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Research article

On the edge metric dimension of graphs

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Abstract: Let $G = (V, E)$ be a connected graph of order n. $S \subseteq V$ is an edge metric generator of G if any pair of edges in E can be distinguished by some element of S. The edge metric dimension $edim(G)$ of a graph G is the least size of an edge metric generator of G. In this paper, we give the characterization of all connected bipartite graphs with $edim = n - 2$, which partially answers an open problem of Zubrilina (2018). Furthermore, we also give a sufficient and necessary condition for $edim(G) = n - 2$, where G is a graph with maximum degree $n - 1$. In addition, the relationship between the edge metric dimension and the clique number of a graph G is investigated by construction.

Keywords: edge metric dimension; clique number; bipartite graphs
Mathematics Subject Classification: 05C40

1. Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. We follow the notation and terminology of Bondy [1] and Diestel [2]. A *generator* of a metric space is a set S of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of S. Nowadays there exist some different kinds of metric generators in graphs (or networks), each one of them studied in theoretical and applied ways, according to its popularity or to its applications. Nevertheless, there exist quite a lot of other points of view which are still not completely taken into account while describing a graph throughout these metric generators. Inspired by this, Kelenc et al. [4] proposed the concept of edge metric generator of a graph. Since then, there are some results about it, the details refer to [3, 5, 7].

Let $G = (V, E)$ be a graph. For every vertex $v \in V$, the *open neighborhood* of v is $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is defined as $d_G(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The distance between the vertex $w \in V$ and the subgraph $H \subseteq G$ is defined as $d_G(w,\ H) = \min\{d_G(w,\ v)|v \in V(H)\}$. Particularly, when H is an edge $e = uv \in E$, then $d_G(w, e) = \min\{d_G(w, u), d_G(w, v)\}$. A vertex $w \in V$ *distinguishes* two edges $e_1, e_2 \in E$ if $d_G(e_1, w) \neq d_G(e_2, w)$. We say $S \subseteq V(G)$ an *edge metric generator* of G if every pair of edges in E can be distinguished by some element of S. The *edge metric dimension* $edim(G)$ of G is the smallest cardinality of an edge metric generator in G. An *edge metric basis* for G is an edge metric generator of G with cardinality $edim(G)$. For any connected graph G of order n, $1 \leq edim(G) \leq n-1$ are natural bounds on the edge metric dimension. In [4], Kelenc et al. proved that $edim(G) = n-2$ when G is a wheel graph of order $n \geq 6$ or a fan graph of order $n \geq 5$. And further, they also gave the lower bound of edge metric dimension $edim(G) \geq n-2$ for any connected graph G with $\Delta = n-1$, where n is the order of G. And there is no more results for the graph with edge metric dimension of a given other value. So it is interesting to characterize graphs with large edge metric dimension such as $edim = n-1$ and $edim = n-2$. Recently, Zubrilina [6] classified the graphs on n vertices for which $edim(G) = n-1$. They also proposed an open problem as follows.

Problem 1. For which graphs G of order n is $edim(G) = n-2$?

In this paper, we consider the first step for this problem on connected bipartite graphs. And then we classify the class of graphs of $\Delta = n-1$ with $edim = n-2$.

In [8], Zhu et al. constructed connected graphs G of order n such that $edim(G) = n-2$ and $edim(G-e) = n-1$, where $e \in E(G)$. This implies that for any subgraph $H \subseteq G$, the different of the edge metric dimension between the subgraph H and G is not clear. Therefore, we first investigate the relation between $edim(H)$ and $edim(G)$ for any induced subgraph $H \subseteq G$. And further, we also consider the relation between $edim(G)$ and clique number $\omega(G)$ of a graph G.

To end this section, we list some known results which will be used in the sequel.

Proposition 1.1. [4] For any complete bipartite graph $K_{r,t}$ different from $K_{1,1}$, $edim(K_{r,t}) = dim(K_{r,t}) = r+t-2$.

Proposition 1.2. [4] Let G be a connected graph of order n. If there is a vertex $v \in V(G)$ of degree $n-1$, then either $edim(G) = n-1$ or $edim(G) = n-2$.

Proposition 1.3. [4] Let G be a connected graph of order n. If there are two vertices $u, v \in V(G)$ of degree $n-1$, then $edim(G) = n-1$.

2. Graphs with *edim=n−2*

Given two graphs G and H, the *join*, denoted by $G \vee H$, is the graph obtained from G and H by adding all the possible edges between a vertex of G and a vertex of H. The wheel graph $W_{1,n}$ is isomorphic to $C_n \vee K_1$, the fan graph $F_{1,n}$ is isomorphic to $P_n \vee K_1$ and the star graph $S_{1,n}$ is isomorphic to $\overline{K_n} \vee K_1$. The vertex of degree n in the star graph $S_{1,n}$ is denoted as its center vertex.

Proposition 2.1. Let $W_{1,n-1} = C_{n-1} \vee K_1$ be a wheel graph and G be a graph obtained from $W_{1,n-1}$ by deleting any edges on the cycle C_{n-1}, where $n \geq 6$. Then $edim(G) = n-2$.

Proof. Suppose $n \geq 6$ and $V(W_{1,n-1}) = \{x, v_1, v_2, \ldots, v_{n-2}, v_{n-1}\}$, where the vertex x has degree $n-1$ and the vertices $v_1, v_2, \ldots, v_{n-2}, v_{n-1}$ induce the cycle C_{n-1}.

Let S be an edge metric generator of G. If there exist $v_i, v_j \in (V(C_{n-1}) \setminus S)$, then the edges xv_i and xv_j can not be distinguished by any vertex $w \in V(G) \setminus \{v_i, v_j\}$ because $d_G(xv_i, w) = d_G(xv_j, w) = 1$ or $d_G(xv_i, w) = d_G(xv_j, w) = 0$. So $|V(C_{n-1}) \setminus S| \leq 1$ and then $|S| \geq n-2$. From the arbitrariness of S, we have $edim(G) \geq n-2$.

Next, it suffices to show that $edim(G) \leq n-2$. Set $S_0 = \{v_1, v_2, \ldots, v_{n-2}\}$. We show that S_0 is an edge metric generator of G. Let $e = uw$ and $f = yz$ be two non-adjacent edges in G. Then $|S_0 \cap \{u, w, y, z\}| \geq 2$. Without loss of generality, we assume that $u \in S_0$. Since $d_G(e, u) = 0$, $d_G(f, u) \geq 1$, then u distinguishes edges e and f. Let $e' = tu_1$ and $f' = tu_2$ be two adjacent edges in G. If $u_1 \in S_0$ (or $u_2 \in S_0$), then e' and f' can be distinguished by u_1 (or u_2). Otherwise, we have $\{u_1, u_2\} = \{x, v_{n-1}\}$ and $t \in \{v_1, v_{n-2}\}$. First, we consider the case $t = v_1$. Since $n \geq 6$, then $S_0 \setminus \{x, v_{n-1}, v_{n-2}, v_1, v_2\} \neq \emptyset$. It is easy to verify that $\{d_G(e', p), d_G(f', p)\} = \{1, 2\}$ for any vertex $p \in (S_0 \setminus \{x, v_{n-1}, v_{n-2}, v_1, v_2\})$, which implies that p distinguishes edges e' and f'. Then we consider the case $t = v_{n-2}$. Since $n \geq 5$, then $S_0 \setminus \{x, v_{n-1}, v_{n-2}, v_{n-3}, v_1\} \neq \emptyset$. Similarly, we have $\{d_G(e', q), d_G(f', q)\} = \{1, 2\}$ for any vertex $q \in (S_0 \setminus \{x, v_{n-1}, v_{n-2}, v_{n-3}, v_1\})$ and then q distinguishes edges e' and f'. From the arbitrariness of e, f and e', f', we have that S_0 is an edge metric generator of G and $edim(G) \leq n-2$, as desired. □

In [4], Kelenc et al. compute the edge metric dimension for wheel graphs and fan graphs, which implies that the wheel graphs of order $n \geq 6$ and the fan graphs of order $n \geq 5$ have edge metric dimension $edim = n-2$. Actually, Proposition 2.1 generalized these results.

As is known to us, complete bipartite graph with n vertices has edge metric dimension $n-2$. In the following theorem, we characterize all connected bipartite graphs with edge metric dimension $n-2$, which partially answers the open problem 1.

Theorem 2.1. *Let G be a connected bipartite graph of order $n \geq 3$. Then $edim(G) = n-2$ if and only if $G = K_{r,t}$, i.e., G is a complete bipartite graph.*

Proof. Let $G = G[U, V]$ be a connected bipartite graph with $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_t\}$. If $G = K_{r,t}$ is a complete bipartite graph, then $edim(G) = n-2$ by Proposition 1.1.

Conversely, without loss of generality, we assume that $r \geq t$. If $t = 1$, then $G = K_{1,n-1} = S_{1,n-1}$ is a star graph and $edim(G) = n-2$. Now on we assume $t \geq 2$. Denote by H a maximum complete bipartite induced subgraph of G. Obviously, $|E(H)| \geq 2$ since G is connected. Let $H_U = V(H) \cap U = \{u_{p+1}, u_{p+2}, \ldots, u_r\}$ and $H_V = V(H) \cap V = \{v_{q+1}, v_{q+2}, \ldots, v_t\}$ where p, q are integers and $0 \leq p \leq r-1$, $0 \leq q \leq t-1$. Next it suffices to show that G has an edge metric generator of size $n-3$ if G is not complete bipartite, which implies that $H \neq G$ and $p + q \geq 1$.

If $\min\{|H_U|, |H_V|\} \geq 2$, then set $S = V(G) \setminus \{u_{p+1}, v_{q+1}, x\}$ such that there exists one vertex $h \in H_V \setminus \{v_{q+1}\}$, $hx \notin E(G)$ and $x \in V(G) \setminus (H_U \cup H_V)$. On one hand, $V(G) \setminus (H_U \cup H_V) \neq \emptyset$ since $p+q \geq 1$. Without loss of generality, set $x = u_1$. On the other hand, there exists some vertex $h \in H_V$ such that $hx \notin E(G)$ because H is a maximum complete bipartite induced subgraph of G. So an appropriate v_{q+1} can be chosen after relabelling the vertices in H_V if necessary. Next we will prove that S is an edge metric generator of G. For any non-adjacent edges $e_1, e_2 \in E(G)$, we have either e_1 or e_2 has one end vertex $y \in S$ and then $d_G(e_1, y) = 0, d_G(e_2, y) \geq 1$ or $d_G(e_2, y) = 0, d_G(e_1, y) \geq 1$. So y distinguishes edges e_1 and e_2. For any adjacent edges $e_1 = zw, e_2 = zg$ of G, if $w \in S$ (or $g \in S$), then e_1, e_2 can be distinguished by w (or g). Otherwise, we have $\{w, g\} = \{u_1, u_{p+1}\}$ and $z \in V \setminus \{h\}$. Since $hu_{p+1} \in E(G)$ and $hx = hu_1 \notin E(G)$, then $\{d_G(e_1, h), d_G(e_2, h)\} = \{1, 2\}$ and h distinguishes edges e_1 and e_2. Therefore, S is an edge metric generator of G and $edim(G) \leq n-3$.

If $\min\{|H_U|, |H_V|\} = 1$, then without loss of generality, we assume that $H_V = \{v_t\}$. Take $S = V(G) \setminus \{u_{p+1}, v_t, x\}$ such that there exists one vertex $h \in H_U \setminus \{u_{p+1}\}$, $hx \notin E(G)$ and $x \in V \setminus H_V$. On one hand, $V \setminus H_V \neq \emptyset$ since $r \geq t \geq 2$. On the other hand, there exists one vertex $h \in H_U$ such that $hx \notin E(G)$ because H is a maximum complete bipartite induced subgraph of G. So an appropriate u_{p+1} can be chosen after relabelling the vertices in H_U if necessary. With a similar analysis as in the case $\min\{|H_U|, |H_V|\} \geq 2$, we also have S is an edge metric generator of G, which implies that $edim(G) \leq n - 3$. □

On one hand, Theorem 2.1 characterizes all connected bipartite graphs with edge metric dimension $n - 2$. On the other hand, Proposition 2.1 implies that there are connected non-bipartite graphs with edge metric dimension $n-2$. So we will try to give a characterization of general connected graphs with edge metric dimension $n - 2$.

Theorem 2.2. *Let G be a connected graph of order $n \geq 3$. If $H = H[U, V]$ is a complete bipartite spanning subgraph of G such that $|U| = s, |V| = t$, $s, t \geq 1$, then $edim(G) = n - 2$ if and only if there is a vertex $u \in U$ such that for any $x \in N(u) \cap U$, $U \setminus (N(x) \cup N(u)) \neq \emptyset$ if $N(u) \cap U \neq \emptyset$ and there is a vertex $v \in V$ such that for any $y \in N(v) \cap V$, $V \setminus (N(y) \cup N(v)) \neq \emptyset$ if $N(v) \cap V \neq \emptyset$.*

Proof. Let $H = H[U, V]$ be a complete bipartite spanning subgraph of G such that $|U| = s, |V| = t$, $s, t \geq 1$. Obviously, $s + t = n$.

First, we show that $edim(G) \geq n - 2$. Let S be an edge metric generator of G. Suppose that there are two vertices $u_1, u_2 \in U \setminus S$. For each vertex $v \in V$, we have $d_G(u_1v, w) = d_G(u_2v, w) = 1$ for any vertex of $w \in V(G) \setminus \{u_1, u_2, v\}$, which contradicts to that S is an edge metric generator of G. Therefore, $|U \setminus S| \leq 1$ and $|V \setminus S| \leq 1$. Thus $|S| \geq n - 2$ and then $edim(G) \geq n - 2$.

Next, we show that $edim(G) \leq n - 2$ if and only if there is a vertex $u \in U$ such that for any $x \in N(u) \cap U$, $U \setminus (N(x) \cup N(u)) \neq \emptyset$ if $N(u) \cap U \neq \emptyset$ and there is a vertex $v \in V$ such that for any $y \in N(v) \cap V$, $V \setminus (N(y) \cup N(v)) \neq \emptyset$ if $N(v) \cap V \neq \emptyset$.

Without loss of generality, we assume that $s \geq t$. For the case $t = 1$, we have $s = n - t \geq 2$. Set $V = \{v_0\}$. If there is a vertex $u \in U$ such that for any $x \in N(u) \cap U$, $U \setminus (N(x) \cup N(u)) \neq \emptyset$ if $N(u) \cap U \neq \emptyset$, then $S = V(G) \setminus \{u, v_0\}$ is an edge metric generator of G. On one hand, any pair of non-adjacent edges can be distinguished by some element of S since $|S| = n - 2$. On the other hand, we consider two adjacent edges $e_1 = zw_1$ and $e_2 = zw_2$. If $S \cap \{w_1, w_2\} \neq \emptyset$, then e_1, e_2 can be distinguished by any element of $S \cap \{w_1, w_2\}$. If $S \cap \{w_1, w_2\} = \emptyset$, then $\{w_1, w_2\} = \{u, v_0\}$ and $\{e_1, e_2\} = \{zu, zv_0\}$. Obviously, $z \in N(u) \cap U$ since $V = \{v_0\}$. For any element $p \in U \setminus (N(z) \cup N(u))$, we have $d_G(zv_0, p) = 1$, $d_G(zu, p) = 2$ and then each vertex in $U \setminus (N(z) \cup N(u))$ can distinguish edges e_1, e_2. Thus $S = V(G) \setminus \{u, v_0\}$ is an edge metric generator of G and then $edim(G) \leq n - 2$. Conversely, if $edim(G) \leq n - 2$, then there is an edge metric generator S of G such that $|S| = n - 2$ and $|U \setminus S| = 1$, $|V \setminus S| = 1$. Obviously, $V \setminus S = \{v_0\}$. Now we suppose $S = V(G) \setminus \{u, v_0\}$ and there is a vertex $x \in N(u) \cap U$ such that $U \setminus (N(x) \cup N(u)) = \emptyset$. Consider the edges $e_1 = xu, e_2 = xv_0$. For any element $z \in S$, we have $d_G(xu, z) = d_G(xv_0, z) = 1$ which contradicts to that S is an edge metric generator of G. Therefore, there is a vertex $u \in U$ such that for any $x \in N(u) \cap U$, $U \setminus (N(x) \cup N(u)) \neq \emptyset$ if $N(u) \cap U \neq \emptyset$.

Next we consider the case $s \geq t \geq 2$. If there is a vertex $u \in U$ such that for any $x \in N(u) \cap U$, $U \setminus (N(x) \cup N(u)) \neq \emptyset$ if $N(u) \cap U \neq \emptyset$ and there is a vertex $v \in V$ such that for any $y \in N(v) \cap V$, $V \setminus (N(y) \cup N(v)) \neq \emptyset$ if $N(v) \cap V \neq \emptyset$, then $S = V(G) \setminus \{u, v\}$ is an edge metric generator of G. Once $G[U] = \overline{K_s}$, then we choose any element of U as u, so as for v. For any pair of non-adjacent edges, they can be distinguished by some element of S since $|S| = n-2$. For two adjacent edges $e_1 = zw_1$ and $e_2 = zw_2$, if $S \cap \{w_1, w_2\} \neq \emptyset$, then e_1, e_2 can be distinguished by any element of $S \cap \{w_1, w_2\}$, otherwise $\{w_1, w_2\} = \{u, v\}$ and $\{e_1, e_2\} = \{zu, zv\}$. If $z \in U$, then for any element $p \in U \setminus (N(z) \cup N(u))$, we have $d_G(zv, p) = 1$, $d_G(zu, p) = 2$ and then each vertex in $U \setminus (N(z) \cup N(u))$ can distinguish edges e_1, e_2. For the case $z \in V$, edges e_1, e_2 can be distinguished by similar analysis. Thus $S = V(G) \setminus \{u, v\}$ is an edge metric generator of G and then $edim(G) \leq n-2$. Conversely, if $edim(G) \leq n-2$, then there is an edge metric generator S of G such that $|S| = n-2$ and $|U \setminus S| = 1$, $|V \setminus S| = 1$. Now we suppose $S = V(G) \setminus \{u, v\}$. Without loss of generality, we assume that there is a vertex $x \in N(u) \cap U$ such that $U \setminus (N(x) \cup N(u)) = \emptyset$. Consider the edges $e_1 = xu, e_2 = xv$. Each element of S has distance 1 to edges both e_1 and e_2, which comes to a contradiction. Therefore, there is a vertex $u \in U$ such that for any $x \in N(u) \cap U$, $U \setminus (N(x) \cup N(u)) \neq \emptyset$ if $N(u) \cap U \neq \emptyset$. Similarly, there is a vertex $v \in V$ such that for any $y \in N(v) \cap V$, $V \setminus (N(y) \cup N(v)) \neq \emptyset$ if $N(v) \cap V \neq \emptyset$ and thus we complete the proof. □

One can check that Theorem 2.2 implies the results of Proposition 1.2 [4] and Proposition 1.3 [4]. Furthermore, the following corollary of Theorem 2.2 gives a sufficient and necessary condition to determine the value of $edim(G)$ for the graph G with n vertices and maximum degree $\Delta = n-1$.

Corollary 2.1. Let G be a connected graph of order n. If there is a vertex $v \in V(G)$ of degree $n-1$, then $edim(G) = n-2$ if there is a vertex $u \in V(G) \setminus \{v\}$ such that for any $x \in N(u) \setminus \{v\}$, $(V(G) \setminus \{v\}) \setminus (N(x) \cup N(u)) \neq \emptyset$ if $N(u) \setminus \{v\} \neq \emptyset$ and otherwise $edim(G) = n-1$.

3. The edge metric dimension and the clique number

Let $n > k \geq 1$ be positive integers. Denote by $\mathcal{G}$ the graph class consists of graphs G_n^k which are obtained by identifying one vertex of K_k and the center vertex of $S_{1,n-k}$, see Figure 1. Obviously, each element of $\mathcal{G}$ is a connected graph of order n.

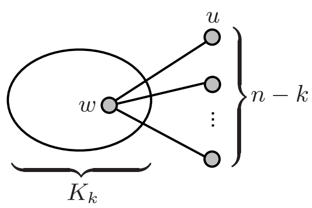

Figure 1. The graph G_n^k of $\mathcal{G}$.

Theorem 3.1. *For any positive integer $p \geq 1$, there exists a connected graph G such that $edim(G) - \omega(G) = p$.*

Proof. Let $n > k \geq 1$ be positive integers such that $p = n-k-2$. Now we prove that $edim(G_n^k) - \omega(G_n^k) = p$ for any graph $G_n^k \in \mathcal{G}$. Obviously, it suffices to show that $edim(G_n^k) = n-2$.

Denote by w as the vertex with degree $n-1$ in G_n^k and u as one of the pendent vertices in G_n^k. Then $edim(G_n^k) \geq n-2$ by Proposition 1.2. Next we will prove $edim(G_n^k) \leq n-2$. Set $S = V(G_n^k) \setminus \{w, u\}$. For any non-adjacent edges $e_1, e_2 \in E(G_n^k)$, we have either e_1 or e_2 has one end vertex $y \in S$ and then $d_{G_n^k}(e_1, y) = 0, d_{G_n^k}(e_2, y) \geq 1$ or $d_{G_n^k}(e_2, y) = 0, d_{G_n^k}(e_1, y) \geq 1$. So y distinguishes edges e_1 and e_2. For any adjacent edges $e_1 = zv, e_2 = zg$ of G_n^k, if $v \in S$ (or $g \in S$), then e_1, e_2 can be distinguished by v (or

g). Otherwise, we have $\{v, g\} = \{w, u\}$, which is impossible. Therefore, S is an edge metric generator of G_n^k and then $edim(G) \leq n - 2$. □

For the complete graph K_n, we have $edim(K_n) = n - 1 = \omega(K_n) - 1$, i.e., $\omega(K_n) - edim(K_n) = 1$. Naturally, what about the value of $\omega(G) - edim(G)$ for any connected graph $G \neq K_n$?

Problem 2. Let $G \neq K_n$ be a connected graph. Is it true that $\omega(G) - edim(G) \leq 1$, i.e., $edim(G) \geq \omega(G) - 1$?

Let $k \geq 2$ be a positive integer. Denote by $\mathcal{H}$ the graph class consists of graphs H_{2k} which are obtained by attaching one end of K_2 to each vertex of K_k. Obviously, each element of $\mathcal{H}$ is a connected graph of order $2k$ and clique number k.

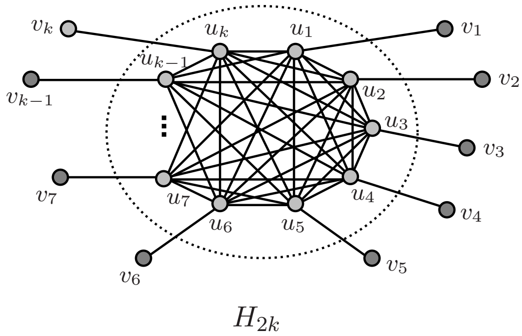

Figure 2. The graph H_{2k} of $\mathcal{H}$.

We claim that $edim(H_{2k}) = k - 1 = \omega(H_{2k}) - 1$, where H_{2k} is depicted in Figure 2. It suffices to show that $S = \{v_1, v_2, \ldots, v_{k-1}\}$ is an edge metric generator of H_{2k}. For any non-adjacent edges $e_1 = w_1w_2, e_2 = w_3w_4$ of H_{2k}, if $S \cap \{w_1, w_2, w_3, w_4\} \neq \emptyset$, then any element of $S \cap \{w_1, w_2, w_3, w_4\}$ can distinguish edges e_1 and e_2. Otherwise, we have e_1 or e_2 or both of them are adjacent to a pendent edge u_iv_i such that $v_i \in S$ and then $d(e_1, v_i) = 1, d(e_2, v_i) = 2$ or $d(e_1, v_i) = 2, d(e_2, v_i) = 1$. So v_i distinguishes edges e_1 and e_2. For any adjacent edges $e_1 = w_5w_6, e_2 = w_5w_7$ of H_{2k}, if $w_6 \in S$ (or $w_7 \in S$), then e_1, e_2 can be distinguished by w_6 (or w_7). Otherwise, we have $\{w_6, w_7\} \subseteq \{u_1, u_2, \ldots, u_k, v_k\}$. Hence, there exists $v_j \in S$ such that $d(e_1, v_j) = 1, d(e_2, v_j) = 2$ or $d(e_1, v_j) = 2, d(e_2, v_j) = 1$. Therefore, S is an edge metric generator of H_{2k} and then $edim(H_{2k}) = k - 1 = \omega(H_{2k}) - 1$. This implies that once the answer of **Problem 2** is true, then the lower bound $edim(G) \geq \omega(G) - 1$ is sharp.

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Conflict of interest

The authors declare no conflict of interest.

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