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Research article

A note on consecutive integers of the form $2^x + y^2$

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Abstract: Let k be a positive integer with $k \geq 2$. Let N_k denote the number of k tuples of consecutive integers with each of them in the form $2^x + y^2$, where x, y are nonnegative integers. In this paper, we investigate the formulas for N_k. Actually, by using some elementary methods, we show that

$$N_k = \begin{cases} +\infty, & \text{if } 2 \leq k \leq 4, \\ 6, & \text{if } k = 5, \\ 3, & \text{if } k = 6, \\ 0, & \text{otherwise.} \end{cases}$$

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1. Introduction

Catalan's conjecture, one of the famous classical problems in number theory, states that the equation

$$x^p - y^q = 1$$

has no solutions in positive integers x and y, other than $3^2 - 2^3 = 1$, where p and q are prime numbers. In 2004, Mihuailesc [1] proved Catalan's conjecture by using the theory of cyclotomic field. On the other hand, some scholars (see [2], and [3–5]) studied the solutions to the general equation

$$a^x - b^y = c, \tag{1.1}$$

where a, b and c are fixed positive integers. In 1936, Pillai conjectured that the number of positive integer solutions (a, b, x, y), with $x \geq 2, y \geq 2$, to Eq (1.1) is finite. This conjecture which is still open for all $c > 1$, amounts to saying that the distance between two consecutive terms in the sequence of all perfect powers tends to infinity. However, it is easy to see that Pillai's conjecture is closely related to the number of consecutive integers tuples of perfect powers. In fact, Catalan's conjecture is equivalent to the statement that no two consecutive integers are perfect powers, other than 2^3 and 3^2. We also easily know that there are no four consecutive integers with each of them being perfect powers, since any set of four consecutive integers must contain one integer of the form $4n + 2$ which can not be a perfect power. Are there three consecutive integers with each of them being perfect powers? In 1962, Chao Ko [6], by supplying a sufficient and necessary condition for the equation $x^p - y^q = 1$ to be solvable with positive integers x and y, showed that no three consecutive integers are powers of other positive integers,

Let k be an integer with $k \geq 2$. In this paper, let us turn our attention to the number of k tuples of consecutive integers such that each of them is the sum of two perfect powers. For any positive integer n, we call the integer n a 22-*STP* (STP means Sum of Two Powers) number if it can be expressed in the form $2^x + y^2$ with x and y being nonnegative integers. Furthermore, we call a k-tuple $(a_1, a_2, \cdots, a_k)$ *nice* if $a_1, a_2, \cdots, a_k$ are increasingly consecutive integers and each of them is a 22-STP number. Then an interesting question is raised naturally as follows.

Question 1.1. For each integer $k \geq 2$, how many nice k-tuples are there?

In this paper, we mainly study Question 1.1 by utilizing some elementary tools in number theory. In fact, we provide the following theorem, which gives the complete answer to Question 1.1.

Theorem 1.2. *Let k be an integer with $k \geq 2$. Each of the following is true.*

(a) If $2 \leq k \leq 4$, then there exist infinitely many nice k-tuples.

(b) If $k = 5$, then there are only six nice 5-tuples. Moreover, the only six nice 5-tuples can be listed as follows:

$$(1, 2, 3, 4, 5), (2, 3, 4, 5, 6), (8, 9, 10, 11, 12), (9, 10, 11, 12, 13),$$

$$(288, 289, 290, 291, 292), \textit{and } (289, 290, 291, 292, 293).$$

(c) If $k = 6$, then there are only three nice 6-tuples. Moreover, the only three nice 6-tuples can be list as follows:

$$(1, 2, 3, 4, 5, 6), (8, 9, 10, 11, 12, 13), \textit{and } (288, 289, 290, 291, 292, 293).$$

(d) If $k \geq 7$, then there is no nice k-tuple at all.

This paper is organized as follows. First in Section 2, we are mainly dedicated in presenting the proof of Theorem 1.2 by using the method of elementary number theory, especially the tool of modulo cover. In Section 3, we propose a general question for readers who are interested in this topic to do further.

2. Proof of Theorem 1.2

In this section, we are concentration on the proof of Theorem 1.2.

Proof of Theorem 1.2. First of all, it is noticed that Parts (c) and (d) of Theorem 1.2 follows immediately from Part (b). So it is sufficient to show that Parts (a) and (b) are true.

For Part (a), we only need to prove that there exist infinitely nice 4-tuples, since from which one can easily deduce that Part (a) holds for any $k \in \{2, 3\}$. One notes that the integers $y^2 + 2^0, y^2 + 2^1, y^2 + 2^2$ are 22-STP numbers for every nonnegative integer y. So the proof of Part (a) will be done if we show that $y^2 + 3$ is 22-STP finitely often. Now let us consider the diophantine equation $y^2 + 3 = 2^x + y'^2$. We claim that this equation at least has one solution $(y, y') \in \mathbb{N}^2$ for any $x \in \mathbb{N}_{\geq 2}$. In fact, since

$$y^2 + 3 = 2^x + y'^2 \Leftrightarrow y^2 - y'^2 = 2^x - 3 \Leftrightarrow (y + y')(y - y') = 2^x - 3,$$

then one can take $y - y' = 1$ and $y + y' = 2^x - 3$, that is, $(y, y') = (2^{x-1} - 1, 2^{x-1} - 2) \in \mathbb{N}^2$. Thus, this completes the proof of Part (a).

Now we turn our attention to Part (b). First, we make a key table as follows to show the results of $2^x + y^2 \pmod 8$.

$2^x + y^2 \pmod 8$	$x = 0$	$x = 1$	$x = 2$	$x \geq 3$
$y \equiv 1 \pmod 2$	2	3	5	1
$y \equiv 0 \pmod 4$	1	2	4	0
$y \equiv 2 \pmod 4$	5	6	0	4

One knows from the key table that there is no 22-STP number which is congruent to 7 modulo 8. It then follows that any nice 5-tuple must contain an integer with $3 \pmod 8$. Let this integer be denoted by A. Thus by the key table we know that $A = y_0^2 + 2$ for some odd positive integer y_0. So we can list all possible nice 5-tuples containing $A = y_0^2 + 2$ as follows:

(i) $(y_0^2 + 1, y_0^2 + 2, y_0^2 + 3, y_0^2 + 4, y_0^2 + 5)$,

(ii) or $(y_0^2, y_0^2 + 1, y_0^2 + 2, y_0^2 + 3, y_0^2 + 4)$,

(iii) or $(y_0^2 - 1, y_0^2, y_0^2 + 1, y_0^2 + 2, y_0^2 + 3)$,

since $y_0^2 + 2 \equiv 3 \pmod 8$ and any number in all possible nice 5-tuples is not congruent to 7 modulo 8. Next, let's discuss the details case by case.

Case 1. Suppose the nice 5-tuple is $(y_0^2 + 1, y_0^2 + 2, y_0^2 + 3, y_0^2 + 4, y_0^2 + 5)$. Note that $y_0^2 + 2 \equiv 3 \pmod 8$, then $y_0^2 + 5 \equiv 6 \pmod 8$. It follows from the key table that there exists a positive integer z with $z \equiv 2 \pmod 4$ such that $y_0^2 + 5 = z^2 + 2$. So one has that $(z + y_0)(z - y_0) = 3$, which implies that $z + y_0 = 3, z - y_0 = 1$, i.e., $z = 2, y_0 = 1$. It gives the unique nice 5-tuple $(2, 3, 4, 5, 6)$ of this case since that $2 = 2^0 + 1^2, 3 = 2^1 + 1^2, 4 = 2^2 + 0^2, 5 = 2^2 + 1^2, 6 = 2^1 + 2^2$.

Case 2. Suppose the nice 5-tuple is $(y_0^2, y_0^2 + 1, y_0^2 + 2, y_0^2 + 3, y_0^2 + 4)$. First, since y_0^2 is a 22-STP number, then there exist two nonnegative integers s and u such that $y_0^2 = 2^s + u^2$, which one writes as

$$(y_0 - u)(y_0 + u) = 2^s. \tag{2.1}$$

Now we solve Eq (2.1) by dividing s into two subcases.

Subcase 2.1. Let $0 \leq s \leq 4$.

- If $s = 0$, then $y_0 = 1$ and $u = 0$, which gives a nice 5-tuple $(1, 2, 3, 4, 5)$ since $1 = 2^0 + 0^2$.
- If $s = 1$, then it is easy to see that Eq (2.1) has no no integer solution since $y_0 + u$ and $y_0 - u$ have the same parity.
- If $s = 2$, then Eq (2.1) is equivalent to the equations $y_0 + u = 2$ and $y_0 - u = 2$ since $y_0 + u$ and $y_0 - u$ have the same parity, which gives $y_0 = 2$, a contradiction with y_0 being odd.
- If $s = 3$, then Eq (2.1) implies that $y_0 + u = 4$ and $y_0 - u = 2$ since $y_0 + u$ and $y_0 - u$ have the same parity. So $y_0 = 3$. It gives a nice 5-tuple $(9, 10, 11, 12, 13)$ since $9 = 2^3 + 1^2, 10 = 2^0 + 3^2, 11 = 2^1 + 3^2, 12 = 2^3 + 2^2, 13 = 2^2 + 3^2$.
- If $s = 4$, then from Eq (2.1), we have that $y_0 + u = 4$ and $y_0 - u = 4$, or $y_0 + u = 8$ and $y_0 - u = 2$ since $y_0 + u$ and $y_0 - u$ have the same parity. So $y_0 = 4$ or 5. Note that y_0 was odd, then $y_0 = 5$, which gives a possible nice 5-tuple $(25, 26, 27, 28, 29)$. But it is easy to check that 28 is not a 22-STP integer. So the 5-tuple $(25, 26, 27, 28, 29)$ is not nice.

Subcase 2.2. Suppose $s \geq 5$. First, one notes that $\gcd(y_0 + u, y_0 - u) \mid 2y_0$ and y_0 is odd. Then, by (2.1) we have that $y_0 - u = 2, y_0 - u = 2^{s-1}$, which implies that $y_0 = 2^{s-2} + 1$. Let's now focus on the 22-STP number $y_0^2 + 3$. One can write that

$$y_0^2 + 3 = 2^t + w^2 \tag{2.2}$$

for some two nonnegative integers t and w. Now we discuss the details for t by splitting into following cases.

Subcase 2.2.1. Assume $0 \leq t \leq 2$.

- ⋆ If $t = 0$, we then easily find that Eq (2.2) has no integer solution.
- ⋆ If $t = 1$, then Eq (2.2) can be reduced to that $y_0 = 1$ and $w = 1$, a contradiction, since y_0 was odd.
- ⋆ If $t = 2$, then Eq (2.2) is equivalent to that $y_0 + w = 1$ and $y_0 - w = 1$, i.e., $y_0 = 1, w = 0$. This gives the nice 5-tuple $(1, 2, 3, 4, 5)$ which was obtained in Subcase 2.1.

Subcase 2.2.2. Assume $t \geq 3$. Note that $y_0 = 2^{s-2} + 1$ with $s \geq 5$. By substituting $y_0 = 2^{s-2} + 1$ into (2.2), we have that

$$2^{2s-6} + 2^{s-3} - 2^{t-2} = (\frac{w}{2})^2 - 1. \tag{2.3}$$

It yields that $w \equiv 2 \pmod 4$ since $s \geq 5$ and $t \geq 3$. In (2.3), for simplicity, let $s - 3 = c \geq 2, t - 2 = a \geq 1, b = \frac{w}{2}$, then (2.3) becomes that

$$2^{2c} + 2^c + 1 = 2^a + b^2. \tag{2.4}$$

It then follows that

$$(2^c + 1 - b)(2^c + 1 + b) = (2^c + 1)^2 - b^2 = 2^a + 2^c > 0,$$

which implies that $b \le 2^c$. So

$$2^a + 2^c = (2^c + 1)^2 - b^2 \ge (2^c + 1)^2 - (2^c)^2 = 2^{c+1} + 1 > 2^{c+1}.$$

Then $2^a > 2^c$, i.e., $a > c$. Thus we rewrite (2.4) as

$$(2^c + 1 - b)(2^c + 1 + b) = 2^c(2^{a-c} + 1). \tag{2.5}$$

However, one notes that $\gcd(2^c + 1 - b, 2^c + 1 + b) \mid 2b$ and b is odd. Then we deduce that one of $2^c + 1 - b$ and $2^c + 1 + b$ has at most one factor of 2. By (2.5), we then know that 2^{c-1} divides one of $2^c + 1 - b$ and $2^c + 1 + b$. Hence $b \equiv \pm 1 \pmod{2^{c-1}}$. This together with $1 \le b \le 2^c$ gives that $b \in \{1, 2^{c-1} - 1, 2^{c-1} + 1, 2^c - 1\}$.

- ⋆ If $b = 1$, then Eq (2.4) is equivalent to the equation $2^c + 1 = 2^{a-c}$ which has no integer solution since $c \ge 2$ and $a > c$.
- ⋆ If $b = 2^{c-1} - 1$, then Eq (2.4) turns out to be $3 \times 2^{c-2} = 2^{a-c} - 2$. It then follows that $a - c = 3$ and $c - 2 = 1$, i.e., $c = 3$ or $s = 6$. So $y_0 = 2^{s-2} + 1 = 17$. This gives us a new nice 5-tuple $(289, 290, 291, 292, 293)$ since $289 = 2^6 + 15^2, 290 = 2^0 + 17^2, 291 = 2^1 + 17^2, 292 = 2^8 + 6^2, 293 = 2^3 + 17^2$.
- ⋆ If $b = 2^{c-1} + 1$, then Eq (2.4) is simplified to that $2^{c-1} + 2^{c-2} = 2^{a-c}$. Clearly, the later equation has no integer solution.
- ⋆ If $b = 2^c - 1$, then Eq (2.4) is reduced to be $2^{c+1} = 2^c(2^{a-c} + 1)$, a contradiction.

CASE 3. Suppose the nice 5-tuple is $(y_0^2-1, y_0^2, y_0^2+1, y_0^2+2, y_0^2+3)$. Then $(y_0^2, y_0^2+1, y_0^2+2, y_0^2+3, y_0^2+4)$ is also a nice 5-tuple since $y_0^2 + 4 = y_0^2 + 2^2$. It then follows from Case 2 that $y_0 \in \{1, 3, 5, 17\}$. But one checks that $(y_0^2 - 1, y_0^2, y_0^2 + 1, y_0^2 + 2, y_0^2 + 3)$ can not be a nice 5-tuple for each $y_0 \in \{1, 5\}$ since both 0 and 28 are not 22-STP integers. But $y_0 = 3$ or 17 give the new nice 5-tuples $(8, 9, 10, 11, 12)$ and $(288, 289, 290, 291, 292)$, as $8 = 2^2 + 2^2$ and $288 = 2^5 + 16^2$.

Hence, the above cases give the all nice 5-tuples, which are $(1, 2, 3, 4, 5), (2, 3, 4, 5, 6), (8, 9, 10, 11, 12), (9, 10, 11, 12, 13), (288, 289, 290, 291, 292), (289, 290, 291, 292, 293)$.

To here, the proof of Part (b) is complete.

Thus we finish the proof of Theorem 2.2. □

3. A general question

In this section, we will propose one general question. Let a, b be given integers with $a, b \ge 2$. Let n be a positive integer. We call the integer n an *ab-STP* number if it can be expressed in the form $a^x + y^b$ with x and y being nonnegative integers. Furthermore, we call a k-tuple $(a_1, a_2, \cdots, a_k)$ *ab-nice* if $a_1, a_2, \cdots, a_k$ are increasingly consecutive integers and each of them is an *ab*-STP number. Now we propose the following general question which seems little hard, as follows.

Question 3.1. Let a and b be given integers with $a, b \ge 2$. For each integer $k \ge 2$, find all *ab-nice* k-tuples.

4. Conclusions

The gap in integer sequences are wide problems in Number Theory. The gap of primes $|p_n - p_{n+1}|$ is one of the most important topics in analytic Number Theory. In the field of Diophantine analysis, there are many open questions on the gap of the powers $|x^m - y^n|$. In this paper, we considered k-tuples of consecutive integers $(a_1, a_2, \cdots, a_k)$ such that each of them is the sum of two perfect powers. We used some elementary methods in number theory to prove that there exists infinitely many 4-tuples with each elements of the form $2^x + y^2$, no such 7-tuples exists, and such quintuples and sextuples were listed. At the end of this paper, a general question was also proposed for the interested readers there.

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Conflict of interest

The authors declare that there is no conflict of interest.

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