Conclusions

We have introduced the following additive-additive (s, t)-functional inequality (0.1) and have investigated homomorphisms and derivations in complex Banach algebras and unital C^*-algebra, associated to the additive-additive (s, t)-functional inequality (0.1) and the functional inequality (1.1). Moreover, we have introduced the following additive-additive (s, t)-functional inequality (0.2) and have investigated C^*-ternary derivations and C^*-ternary homomorphisms in C^*-ternary algebras, associated to the additive-additive (s, t)-functional inequality (0.2) and the functional inequality (1.2).

Conflict of interests

The authors declare no conflicts of interest in this paper.

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