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Research article

A critical point theorem for a class of non-differentiable functionals with applications

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Abstract: This paper presents a multiplicity theorem for a kind of non-smooth functionals. The proof of this theorem relies on a suitable deformation lemma and the perturbation methods. We also apply this result to prove a multiplicity theorem for elliptic variational-hemivariational inequality problems.

Keywords: critical points; non-smooth functions; deformation; variational-hemivariational inequality; locally Lipschitz continuous
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1. Introduction

The mountain pass theorem of Ambrosetti and Rabinowitz for C^1 functions plays an essential role in the area of nonlinear analysis. We are interested in the monograph [22] of Motreanu and Panagiotopoulos in which they established a new version of the mounbtain pass theorem [22, Theorem 3.2] for the functionals f from Banach space X to $\mathbb{R} \cup \{+\infty\}$ satisfying the following hypothesis:
(H_f) : $f(x) = \Phi(x) + \Psi(x)$ *for all* $x \in X$, *where* $\Phi : X \to \mathbb{R}$ *is locally Lipschitz continuous while* $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ *is convex, proper, and lower semi-continuous.*

We call $x \in X$ is a critical point of f if x solves the following problem:

$$\Phi^0(x; z - x) + \Psi(z) - \Psi(x) \geq 0, \quad \forall\, z \in X, \tag{1.1}$$

where $\Phi^0(x; z - x)$ is the generalized directional derivative of Φ at x in the direction $z - x$ (see [6] for detail).

Recall that f satisfies the $(PS)_c$ condition if any sequence $\{x_n\} \subset X$ for which $\lim_{n\to+\infty} f(x_n) = c$ and

$$\Phi^0(x_n; z - x_n) + \Psi(z) - \Psi(x_n) \geq -\epsilon_n \|z - x_n\|, \quad \forall n \in N, z \in X$$

where $\epsilon_n \to 0^+$, possesses a convergent subsequence.

When $(PS)_c$ holds true at any level c we simply write $(PS)_f$ in place of $(PS)_c$.

Inequality (1.1) is usually called variational-hemivariational inequality, which has been exploited for mathematically formulating several engineering, besides mechanical questions and extensively studied from many points of view in the latest years [1, 22, 23].

Variational-hemivariational inequalities can be studied in the framework of a general critical point theory which combines features of the classical convex analysis and of the theory of generalized gradients for locally Lipschitz functions. Such inequalities represent a very general pattern for several kinds of variational problems. Indeed, if $\Phi \in C^1(X, \mathbb{R})$, the problem (1.1) is reduced to a variational inequality and the relevant critical point theory as well as significant applications are developed in [25]; if $\Psi \equiv 0$, then (1.1) coincides with the problem treated by Chang in [5] which is called differential equations with discontinuous nonlinearities; differential inclusions (see [12]) and special non-smooth problems with constraints (see [13]) can be considered as special cases of variational-hemivariational inequality. Finally, when both $\Phi \in C^1(X, \mathbb{R})$ and $\Psi \equiv 0$, the problem (1.1) becomes the Euler equation $\Phi'(u) = 0$ and the theory is classical. For the new results on this topic, see the excellent overview in [4, 7, 10, 11, 14, 15, 17–19, 26, 27].

Chang in [5] established the critical point theory for non-differentiable functionals and represented some applications to partial differential equations with discontinuous nonlinearities. Marano and Motreanu [20] obtained a critical points theorem which extends the variational principle of Ricceri to variational-hemivariational inequalities and semilinear elliptic eigenvalue problems with discontinuous nonlinearities. The critical point theorem in presence of splitting was established by Brȅzis-Nirenberg [3]. Subsequently Livrea, Marano and Motreanu [16] extended it to Motreanu-Panagiotopoulos' setting under the the following structural hypothesis $(H_f)'$:

$(H_f)'$: *$f(x) = \Phi(x) + \Psi(x)$ for all $x \in X$, where $\Phi : X \to \mathbb{R}$ is locally Lipschitz continuous while $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, and lower semi-continuous, and Ψ is continuous on any nonempty compact set $A \subseteq X$ such that* $\sup_{x \in A} \Psi(x) < +\infty$.

And they applied the conclusions to an elliptic variational-hemivariational inequality.

Motivatied by the above cited papers, we try to prove a multiplicity theorem of functions f fulfilling the structural hypothesis (H_f), mountain pass geometry and the bounded from below conditions. In Section 2, we will recall some basic definitions and preliminaries. The essential tool used in the proof is a general deformation lemma, which will be set forth in Section 3. Section 4 presents our main result, a new critical point theorem.

In the last section we consider an application to the elliptic variational-hemivariational inequality:

(P_λ) : Find $u \in K_\lambda$ such that for all $v \in K_\lambda$,

$$-\int_\Omega \nabla u(x) \nabla (v-u)(x) dx - \int_\Omega a(x) u(x) (v-u)(x) dx \le \lambda \mathcal{G}^0(u; v-u),$$

where $\lambda > 0$, K_λ is convex and closed in $H_0^1(\Omega)$, $g : \mathbb{R} \to \mathbb{R}$ is locally bounded and measurable, and the functions $G : \mathbb{R} \to \mathbb{R}$ and $\mathcal{G} : H_0^1(\Omega) \to \mathbb{R}$ given by

$$G(\xi) = \int_0^\xi -g(t) dt, \ \forall \ \xi \in \mathbb{R}, \quad \mathcal{G}(u) = \int_\Omega G(u(x)) dx, \quad \forall \ u \in H_0^1(\Omega),$$

respectively, are well defined and locally Lipschitz continuous. Some results of [16] are improved.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a reflexive Banach space. Denote by $B(x,\delta) := \{z \in X : \|z-x\| < \delta\}$ as well as $B_\delta := B(0,\delta)$. The symbol $[x,z]$ denote the segment joining x to z, namely $[x,z] := \{(1-t)x + tz : t \in [0,1]\}$, and $(x,z] := [x,z]\backslash\{x\}$. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X and X^*. A functional $\varphi : X \to \mathbb{R}\cup\{+\infty\}$ is proper if $D\varphi = \{x \in X : \varphi(x) < \infty\} \neq \emptyset$. Functional $\Phi : X \to \mathbb{R}$ is called locally Lipchitz continuous if for every $x \in X$ there exists a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x\|z - w\|, \ \forall z, w \in V_x.$$

Let $\Phi^0(x;z)$ be the generalized directional derivative of Φ at x along the direction z, i.e.,

$$\Phi^0(x;z) := \limsup_{w\to x, t\to 0^+} \frac{\Phi(w+tz) - \Phi(w)}{t}.$$

The generalized gradient of the function Φ at x, denoted by $\partial\Phi(x)$, is the set

$$\partial\Phi(x) := \{x^* \in X^* : \langle x^*, z\rangle \leq \Phi^0(x;z), \ \forall z \in X\}.$$

The mapping $z \mapsto \Phi^0(x;z)$ is positively homogeneous and sub-additive, thus, due to the Hahn-Banach theorem, the set $\partial\Phi(x)$ is nonempty. In the sequel, we state the main properties of the generalized directional derivatives and the generalized gradients:

1) For each $x \in X$, $\partial\Phi(x)$ is a nonempty, convex in addition to *weak** compact subset of X^*.
2) For each $x, z \in X$, $\Phi^0(x,z)$ is upper semicontinuous on $X \times X$.
3) For each $x, z \in X$, we have $\Phi^0(x;z) = \max\{\langle x^*, z\rangle; \ x^* \in \partial\Phi(x)\}$.
4) If Φ attains a local minimum or maximum at x, then $0 \in \partial\Phi(x)$.
5) The function $m_\Phi(x) = \min\{\|x^*\|_{X^*}, x^* \in \partial\Phi(x)\}$ exists and is lower semi-continuous.

Let $\Psi : X \to \mathbb{R}\cup\{+\infty\}$ be convex, proper, and lower semi-continuous. Set $D_\Psi = \{x \in X : \Phi(x) < +\infty\}$, then Ψ is continuous in $int(D_\Psi)$ (see [8]). To simplify notation, denote by $\partial\Psi(x)$ the subdifferential of Ψ at the point $x \in X$ in the sense of convex analysis, while $D_{\partial\Psi} = \{x \in X : \partial\Psi(x) \neq \emptyset\}$. By [8], $int(D_\Psi) = int(D_{\partial\Psi})$, $\partial\Psi(x)$ is convex and *weak** closed.

Let f be a function on X satisfying the hypothesis(H_f), $a \in \mathbb{R}$. Define

$$K_a(f) = \{x \in X : f(x) = a, \ x \text{ is a critical point of } f\},$$
$$f^a = \{x \in X : f(x) \geq a\}, f_a = \{x \in X : f(x) \leq a\}.$$

For every $\epsilon, r > 0$, we introduce the set

$$F^r_{a,\epsilon} = \{x \in X : \|x\| \leq r+1, \text{and } |f(x) - a| \leq \epsilon\},$$

it is easy to see that $F^r_{a,\epsilon}$ is closed.

3. A deformation result

In this section we establish a deformation lemma for the functions satisfying the hypothesis (H_f).

Lemma 3.1. *Suppose $x \in int(D_\Psi)$. Then for every $x_n \to x$ in X and every $z_n^* \in \partial\Psi(x_n), n \in N$, there exists $z^* \in \partial\Psi(x)$ as well as a subsequence $\{z_{r_n}^*\}$ of $\{z_n^*\}$ such that $z_{r_n}^* \rightharpoonup z^*$ in X^*.*

For the proof the reader could refer to [21, Remark 2.1].

Lemma 3.2. *Let f be a function satisfying (H_f). Assume that there exist constants $\epsilon > 0, r > 0$ and $a \in \mathbb{R}$ such that $F_{a,\epsilon}^r \neq \emptyset$, $F_{a,\epsilon}^r \subseteq int(D_\Psi)$, and*

$$\inf\{\|x^* + z^*\| : x^* \in \partial\Phi(x), z^* \in \partial\Psi(x), x \in F_{a,\epsilon}^r\} > 2\epsilon.$$

Then for every $x \in F_{a,\epsilon}^r$, there exists $\xi_x \in X$ such that

$$\|\xi_x\| = 1, \langle x^* + z^*, \xi_x\rangle > 2\epsilon, \text{ for all } x^* \in \partial\Phi(x), z^* \in \partial\Psi(x). \tag{3.1}$$

Proof. Since $\inf\{\|x^* + z^*\| : x^* \in \partial\Phi(x),\ z^* \in \partial\Psi(x),\ x \in F_{a,\epsilon}^r\} > 2\epsilon$, there exists an $\epsilon_0 > 0$ such that for every $x \in F_{a,\epsilon}^r, x^* \in \partial\Phi(x), z^* \in \partial\Psi(x)$, we have $\|x^* + z^*\|_{X^*} \geq 2\epsilon + \epsilon_0$.

Fix an $x \in F_{a,\epsilon}^r$, since $\partial\Phi(x)$ and $\partial\Psi(x)$ are nonempty and convex, so is $\partial\Phi(x) + \partial\Psi(x)$. As X is reflexive, $\partial\Phi(x)$ is weak* compact and $\partial\Psi(x)$ is weak* closed, then $\partial\Phi(x) + \partial\Psi(x)$ is closed.

Note that $0 \notin \partial\Phi(x) + \partial\Psi(x)$. By [2, Corollary 3.20], we have $u^* \in \partial\Phi(x), v^* \in \partial\Psi(x)$ satisfying

$$B_{\delta^*} \cap (\partial\Phi(x) + \partial\Psi(x)) = \emptyset, \text{ where } \delta^* = \|u^* + v^*\|_{X^*} > 0.$$

Now the Hahn-Banach theorem provides a point $\xi_x \in X$ with $\|\xi_x\| = 1$ and whenever $x^* \in \partial\Phi(x),\ z^* \in \partial\Psi(x)$,

$$\langle x^* + z^*, \xi_x\rangle \geq \langle w^*, \xi_x\rangle, \ \forall\, w^* \in B_{\delta^*}.$$

Since $\|u^* + v^*\|_{X^*} = \|u^* + v^*\|_{X^*}\|\xi_x\| = \max\left\{\langle w^*, \xi_x\rangle, w^* \in \overline{B_{\delta^*}}\right\}$, the above inequality and Lemma 3.1 lead to

$$\langle x^* + z^*, \xi_x\rangle \geq \|u^* + v^*\|_{X^*} \geq 2\epsilon + \epsilon_0 > 2\epsilon, \ \ \forall\, x^* \in \partial\Phi(x),\ z^* \in \partial\Psi(x).$$

The proof is completed. □

Lemma 3.3 *Under the conditions of Lemma 3.2, for every $x \in F_{a,\epsilon}^r$, there exists a $\delta_x > 0$ such that*

$$\langle x^* + z^*,\ \xi_x\rangle > 2\epsilon, \ \ \forall\, x^* \in \partial\Phi(x'),\ z^* \in \partial\Psi(x''),\ \forall\, x',\ x'' \in B(x, \delta_x), \tag{3.2}$$

where ξ_x is given by Lemma 3.2.

Proof. If the conclusion were false, then we could find $x \in F_{a,\epsilon}^r, \{x_n^{'}\}, \{x_n''\} \subseteq X$ and $\{x_n^*\}, \{z_n^*\} \subseteq X^*$ such that

$$x_n^{'} \to x, x_n^* \in \partial\Phi(x_n'), \ \ \forall\, n \in N; \tag{3.3}$$

$$x_n^{''} \to x, z_n^* \in \partial\Psi(x_n''), \ \ \forall\, n \in N; \tag{3.4}$$

$$\langle x_n^* + z_n^*, \xi_x\rangle \leq 2\epsilon, \ \ \forall\, n \in N. \tag{3.5}$$

Due to the reflexivity of X and (3.3), Proposition 2.1.2 of [22] yields $x^* \in X^*$ such that $x_n^* \rightharpoonup x^*$ in X^*, where a subsequence is considered when necessary, while Proposition 2.1.5 of [22] forces $x^* \subseteq \partial\Phi(x)$. Since $x \in F_{a,\epsilon}^r \subseteq int(D_\Psi) = int(D_{\partial\Psi})$, combining (3.4) with Lemma 3.1, we obtain, up to subsequences, $z_n^* \rightharpoonup z^*$ for some $z^* \in \partial\Psi(x)$. Now from (3.5) it follows, as $n \to +\infty$, $\langle x^* + z^*, \xi_x\rangle \le 2\epsilon$. However, this contradicts (3.1). □

Theorem 3.4 *Let f be a function satisfying (H_f), assume that there exist constants $\epsilon > 0, r > 0$ and $a \in R$ such that $F_{a,\epsilon}^r \neq \emptyset, F_{a,\epsilon}^r \subseteq int(D_\Psi)$, and*

$$\inf\{\|x^* + z^*\| : x^* \in \partial\Phi(x), z^* \in \partial\Psi(x), x \in F_{a,\epsilon}^r\} > 2\epsilon.$$

Then there exists a continuous mapping $\eta : X \to X$ with the following properties:

(1) $\eta : X \to X$ *is a homeomorphism;*
(2) $\eta(x) = x$ *whenever* $|f(x) - a| \ge 2\epsilon$;
(3) $\|\eta(x) - x\| \le 1, \quad \forall x \in X$;
(4) $f(\eta(x)) \le f(x), \quad \forall x \in X$;
(5) $\eta(D_\Psi) \subseteq D_\Psi$;
(6) $f(\eta(x)) \le a - \epsilon,\ \forall\ x \in X$ *provided* $\|x\| \le r$ *and* $f(x) \le a + \epsilon$.

Proof. The family of balls $\mathbb{B} = \{B(x, \delta_x) : x \in F_{a,\epsilon}^r\}$ constructed through Lemma 3.3 represents an open covering of $F_{a,\epsilon}^r$, and the assumptions ensure that $F_{a,\epsilon}^r$ is a nonempty para-compact set because it is closed. So $\mathbb{B}$ possesses an open locally finite refinement $\mathbb{V} = \{V_i; i \in I\}$. Moreover, to each $i \in I$ there corresponds $\xi_i \in X$ such that $\|\xi_i\| = 1$ as well as

$$\langle x^* + z^*, \xi_i\rangle > 2\epsilon, \quad \forall\ x^* \in \partial\Phi(x'), z^* \in \partial\Psi(x''),\ \forall\ x', x'' \in V_i. \tag{3.6}$$

Shrink $\mathbb{V}$ to an open locally finite covering $\mathbb{W} = \{W_i; i \in I\}$ fulfilling for every $i \in I$, $W_i \subseteq V_i$ ([9, Theorems VIII 2.2 and Theorems VII 6.1]) with W_i is convex, and $f|_{W_i}(\cdot)$ is Lipschitz continuous satisfying

$$a - 2\epsilon < f(x) < a + 2\epsilon, \quad \forall\ x \in W_i. \tag{3.7}$$

Set

$$\begin{aligned}
W &= \bigcup_{i\in I} W_i, d_i(x) = d(x, X\backslash W_i),\\
\rho_i(x) &= \frac{d_i(x)}{\sum_{j\in I} d_j(x)},\ \forall\ x \in W,\ i \in I,\\
\Theta(x) &= \begin{cases} \displaystyle\sum_{i\in I} \rho_i(x)\xi_i, & \text{if } x \in W,\\ 0, & \text{otherwise,}\end{cases}\\
l(x) &= \frac{d(x, X\backslash W)}{d(x, X\backslash W) + d(x, F_{a,\epsilon}^r)},\ \forall\ x \in X,\\
V(x) &= l(x)\Theta(x),\ \forall\ x \in X,
\end{aligned}$$

$$V(x) = l(x)\Theta(x), \ \forall \ x \in X.$$

We observe that $V : X \to X$ is locally Lipschitz continuous and

$$\|V(x)\| \le 1, \ \forall \ x \in X. \tag{3.8}$$

The existence-uniqueness theorem for ordinary differential equations provides a mapping $\sigma \in C^0(\mathbb{R} \times X, X)$ such that

$$\frac{d\sigma(t,x)}{dt} = -V(\sigma(t,x)), \ \forall \ (t,x) \in \mathbb{R} \times X, \ \sigma(0,x) = x.$$

We claim that

$$\text{for every } x \in X, \ \text{the function } t \mapsto f(\sigma(t,x)) \text{ is non-increasing on } \mathbb{R}. \tag{3.9}$$

In fact, if $x \in X \setminus W$, $V(x) = 0$ and thus $\sigma(\cdot, x)$ is constant and (3.9) holds true. In the case $x \in W$, we start by noting that $\sigma(\mathbb{R}, x) \subseteq W$. Indeed setting $T = \sup\{t > 0 : \sigma((-t,t),x) \subseteq W\}$, assume by contradiction that $T < +\infty$. Hence we have $W_x = \sigma(T,x) = \lim_{t \to T^-} \sigma(t,x) \in \partial W$. Then the Cauchy problem

$$\frac{d\widetilde{\sigma}(t)}{dt} = -V(\widetilde{\sigma}(t)), \ \forall \ t \in \mathbb{R}, \ \widetilde{\sigma}(T) = W_x$$

admits the constant solution $\widetilde{\sigma}(\cdot) \equiv W_x$, as does $t \mapsto \sigma(t,x)$ for $t \le T$, which is against the uniqueness of solutions.

Fixing $t \in \mathbb{R}$, we know that $\sigma(t,x) \in W$ and due to the local finiteness of W, the set $J = \{i \in I : \sigma(t,x) \in W_i\}$ is finite. It follows that $\widetilde{W} = \cap_{i \in J} W_i$ is a convex, open neighborhood of $\sigma(t,x)$, and there exists $\delta > 0$ such that

$$\sigma((t-\delta, t+\delta), x) \subseteq \widetilde{W}, \ \text{and } \sigma((t-\delta,t+\delta),x) \bigcap (\bigcup_{i \in I \setminus J} W_i) = \emptyset. \tag{3.10}$$

For arbitrary $t', t'' \in (t-\delta, t+\delta)$ with $t' < t''$. Lebourg's mean value theorem provides $y \in (\sigma(t',x), \sigma(t'',x))$, $x^* \in \partial\Phi(y)$, $z^* \in \partial\Psi(y)$ satisfying

$$\begin{aligned} f(\sigma(t',x)) - f(\sigma(t'',x)) &= \langle x^* + z^*, \sigma(t'',x) - \sigma(t',x) \rangle \\ &= -\int_{t'}^{t''} \langle x^* + z^*, V(\sigma(\tau,x)) \rangle d\tau \\ &< -2\epsilon \int_{t'}^{t''} l(\sigma(\tau,x)) \sum_{j \in J} \rho_j(\sigma(\tau,x)) d\tau \\ &= -2\epsilon \int_{t'}^{t''} l(\sigma(\tau,x)) d\tau, \end{aligned}$$

where (3.6) and (3.10) have been used. Given $p, q \in [t, t+1]$ with $p < q$, a standard compactness argument and the above estimate enable us to find $t_1, t_2, ..., t_s \in [t, t+1]$ with $p = t_1 < t_2 < ... < t_s = q$ such that

$$f(\sigma(t_i,x)) - f(\sigma(t_{i-1},x)) < -2\epsilon \int_{t_{i-1}}^{t_i} l(\sigma(\tau,x)) d\tau,$$

for all $i = 1, 2, ..., s$. It turns out that

$$
\begin{aligned}
f(\sigma(q,x)) - f(\sigma(p,x)) &= \sum_{i=1}^{s}[f(\sigma(t_i,x)) - f(\sigma(t_{i-1},x))] \\
&< -2\epsilon \int_p^q l(\sigma(\tau,x))d\tau < 0,
\end{aligned}
\tag{3.11}
$$

which establish (3.9). Now we define

$$\eta(x) = \sigma(1,x), \ \forall\, x \in X.$$

From the general theory of ordinary differential equations it is well known that $\eta : X \to X$ is a homeomorphism.

Since (3.7) renders $\{x \in X : |f(x) - a| \geq 2\epsilon\} \subseteq X \setminus W$ when $x \in X \setminus W$, $V(x) = 0$, thus $\sigma(\cdot, x)$ is constant, and $\eta(x) = \sigma(1,x) = \sigma(0,x) = x$. We then deduce property (2).

From (3.8), for all $x \in X$, we have

$$
\begin{aligned}
\|\eta(x) - x\| = \|\sigma(1,x) - \sigma(0,x)\| &= \| \int_0^1 V(\sigma(\tau,x))d\tau\| \\
&\leq \int_0^1 \|V(\sigma(\tau,x))\|d\tau \leq 1,
\end{aligned}
$$

i.e., property (3) holds true.

Since $f(\eta(x)) = f(\sigma(1,x)) \leq f(\sigma(0,x)) = f(x)$, for all $x \in X$, the property (4) holds true.

For every $x \in D_\Psi$, there is $f(x) < +\infty$. Since (4) holds, $f(\eta(x)) \leq f(x) < +\infty$, so $\eta(x) \in D_\Psi$. i.e. $\eta(D_\Psi) \subseteq D_\Psi$, (5) holds true.

In order to prove (6), let $x \in X$ with $\|x\| \leq r$ and $f(x) \leq a + \epsilon$. If $f(x) \leq a - \epsilon$, (6) follows from (4) immediately. In case $a - \epsilon < f(x) \leq a + \epsilon$, we argue by contradiction. Suppose

$$a - \epsilon < f(\eta(x)) = f(\sigma(1,x)) \leq f(\sigma(t,x)) \leq f(x) \leq a + \epsilon, \ \forall\, t \in [0,1]. \tag{3.12}$$

In addition, through (3.8), for all $t \in [0,1]$ we have

$$
\begin{aligned}
\|\sigma(t,x)\| \leq \|x\| + \|\sigma(t,x) - x\| &\leq r + \| \int_0^t \frac{d\sigma(\tau,x)}{d\tau} d\tau\| \\
&\leq r + \int_0^t \|V(\sigma(\tau,x))\|d\tau \leq r + 1.
\end{aligned}
$$

Consequently, $\sigma([0,1],x) \subseteq F^r_{a,\epsilon}$, which forces $l(\sigma(\cdot,x))|_{[0,1]} \equiv 1$. Then (3.11) with $p = 0$ and $q = 1$ reads as

$$f(\eta(x)) - f(x) < -2\epsilon. \tag{3.13}$$

Combining (3.12) and (3.13) gives

$$a - \epsilon < f(\eta(x)) < f(x) - 2\epsilon \leq a - \epsilon,$$

which is a contradiction. □

4. Existence of critical points

Fix $v_0,\ v_1 \in D_\Psi$. Consider the following set of paths

$$\Gamma = \{\gamma \in C^0([0,1], X) : \gamma(0) = v_0,\ \gamma(1) = v_1\}, \tag{4.1}$$

and a function $f : X \to \mathbb{R} \cup \{+\infty\}$ which verifies hypothesis (H_f). Set

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} f(\gamma(t)). \tag{4.2}$$

Theorem 4.1 *Suppose $f : X \to \mathbb{R} \cup \{+\infty\}$ satisfies (H_f) and $(PS)_f$. Assume in addition that*

(i_1) *f is bounded below and coercive, and put $\alpha = \inf_{x\in X} f(x)$;*
(i_2) *$\alpha < \max\{f(v_0), f(v_1)\} \leq c$, $v_0 \neq v_1$ and for every $\gamma \in \Gamma$, there exists $t \in (0,1)$ such that $f(\gamma(t)) \geq \max\{f(v_0), f(v_1)\}$;*
(i_3) *for every $a \in \mathbb{R}$, there exist $r > 0$ and $\epsilon_0 > 0$ such that $F^r_{a,\epsilon_0} \subseteq int(D_\Psi)$.*

Then the function f possesses at least two critical points.
Proof. Since f is coercive, for every $x \in X$ with $c - 1 \leq f(x) \leq c + 1$, there exists a constant k such that

$$\|x\| \leq k. \tag{4.3}$$

From (i_3), for c and $k > 0$, there is $\epsilon_0 > 0$ such that

$$F^k_{c,\epsilon_0} \subseteq int(D_\Psi). \tag{4.4}$$

Without loss of generality, we assume $\epsilon_0 < 1$. Let

$$c_\epsilon = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} [f(\gamma(t)) + \epsilon d(t)], \tag{4.5}$$

where $0 < \epsilon < \frac{1}{2}\epsilon_0$ is arbitrary, and $d(t) = \min\{t, 1-t\},\ t \in [0,1]$.

From (i_2), we can easily verify that

$$c \leq c_\epsilon < c + \epsilon.$$

Since for every $\gamma \in \Gamma$, there exists $t_0 \in (0,1)$ such that $f(\gamma(t_0)) \geq \max\{f(v_0), f(v_1)\}$, one has

$$\sup_{t\in[0,1]} (f(\gamma(t)) + \epsilon d(t)) \geq f(\gamma(t_0)) + \epsilon d(t_0) > f(\gamma(t_0)) \geq \max\{f(v_0), f(v_1)\},$$

thus $c_\epsilon > \max\{f(v_0), f(v_1)\}$ for every $\epsilon \in (0, \frac{1}{2}\epsilon_0)$.

We claim that for every $\epsilon \in (0, \frac{1}{2}\epsilon_0)$ satisfying that $\epsilon < \frac{1}{2}(c_\epsilon - \max\{f(v_0), f(v_1)\})$ there holds

$$\inf\{\|x^* + z^*\| : x^* \in \partial\Phi(x),\ z^* \in \partial\Psi(x),\ x \in F^k_{c_\epsilon,\epsilon}\} \leq 2\epsilon. \tag{4.6}$$

Due to the definition of $\widehat{\epsilon}$, for every $\epsilon \in (0, \widehat{\epsilon})$, we have

$$c - \epsilon_0 < c - \epsilon < c_\epsilon - \epsilon \leq f(x) \leq c_\epsilon + \epsilon < c + 2\epsilon < c + \epsilon_0$$

and $\|x\| \leq k$. It is straightforward to verify that $F^k_{c_\epsilon,\epsilon} \subseteq int(D_\Psi)$.

To show (4.6), we argue by contradiction. If it was not true, then we would find $\epsilon \in (0, \widehat{\epsilon})$ for which Theorem 3.4 can be applied with $a = c_\epsilon$ and $r = k$. So there would exist a continuous mapping $\eta : X \to X$ with the properties (1)–(6) formulated in Theorem 3.4. By the definition of $\widehat{\epsilon}$, there is $\gamma_\epsilon \in \Gamma$ such that

$$c_\epsilon \leq \sup_{t\in[0,1]} [f(\gamma_\epsilon(t)) + \epsilon d(t)] < c_\epsilon + \epsilon,$$

which easy to verify that

$$c_\epsilon - \epsilon < \sup_{t\in[0,1]} f(\gamma_\epsilon(t)) < c_\epsilon + \epsilon,$$

and use the definition of $\widehat{\epsilon}$ again, it is straightforward to verify that $\eta(\gamma_\epsilon(\cdot)) \in \Gamma$ for every ϵ. Hence, in view of (4.5), we may consider a sequence $\{s_\epsilon\}$ in $[0, 1]$ such that

$$c_\epsilon - \epsilon < f(\eta(\gamma_\epsilon(s_\epsilon))), c_\epsilon - \epsilon < f(\gamma_\epsilon(s_\epsilon)) < c_\epsilon + \epsilon. \tag{4.7}$$

Since $c_\epsilon + \epsilon < c + 2\epsilon < c + 1$ and $c_\epsilon - \epsilon > c - \epsilon > c - 1$, it implies that $\|\gamma_\epsilon(s_\epsilon)\| \leq k$.

Exploiting (4.7) and property (6), we achieve the contradiction

$$c_\epsilon - \epsilon < f(\eta(\gamma_\epsilon(s_\epsilon))) \leq c_\epsilon - \epsilon.$$

thereby (4.6) holds true.

By virtue of (4.6), for every $x \in X$ and all $n \in N$ sufficiently large, there exists $x_n \in F^k_{c_{\frac{1}{n}}, \frac{1}{n}}$, $x_n^* \in \partial\Phi(x_n)$, $z_n^* \in \partial\Psi(x_n)$ such that

$$\|x_n^* + z_n^*\| < \frac{3}{n},$$

and

$$c - \frac{1}{n} < c_{\frac{1}{n}} - \frac{1}{n} \leq f(x_n) \leq c_{\frac{1}{n}} + \frac{1}{n} < c + \frac{2}{n}.$$

This guarantees that

$$\|x_n\| \leq k + 1, c - \frac{1}{n} < f(x_n) < c + \frac{2}{n},$$

and

$$\begin{aligned} \Phi^0(x_n; x - x_n) + \Psi(x) - \Psi(x_n) &\geq \langle x_n^* + z_n^*, x - x_n \rangle \\ &\geq -\|x_n^* + z_n^*\|\|x - x_n\| \\ &> -\frac{3}{n}\|x - x_n\|. \end{aligned}$$

Since f satisfies the $(PS)_f$ condition, there is an $\overline{x} \in X$ such that $x_n \to \overline{x}$ in X, where a subsequence is considered when necessary. At this point, $\overline{x}$ is a critical point of f, and $\overline{x} \in K_c(f)$.

Next we prove that f possesses a global minimum point $x_0 \in X$. Since by (i_1) and the condition $(PS)_f$, each minimizing sequence for f possesses a convergent subsequence (see [16]), the function f must attain its minimum at some point $x_0 \in X$.

Due to $f(x_0) = \alpha < c = f(\bar{x})$, $x_0 \neq \bar{x}$, which completes the proof. □

Remark 4.2 By the above proof, one can find that under the conditions of Theorem 4.1, when $a \geq \alpha$, there exist $r > 0$, $\epsilon > 0$ such that $F^r_{a,\epsilon} \neq \emptyset$. By the coercivity of f, if Ψ is convex and continuous, the condition (i_3) obviously holds.

5. An application

In this section we use Theorem 4.1 to discuss an elliptic variational-hemivariational inequality in the sense of Panagiotopoulos [24].

Let Ω be a nonempty, bounded, open subset of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. Denote by $H_0^1(\Omega)$ the usual Sobolev space with norm

$$\|u\| = (\int_\Omega |\nabla u(x)|^2 dx)^{\frac{1}{2}}.$$

It's well known that for $p \in [1, 2^*]$, $2^* = 2N/(N-2)$, there exists a positive constant C_p such that

$$\|u\|_{L^p(\Omega)} \leq C_p\|u\|, \quad u \in H_0^1(\Omega). \tag{5.1}$$

Given a function $a \in L^\infty(\Omega)$ satisfying $a(x) \geq 0$ for a.e. $x \in \Omega$. Let

$$\beta = ess \inf_{x\in\Omega} a(x) \geq 0.$$

If $g : \mathbb{R} \to \mathbb{R}$ satisfies the condition

(g_1) g is locally bounded and measurable.

Then the functions $G : \mathbb{R} \to \mathbb{R}$ and $\mathcal{G} : H_0^1(\Omega) \to \mathbb{R}$ given by

$$G(\xi) = \int_0^\xi -g(t)dt, \ \forall \ \xi \in \mathbb{R},$$
$$\mathcal{G}(u) = \int_\Omega G(u(x))dx, \quad \forall \ u \in H_0^1(\Omega),$$

respectively, are well defined and locally Lipschitz continuous. So it makes sense to consider their generalized directional derivatives G^0 and $\mathcal{G}^0$. On account of [22, formula(9), P.84] one has

$$\mathcal{G}^0(u; v) \leq \int_\Omega G^0(u(x); v(x))dx, \ u, \ v \in H_0^1(\Omega). \tag{5.2}$$

We will further assume

(g_2) $\lim_{t\to 0} \frac{g(t)}{t} = 0$;

(g_3) $\limsup_{|t|\to+\infty} \frac{g(t)}{t} \leq 0$;

(g_4) there exists a $\xi_0 \in \mathbb{R}$ such that $G(\xi_0) < 0$.

Through (g_3) for every $\epsilon > 0$ there exists a constant $r > 0$ such that

$$g(t) \leq \epsilon t, \quad \text{for all} \ |t| \geq r. \tag{5.3}$$

Since g is locally bounded, we also have

$$M = \sup_{t\in[-r,r]} |g(t)| < +\infty. \tag{5.4}$$

Let $\lambda > 0$, $\mu(\Omega)$ be the Lebesgue measure of Ω. Define

$$r_\lambda = \sqrt{4\lambda + 2Mr\mu(\Omega)}.$$

A set $K_\lambda \subseteq H_0^1(\Omega)$ is called of type (K_λ^g) provided

(K^g_λ) : K_λ is convex and closed in $H^1_0(\Omega)$. Moreover, $\overline{B_{r_\lambda}} \subseteq K_\lambda$.

Given $\lambda > 0$ and K_λ satisfying (K^g_λ), consider the elliptic variational-hemivariational inequality problems:

(P_λ) : Find $u \in K_\lambda$ such that for all $v \in K_\lambda$,

$$-\int_\Omega \nabla u(x)\nabla(v-u)(x)dx - \int_\Omega a(x)u(x)(v-u)(x)dx \leq \lambda\mathcal{G}^0(u; v-u).$$

Due to (5.2), any solution u of (P_λ) also fulfills the inequality

$$\begin{aligned}&-\int_\Omega \nabla u(x)\nabla(v-u)(x)dx - \int_\Omega a(x)u(x)(v-u)(x)dx\\ &\leq \lambda\int_\Omega G^0(u(x);(v-u)(x))dx, \quad \text{for all} \quad v \in K_\lambda.\end{aligned}$$

If g is continuous and $K_\lambda = H^1_0(\Omega)$, the function $u \in H^1_0(\Omega)$ turns out a weak solution to the Dirichlet problem

$$\begin{cases} -\triangle u + a(x)u = \lambda g(u), \text{ in } \Omega,\\ u = 0, \text{ on } \partial\Omega,\end{cases}$$

which has been previously investigated in [3, 14] under more restrictive conditions.

Theorem 5.1 *Suppose* $(g_1)-(g_4)$ *hold true. Then, for every* λ *sufficiently large, problem* (P_λ) *possesses at least two solutions.*

Proof . Let $X = H^1_0(\Omega)$, $p \in (2, 2^*)$. Define a functional $f(u) = \Phi(u) + \Psi(u)$ on X as follows:

$$\Phi(u) = \frac{1}{2}\int_\Omega(|\nabla u(x)|^2 + a(x)u(x)^2)dx + \lambda\mathcal{G}(u)$$

as well as

$$\Psi(u) = \begin{cases} 0, & \text{if } u \in K_\lambda,\\ +\infty, & \text{otherwise,}\end{cases}$$

where $\lambda > 0$ and $K_\lambda \subseteq H^1_0(\Omega)$ is of type (K^g_λ). Owing to (g_1) the function $\Phi : X \to \mathbb{R}$ turns out locally Lipschitz continuous. Consequently, f satisfies condition (H_f).

We shall prove that f is bounded from below and coercive.

By (5.3) and (5.4), one has

$$\int_0^\xi g(t)dt \leq Mr + \frac{\epsilon}{2}\xi^2, \ \forall\, \xi \in \mathbb{R}. \tag{5.5}$$

Which clearly implies

$$\mathcal{G}(u) \geq -Mr\mu(\Omega) - \frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)}, \ \forall\, u \in X.$$

Then we obtain

$$\begin{aligned} f(u) \geq \Phi(u) &\geq \frac{1}{2}\|u\|^2 + \frac{\beta}{2}\|u\|^2_{L^2(\Omega)} - \frac{\lambda\epsilon}{2}\|u\|^2_{L^2(\Omega)} - \lambda Mr\mu(\Omega)\\ &= \frac{\epsilon}{2}\|u\|^2 + \frac{1}{2}(\beta - \epsilon\lambda)\|u\|^2_{L^2(\Omega)} - \lambda Mr\mu(\Omega).\end{aligned}$$

Setting $\epsilon \in (0, \frac{\beta}{\lambda})$, then we have

$$f(u) \geq \frac{1}{2}\|u\|^2 - \lambda Mr\mu(\Omega), \ \forall \ u \in X, \tag{5.6}$$

which shows the claim.

Let us next show that the function f satisfies condition $(PS)_f$. Pick a sequence $\{u_n\} \subseteq X$ such that $\{f(u_n)\}$ is bounded and

$$\Phi^0(u_n; v - u_n) + \Psi(v) - \Psi(u_n) \geq -\epsilon_n\|v - u_n\|. \tag{5.7}$$

for all $n \in N, v \in X$, where $\epsilon_n \to 0^+$.

By (5.7) one evidently has $\{u_n\} \subseteq K_\lambda$, and $\{f(u_n)\}$ is bounded. Since f is coercive, the sequence $\{u_n\}$ turns out bounded. Passing to a subsequence if necessary, we suppose $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^2(\Omega)$. The point u belongs to K_λ because this set is weakly closed.

Exploiting (5.7) with $v = u$, we then get

$$\begin{aligned}\int_\Omega \nabla u_n(x)\nabla(u - u_n)(x)dx + \int_\Omega a(x)u_n(x)(u - u_n)(x)dx \\ + \lambda\mathcal{G}^0(u_n; u - u_n) \geq -\epsilon_n\|u - u_n\|,\end{aligned} \tag{5.8}$$

for all $n \in N$.

From $u_n \rightharpoonup u$ in X it follows

$$\lim_{n\to+\infty} \int_\Omega a(x)u_n(x)(u - u_n)(x)dx = 0. \tag{5.9}$$

The upper semi-continuity of $\mathcal{G}^0$ on $L^2(\Omega) \times L^2(\Omega)$ forces

$$\limsup_{n\to+\infty} \mathcal{G}^0(u_n; u - u_n) \leq \mathcal{G}^0(u; 0) = 0. \tag{5.10}$$

Taking account of (5.9), (5.10) besides $\{\|u - u_n\|\}$ is bounded, and letting $n \to +\infty$, inequality (5.8) yields

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n(x)|^2 dx \leq \int_\Omega \nabla|u(x)|^2 dx.$$

Hence, thanks to [2, Proposition III.3], $u_n \to u$ in X. i.e., $(PS)_a$ holds.

By (g_4), we can construct an $u_0 \in X$ such that $\mathcal{G}(u_0) < 0$. Moreover, $u_0 \in \overline{B_{r_\lambda}}$ for any $\lambda \geq \frac{1}{4}\|u_0\|^2$. Therefore, $\inf_{u\in X} f(u) \leq f(u_0) < 0$ provided

$$\lambda > \max\Big\{\frac{1}{4}\|u_0\|^2, -\frac{1}{2\mathcal{G}(u_0)} \int_\Omega (|\nabla u_0(x)|^2 + a(x)u_0(x)^2)dx\Big\},$$

while $f(0) = \lambda\mathcal{G}(0) = 0$.

Our next objective is to verify (i_1). From (g_2), there exists $\sigma \in (0, r)$ such that

$$\int_{|u(x)|<\sigma} [\int_0^{u(x)} g(t)dt]dx \leq \frac{\epsilon}{2} \int_\Omega |u(x)|^2 dx. \tag{5.11}$$

Due to (5.5), one has

$$G(\xi) \geq -Mr - \frac{\epsilon}{2}\xi^2 \geq -(\frac{Mr}{\sigma^p} + \frac{\epsilon}{2\sigma^{p-2}})|\xi|^p,$$

provided $|\xi| \geq \sigma$.
The Sobolev embedding theorem gives

$$\int_{|u(x)|\geq\sigma} G(u(x))dx \geq -(\frac{Mr}{\sigma^p} + \frac{\epsilon}{2\sigma^{p-2}})\|u\|^p_{L^p(\Omega)} \geq -C^*\|u\|^p, \tag{5.12}$$

where $C^* = (\frac{Mr}{\sigma^p} + \frac{\epsilon}{2\sigma^{p-2}})C_p^p$. Then by (5.11) (5.12) and (5.1) we get

$$\begin{aligned} \mathcal{G}(u) &= \int_{|u(x)|<\sigma} [\int_0^{u(x)} -g(t)dt]dx + \int_{|u(x)|\geq\sigma} G(u(x))dx \\ &\geq -\frac{\epsilon}{2}C_2^2\|u\|^2 - C^*\|u\|^p \\ &= -\|u\|^2(\frac{\epsilon}{2}C_2^2 + C^*\|u\|^{p-2}), \forall u \in X. \end{aligned} \tag{5.13}$$

Let us next prove that for a suitable constant $\theta > 0$,

$$\int_\Omega (|\nabla u(x)|^2 + a(x)u(x)^2)dx \geq \theta \int_\Omega |\nabla u(x)|^2 dx, \ \forall\ u \in X. \tag{5.14}$$

Indeed, if it's not true, there exists a sequence $\{u_n\} \subseteq X$ enjoying the properties

$$\begin{gathered} \|u_n\| = 1,\ n \in N, \\ \int_\Omega (|\nabla u_n(x)|^2 + a(x)u_n(x)^2)dx < \frac{1}{n},\ \forall\ n \in N. \end{gathered} \tag{5.15}$$

Passing to a subsequence if necessary, we may suppose $u_n \rightharpoonup u$ in X as well as $u_n \to u$ in $L^2(\Omega)$. Thus, letting $n \to +\infty$ in (5.15) yields

$$\int_\Omega (|\nabla u(x)|^2 + a(x)u(x)^2)dx \leq 0. \tag{5.16}$$

Using the sobolev embedding theorem and $\beta = ess\inf_{x\in\Omega} a(x) \geq 0$ we obtain

$$(\frac{1}{C_2^2} + \beta)\|u\|^2_{L^2(\Omega)} \leq \int_\Omega (|\nabla u(x)|^2 + a(x)u(x)^2)dx. \tag{5.17}$$

Gathering (5.16) and (5.17) together, leads to $u = 0$. By (5.15) this forces $u_n \to 0$ in X, against to $\|u_n\| = 1, \forall n \in N$.

Combining (5.14) with (5.13), provides

$$f(u) \geq \|u\|^2(\frac{\theta}{2} - \lambda(\frac{\epsilon}{2}C_2^2 + C^*\|u\|^{p-2})),\ \forall\ u \in X. \tag{5.18}$$

Pick $\epsilon > 0$ and $R \in (0, \frac{1}{2}\|u_0\|)$ sufficiently small such that

$$\frac{\theta}{2} - \lambda(\frac{\epsilon}{2}C_2^2 + C^*R^{p-2}) > 0.$$

Then by (5.18) we have

$$f(u) \geq 0, \ \forall \, u \in \overline{B_R}. \tag{5.19}$$

Furthermore, it is easy to prove that $R < \frac{1}{2}\|u_0\| < r_\lambda$.
Now, let $v_0 = 0, \ v_1 = u_0$. Define

$$\Gamma = \{\gamma \in C^0([0,1], X) : \gamma(0) = v_0, \ \gamma(1) = v_1\},$$
$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} f(\gamma(t)).$$

Thanks to (5.19) and the definition of c, one has

$$c \geq 0 = \max\{f(v_0), \ f(v_1)\},$$

and for every $\gamma \in \Gamma$, there exists a $t \in (0,1)$ such that $\gamma(t) \in X$ and $\|\gamma(t)\| = R$. Then by (5.19) again, we obtain $f(\gamma(t)) \geq 0$. Hence hypothesis (i_1) of Theorem 4.1 is fulfilled.

Finally, let us prove that (i_3) holds. Since f is bounded below, put $\alpha = \inf_{x \in X} f(x)$, then $\alpha < 0 \leq c$. For every $a \geq \alpha$ suppose that $a < \lambda$, then there exist $r > 0$ and $\epsilon_0 > 0$ such that

$$F^r_{a,\epsilon_0} \subseteq int(D_\Psi). \tag{5.20}$$

Indeed, there is $\epsilon_0 > 0$ such that $a + \epsilon_0 \leq \lambda < 2\lambda$.

Inequality (5.6) ensures that

$$\{u \in X : f(u) \leq a + \epsilon_0\} \subseteq \{u \in X : f(u) \leq \lambda\}$$
$$\subsetneqq \{u \in X : \|u\| < r_\lambda\} \subseteq \overline{B_{r_\lambda}} \subseteq D_\Psi.$$

So we immediately have $\{u \in X : f(u) \leq a + \epsilon_0\} \subseteq int(D_\Psi)$.

Since f is coercive, there exists $r > 0$ such that every $u \in X$ satisfies $a - \epsilon_0 \leq f(u) \leq a + \epsilon_0$, and $\|u\| \leq r + 1$, which leads to (5.20), i.e., condition (i_3) holds true.

We are now in a position to apply Theorem 4.1. By this theorem, there exist at least two points $u_1, u_2 \in X$ such that

$$\Phi^0(u_i; v - u_i) + \Psi(v) - \Psi(u_i) \geq 0, \ \forall \, v \in X, \ i = 1, 2.$$

The choice of Ψ gives both $u_i \in K_\lambda$ and $\Phi^0(u_i; v - u_i) \geq 0, \ v \in K_\lambda, \ i = 1, 2$. Namely, u_1, u_2 are solutions to the problem (P_λ). □

Example 5.2 The aim of this example is to exhibit a nontrivial case of set in $H^1_0(\Omega)$ of type (K^g_λ). Let $h : H^1_0(\Omega) \to \mathbb{R}$ be a weakly continuous and convex function. For $\bar{r} > 0$ fixed, $\lambda > 0$, put

$$\bar{r}_\lambda = \sqrt{4\lambda + 2M\bar{r}\mu(\Omega)},$$

with the same notation as before. The ball $\bar{B}(0, \bar{r}_\lambda)$ is a weakly compact subset of $H^1_0(\Omega)$, since h is weakly continuous, there exists $u_0 \in \bar{B}(0, \bar{r}_\lambda)$ such that

$$\gamma = \max_{u \in \bar{B}(0, \bar{r}_\lambda)} h(u) = h(u_0),$$

i.e., $h_{\bar{B}(0,\bar{r}_\lambda)}$ admits a global maximum. Then the set

$$K_\lambda := \{u \in H_0^1(\Omega) \ : \ h(u) \leq \gamma + 1\}$$

is a subset of $H_0^1(\Omega)$ of type (K_λ^g).

Example 5.3 There exist functionals satisfying the conditions of Theorem 5.1. For example

$$g(t) = \begin{cases} |t|(1 - e^{-t^2}), & |t| \leq 1, \\ t(e^{-t^2} - 1), & |t| > 1. \end{cases}$$

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Conflict of interest

The authors declare no conflict of interest.

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